The subset R of $\mathbb{R}^n$ is a *closed rectangle* if there are n non-empty closed intervals $\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$ so that

$$R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n].$$

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The volume $\mu(R)$ of the rectangle R, in both cases, is

$$\mu(R) := (b_1 - a_1) \cdot (b_2 - a_2) \cdot \cdots \cdot (b_n - a_n).$$

Definition 1. *The subset S of $\mathbb{R}^n$ has* (Lebesgue) measure zero *if for every $\epsilon > 0$ there exist open rectangles $R_1, R_2, R_3, \ldots$ so that*

$$S \subseteq \bigcup_{i=1}^{\infty} R_i \quad \textit{and} \quad \sum_{i=1}^{\infty} \mu(R_i) < \epsilon.$$

Note that if $T \subseteq S$ and S has measure zero then T has measure zero.

Lemma 1. *If $S_1, S_2, S_3, S_4, \ldots$ are countably many sets of measure zero then the set*

$$S = \bigcup_{i \in \mathbb{N}} S_i$$

has measure zero.

Proof. Let $\epsilon > 0$ be given. There exists, for every $i \in \mathbb{N}$, a set $\mathcal{R}_i$ of open rectangles with

$$S_i \subseteq \bigcup \mathcal{R}_i \quad \text{and} \quad \sum_{R \in \mathcal{R}_i} \mu(R) < \frac{\epsilon}{2^i}.$$

Then

$$S \subseteq \bigcup_{i \in \mathbb{N}} \bigcup \mathcal{R}_i \quad \text{and} \quad \sum_{i \in \mathbb{N}} \sum_{R \in \mathcal{R}_i} \mu(R) < \sum_{i \in \mathbb{N}} \frac{\epsilon}{2^i} = \epsilon.$$

□

Lemma 2. *Let P be a partition of the rectangle R of $\mathbb{R}^n$. Then the set of points on the boundaries of the rectangles in the partition P has measure zero.*

Proof. Let R be an $(n-1)$-dimensional rectangle in $\mathbb{R}^n$. Then for every $\epsilon > 0$ there exists an n-dimensional rectangle R' of $\mathbb{R}^n$ with $\mu(R') < \epsilon$ and with $R \subseteq R'$. Hence R has measure zero.

The set of points on the boundaries of the rectangles in the partition P can be covered by finitely many $(n-1)$-dimensional rectangles. The Lemma follows now from Lemma 1. □

Definition 2. *The function* $f : \mathbb{R}^n \to \mathbb{R}$ *has* bounded support *if there exists a rectangle* R *so that* $f(x_1, x_2, \dots, x_n) = 0$ *whenever* $(x_1, x_2, \dots, x_n) \notin R$.

Definition 3. *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be a function. The* oscillation of the function f at $a \in \mathbb{R}^n$, *denoted by* $\mathbf{O}(f, a)$, *is*

$$\mathbf{O}(f,a) := \inf_U \Big\{ \sup\{|f(x) - f(y)| : x, y \in U\} : \textit{where } U \textit{ is a neighborhood of } a \Big\}.$$

Note that $\mathbf{O}(f, a) \geq 0$.

Lemma 3. *Let* $\mathbf{O}(f, a) < \epsilon$. *There exists an open rectangle* R *containing* a *so that*

$$\sup\{f(x) : x \in R\} - \inf\{f(x) : x \in R\} < \epsilon. \tag{1}$$

Proof. If $\mathbf{O}(f, a) < \epsilon$ then there exists an open rectangle R so that

$$\sup\{|f(x) - f(y)| : x, y \in R\} = \epsilon - \delta \quad \text{for some } \delta > 0$$

and hence $f(x) - f(y) \leq \epsilon - \delta$ for all $x, y \in R$ which in turn implies (1). □

Lemma 4. *Let* $\epsilon > 0$. *If for every open rectangle* R *containing* a *there are points* $x, y \in R$ *with* $f(x) - f(y) > \epsilon$ *then* $\mathbf{O}(f, a) \geq \epsilon$.

If $\mathbf{O}(f, a) > \epsilon$ *then for every open rectangle* R *containing* a *there are points* $x, y \in R$ *with* $f(x) - f(y) \geq \epsilon$.

Proof. For every open rectangle R containing a there are points $x, y \in R$ with $f(x) - f(y) > \epsilon$ implies that for every open rectangle R containing a there are points $x, y \in R$ with $|f(x) - f(y)| > \epsilon$ implies that $\sup\{|f(x) - f(y)| : x, y \in R\} > \epsilon$ for every open rectangle R containing a implies that $\mathbf{O}(f, a) \geq \epsilon$.

$\mathbf{O}(f, a) > \epsilon$ implies that $\sup\{|f(x) - f(y)| : x, y \in R\} > \epsilon$ for every open rectangle R containing a implies that for every open rectangle R containing a there are points $x, y \in R$ with $|f(x) - f(y)| > \epsilon$ implies that for every open rectangle R containing a there are points $x, y \in R$ with $f(x) - f(y) > \epsilon$. □

Lemma 5. *Let* $\epsilon > 0$ *and* $\mathbf{O}(f,a) \geq \epsilon$. *Then for every open rectangle containing* a

$$\sup\{f(x) : x \in R\} - \inf\{f(x) : x \in R\} \geq \epsilon.$$

Proof. If the Lemma does not hold then there exists an open rectangle R containing a with

$$\sup\{f(x) : x \in R\} - \inf\{f(x) : x \in R\} = \epsilon - \delta \text{ for some } \delta > 0.$$

Hence $f(x) - f(y) \leq \epsilon - \delta$ for all points $x, y \in R$ which in turn implies $\sup\{|f(x) - f(y)| < \epsilon$ in contradiction to $\mathbf{O}(f,a) \geq \epsilon$. □

Lemma 6. *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be a function. Then* $\mathbf{O}(f,a) = 0$ *if and only if* f *is continuous at* a.

Proof. Let f be continuous at a and $\epsilon > 0$. There exists an open rectangle R containing a so that $|f(x) - f(a)| < \epsilon/2$ for all $x \in R$. Hence for all $x, y \in R$

$$|f(x) - f(y)| = |f(x) - f(a) + f(a) - f(x)| \leq$$
$$|f(x) - f(a)| + |f(a) - f(y)| \leq \epsilon/2 + \epsilon/2 = \epsilon.$$

Hence $\sup\{|f(x) - f(y)| : x, y \in R\} \leq \epsilon$ which in turn implies that $\mathbf{O}(f,a) \leq \epsilon$. Because $\epsilon > 0$ was chosen arbitrary it follows that $\mathbf{O}(f,a) = 0$.

Let $\mathbf{O}(f,a) = 0$ and $\epsilon > 0$ and hence $\mathbf{O}(f,a) < \epsilon$. There exists, according to Lemma 3, an open rectangle R containing a so that

$$\sup\{f(x) : x \in R\} - \inf\{f(x) : x \in R\} < \epsilon.$$

Hence $|f(x) - f(a)| < \epsilon$ for all $x \in R$. □

Lemma 7. *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be a function and* $\epsilon > 0$ *and* S_ϵ *the set of points* a *for which* $\mathbf{O}(f,a) \geq \epsilon$. *The set* S_ϵ *is closed.*

Proof. Let a be an accumulation point of S_ϵ. Assume for a contradiction that $a \notin S_\epsilon$, that is $\mathbf{O}(f,a) < \epsilon$. Then there exists an open rectangle R so that

$$\sup\{|f(x) - f(y)| : x, y \in R\} = \epsilon - \delta \text{ for some } \delta > 0$$

and hence

$$f(x) - f(y) \leq \epsilon - \delta \text{ for all } x, y \in R. \tag{2}$$

Because a is an accumulation point of S_ϵ there exists an element, say $b \in S_\epsilon$, with $b \in R$. It follows from Lemma 4 and from $\mathbf{O}(f, b) > \epsilon - \delta/2$ that there are points $x, y \in R$ with $f(x) - f(y) \geq \epsilon - \delta/2$ in contradiction to Inequality (2). □

Theorem 1 (Lebesgue's Theorem for Riemann Integrability)**.**
Let $f : \mathbb{R}^n \to \mathbb{R}$ be a bounded function with bounded support. Then f is Riemann integrable if and only if the set of points in $\mathbb{R}^n$ at which f is discontinuous has (Lebesgue) measure zero.

Proof. Let S be the set of discontinuities of f and R a closed rectangle so that $f(x_1, x_2, \ldots, x_n) = 0$ whenever $(x_1, x_2, \ldots, x_n) \notin R$. Let $|f| < M \in \mathbb{R}$.

Assume first that S has measure zero. We have to show that f is Riemann integrable.

Let $\epsilon > 0$ be given. We have to find a partition $P = (P_1, P_2, \ldots, P_n)$ for which the difference of the upper sum and the lower sum is smaller than ϵ.

Let S_ϵ be the set of points a in $\mathbb{R}^n$ for which the oscillation

$$\mathbf{O}(f, a) \geq \frac{\epsilon}{2\mu(R)}.$$

Let S_ϵ be the set of points a in $\mathbb{R}^n$ for which the oscillation $\mathbf{O}(f, a) \geq \epsilon$. It follows that S_ϵ is bounded because $S_\epsilon \subseteq R$ and because it is closed according to Lemma 7 it is compact. Also $S_\epsilon \subseteq S$ according to Lemma 6. Hence S_ϵ has measure zero.

Let $R_1, R_2, R_3, \ldots$ be open rectangles so that

$$S_\epsilon \subseteq \bigcup_{i=1}^{\infty} R_i \quad \text{and} \quad \sum_{i=1}^{\infty} \mu(R_i) < \frac{\epsilon}{4M}.$$

Because S_ϵ is compact there exist finitely many of those rectangles, say $\{T_1, T_2, \ldots, T_m\}$, with $S_\epsilon \subseteq T_1 \cup T_2 \cup \cdots \cup T_m := T$. The set T is open and hence the complement $\overline{T} := R \setminus T$ of T in R is closed and bounded and hence compact.

Let $b \in \overline{T}$ and hence not in S_ϵ. According to Lemma 3 there exists an open rectangle R_b containing b so that

$$\sup\{f(x) : x \in R_b\} - \inf\{f(x) : x \in R_b\} < \frac{\epsilon}{2\mu(R)}.$$

The set of rectangles $\{R_b : b \in \overline{T}\}$ is an open cover of the compact set $\overline{T}$ and hence there is a finite subcover, say $\{T'_1, T'_2, \ldots, T'_r\}$, of open rectangles of $\overline{T}$. Let

$$\mathcal{A} := \{T_1, T_2, \ldots, T_m\} \cup \{T'_1, T'_2, \ldots, T'_r\}.$$

Then $\mathcal{A}$ is an open cover of R.

Let P be a partition of the rectangle R so that every rectangle of the partition P has the property that if it has a point in common with some rectangle $A \in \mathcal{A}$ then it is a subset of the closure of A. Let $\mathcal{A}_1$ be the set of rectangles in $\mathcal{A}$ which are a subset of the closure of a rectangle in the set $\{T_1, T_2, \ldots, T_m\}$. Let $\mathcal{A}_2$ be the rectangles in $\mathcal{A}$ not in $\mathcal{A}_1$. Note that every rectangle in $\mathcal{A}_2$ is a subset of the closure a rectangle in the set $\{T'_1, T'_2, \ldots, T'_r\}$. Then

$$U(P,f) - L(P,f) \leq \sum_{A \in \mathcal{A}_1} 2M\mu(A) + \sum_{A \in \mathcal{A}_2} \frac{\epsilon}{2\mu(R)}\mu(A) =$$

$$2M \sum_{A \in \mathcal{A}_1} \mu(A) + \frac{\epsilon}{2\mu(R)} \sum_{A \in \mathcal{A}_2} \mu(A) \leq 2M\frac{\epsilon}{4M} + \frac{\epsilon}{2\mu(R)}\mu(R) = \epsilon.$$

For the other direction we assume that f is Riemann integrable. We have to show that S has measure zero.

For $r > 0$ let S_r be the set of elements $a \in S$ with $\mathbf{O}(f,a) \geq r$. It follows from Lemma 6 that

$$S = \bigcup_{n \in \mathbb{N}} S_{\frac{1}{n}}.$$

Hence it follows from Lemma 1 that it suffices to prove that S_r has measure zero for $r > 0$. Let $\epsilon > 0$ be given and assume for a contradiction that whenever $\{R_i : i \in \mathbb{N}\}$ is a countable set of rectangles which cover S_r then

$$\sum_{i \in \mathbb{N}} \mu(R_i) \geq \epsilon.$$

Let P be a partition of R. Let $\mathcal{A}$ be the set of rectangles of P which contain an element of S_r in the interior. It follows from Lemma 2 that

$$\sum_{A \in \mathcal{A}} \mu(A) \geq \frac{\epsilon}{2}.$$

It follows from Lemma 5 that

$$\sup\{f(x) : x \in A\} - \inf\{f(x) : x \in A\} \geq r$$

for all $A \in \mathcal{A}$. Hence

$$U(P,f) - L(P,f) = \sum_{A \in P} \big(\sup\{f(x) : x \in A\} - \inf\{f(x) : x \in A\}\big)\mu(A) \geq$$
$$\sum_{A \in \mathcal{A}} \big(\sup\{f(x) : x \in A\} - \inf\{f(x) : x \in A\}\big)\mu(A) \geq$$
$$\sum_{A \in \mathcal{A}} r\mu(A) = r \sum_{A \in \mathcal{A}} \mu(A) \geq \frac{r\epsilon}{2}.$$

In contradiction to the assumption that f is Riemann integrable we obtained that $U(P,f) - L(P,f) \geq r\epsilon/2$ for every partition P.

□