Theorem 0.1 (Abel's Theorem for powerseries). *Suppose that* $R \in (0, \inf)$ *is the radius of convergence for the powerseries* $\sum_{k=0}^{\infty} a_k(x-a)^k$. *Then:*

- ***(a)*** *If* $\sum_{k=0}^{\infty} R^k$ *converges, then* $\sum_{k=0}^{\infty} a_k(x-a)^k$ *converges uniformly on* $[a-R+\epsilon, a+R]$ *for any* $\epsilon > 0$.
- ***(b)*** *If* $\sum_{k=0}^{\infty}(-R)$ *converges, then* $\sum_{k=0}^{\infty} a_k(x-a)^k$ *converges uniformly on* $[a-R, a+R-\epsilon]$ *for any* $\epsilon > 0$.

Abel's Theorem follows from the following proposition:

Proposition 0.1. *Let* r, s *be two real numbers with* $r < s$. *If* $\sum_{k=0}^{\infty} a_k(x-a)^k$ *converges at* r *and at* s *then it converges uniformly on the interval* $[r, s]$.

Proof.
CLAIM 1: If the powerseries $\sum_{k=0}^{\infty} a_k x^k$ converges at 1 then it converges uniformly on the interval $[0, 1]$.
Proof of CLAIM 1: Let $\epsilon > 0$ be given. According to the Cauchy Criterion for series we have to prove that there exists a number n^* so that for all $m \geq n^*$ and $k \geq 0$ and $x \in [0, 1]$

$$\left|a_m x^m + a_{m+1}x^{m+1} + a_{m+2}x^{m+2} + \cdots + a_{m+k}x^{m+k}\right| < \epsilon.$$

The powerseries $\sum_{k=0}^{\infty} a_k$ converges and hence, according to Cauchy's Criterion, there exists an index n^* so that for all $m \geq n^*$ and $k \geq 0$

$$\left|a_m + a_{m+1} + a_{m+2} + \cdots + a_{m+k}\right| < \epsilon.$$

Let

$$s_0 := a_m, s_1 := a_m + a_{m+1}, s_2 := a_m + a_{m+1} + a_{m+2}$$

and in general

$$s_k := a_m + a_{m+1} + a_{m+2} + \cdots + a_{m+k}.$$

Then:

$$\begin{aligned}
&a_m x^m + a_{m+1}x^{m+1} + a_{m+2}x^{m+2} + \cdots + a_{m+k}x^{m+k} = \\
&s_0 x^m + (s_1 - s_0)x^{m+1} + (s_2 - s_1)x^{m+2} + \cdots + (s_k - s_{k-1})x^{m+k} = \\
&s_0(x^m - x^{m+1}) + s_1(x^{m+1} - x^{m+2}) + s_2(x^{m+2} - x^{m+3}) + \ldots \\
&\qquad\qquad \cdots + s_{k-1}(x^{m+k-1} - x^{m+k}) + s_k x^{m+k} = \\
&x^m(1-x)\big(s_0 + s_1 x + x_2 x^2 + \cdots + s_{k-1}x^{k-1}\big) + s_k x^{m+k}.
\end{aligned}$$

Then, using the triangle inequality and the fact that $x \geq 0$ and $1 - x \geq 0$:

$$\begin{aligned}
&\big|a_m x^m + a_{m+1} x^{m+1} + a_{m+2} x^{m+2} + \cdots + a_{m+k} x^{m+k}\big| = \\
&\big|x^m(1-x)\big(s_0 + s_1 x + x_2 x^2 + \cdots + s_{k-1} x^{k-1}\big) + s_k x^{m+k}\big| \leq \\
&x^m(1-x)\big(|s_0| + |s_1| x + |s_2| x^2 + \cdots + |s_{k-1}| x^{k-1}\big) + |s_k| x^{m+k} < \\
&x^m(1-x)\big(\epsilon + \epsilon x + \epsilon x^2 + \cdots + \epsilon x^{k-1}\big) + \epsilon x^{m+k} = \\
&\epsilon x^m(1-x)\big(1 + x + x^2 + \cdots + x^{k-1}\big) + \epsilon x^{m+k} =^* \\
&\epsilon\big(x^m(1-x^k) + x^{m+k}\big) = \epsilon x^m \leq \epsilon.
\end{aligned}$$

Note that equality $=^*$ holds in case $x < 1$ because then

$$1 + x + x^2 + \cdots + x^{k-1} = \frac{1 - x^k}{1 - x}.$$

If $x = 1$ then $1 - x = 0$ and then we replace the last line by $=^* \epsilon x^{m+k} = \epsilon$.

End of proof of CLAIM 1.

CLAIM 2: If the powerseries $\sum_{k=0}^{\infty} a_k x^k$ converges at $s \neq 0$ then it converges uniformly on the interval $[0, s]$ if $s > 0$ and on the interval $[s, 0]$ if $s < 0$.
Proof of CLAIM 2: Given $\epsilon > 0$. The series $\sum_{k=0}^{\infty} (a_k s^k) x^k$ converges at $x = 1$ and hence is uniformly convergent on $[0, 1]$ according to Claim 1. Hence there exists an n^* so that for all $m > n \geq n^*$ and all $x \in [0, 1]$

$$\big|a_n s^n x^n + a_{n+1} s^{n+1} x^{n+1} + \cdots + a_m s^m x^m\big| < \epsilon.$$

If $x \in [0, s]$ for $s > 0$, or in $[s, 0]$ for $s < 0$, then $x/s \in [0, 1]$ and hence for all $m > n \geq n^*$

$$\begin{aligned}
&\big|a_n x^n + a_{n+1} x^{n+1} + \cdots + a_m x^m\big| = \\
&\big|a_n s^n (x/s)^n + a_{n+1} s^{n+1} (x/s)^{n+1} + \cdots + a_m s^m (x/s)^m\big| < \epsilon.
\end{aligned}$$

End of proof of CLAIM 2.

CLAIM 3: If the powerseries $\sum_{k=0}^{\infty} a_k x^k$ converges at $r < 0$ and at $s > 0$ then it converges uniformly on the interval $[r, s]$.

Proof of CLAIM 3: Given $\epsilon > 0$. According to Claim 2 There exists an index n_1 so that for all $n > m \geq n_1$ and all $x \in [r, 0]$

$$\left| a_m x^m + a_{m+1} x^{m+1} + a_{m+2} x^{m+2} + \cdots + a_{m+k} x^{m+k} \right| < \epsilon.$$

There exists an index n_2 so that for all $x \in [0, s]$ and all $n > m \geq n_2$

$$\left| a_m x^m + a_{m+1} x^{m+1} + a_{m+2} x^{m+2} + \cdots + a_{m+k} x^{m+k} \right| < \epsilon.$$

Hence, for all $m > n \geq \max\{n_1, n_2\}$ and all $x \in [r, s]$

$$\left| a_m x^m + a_{m+1} x^{m+1} + a_{m+2} x^{m+2} + \cdots + a_{m+k} x^{m+k} \right| < \epsilon.$$

End of proof of CLAIM 3.

CLAIM 4: If the powerseries $\sum_{k=0}^{\infty} a_k x^k$ converges uniformly on $[r, s]$ and if $r \leq r_1 < s_1 \leq s$ then it converges uniformly on the interval $[r_1, s_1]$.
Proof of CLAIM 4: Given $\epsilon > 0$. Because the powerseries $\sum_{k=0}^{\infty} a_k x^k$ converges uniformly on the interval $[r, s]$ there exists an index n^* so that for all $x \in [r, s]$ and all $n > m \geq n^*$

$$\left| a_m x^m + a_{m+1} x^{m+1} + a_{m+2} x^{m+2} + \cdots + a_{m+k} x^{m+k} \right| < \epsilon. \tag{1}$$

It follows that inequality (1) holds for all $x \in [r_1, s_1]$.

End of proof of CLAIM 4.

It is easy to see that Proposition 0.1 follows from CLAIMS 2,3 and 4.

□