Lemma 1. *If $f \in \mathcal{R}$ in the interval $[a,b]$ then for every $\epsilon > 0$ there exists a $\delta > 0$ so that for all partitions $P := \{a = x_0, x_1, \ldots, x_n = b\}$ with $\mu(P) < \delta$ and all $S := \{s_i \in [x_{i-1}, x_i] : 1 \leq i \leq n\}$ and $T := \{t_i \in [x_{i-1}, x_i] : 1 \leq i \leq n\}$*

$$\sum_{i=1}^{n} |f(s_i) - f(t_i)|\Delta x_i < \epsilon.$$

Proof. For all $1 \leq i \leq n$ let s_i' and t_i' be such that $\{s_i', t_i'\} := \{s_i, t_i\}$ and

$$f(s_i') - f(t_i') \geq 0. \tag{1}$$

Then

$$|f(s_i') - f(t_i')| = |f(s_i) - f(t_i)|. \tag{2}$$

There exists a $\delta > 0$ so that $|S(P, f) - \int f| < \epsilon$ for all partitions P with $\mu(P) < \delta$. In particular then

$$\left|\sum f(s_i')\Delta x_i - \int f\right| < \epsilon/2 \quad \text{and} \quad \left|\sum f(t_i')\Delta x_i - \int f\right| < \epsilon/2. \tag{3}$$

We obtain

$$\left|\sum f(s_i')\Delta x_i - \sum f(t_i')\Delta x_i\right| < \epsilon. \tag{4}$$

Then, from Inequalities (1), (2) and (4):

$$\sum_{i=1}^{n} |f(s_i) - f(t_i)|\Delta x_i = \sum_{i=1}^{n} |f(s_i') - f(t_i')|\Delta x_i = \left|\sum_{i=1}^{n} \left(f(s_i') - f(t_i')\right)\Delta x_i\right| = \left|\sum f(s_i')\Delta x_i - \sum f(t_i')\Delta x_i\right| < \epsilon.$$

□

Theorem 1. *Let $f \in \mathcal{R}$ and $\alpha' \in \mathcal{R}$ on $[a,b]$, then $f \in \mathcal{R}(\alpha)$, and*

$$\int f d\alpha = \int f(x)\alpha'(x)dx.$$

Proof. Let $|f| \le M$ and $\epsilon > 0$ be given. According to previous lemma we have $f\alpha' \in \mathcal{R}$. There exists a $\delta > 0$ so that for all partitions P with $\mu(P) < \delta$ and all choices of s_i and t_i in the intervals of P

$$\left|\sum f(s_i)\alpha'(s_i)\Delta x_i - \int f\alpha'\right| < \epsilon/2 \quad \text{and} \quad \sum_{i=1}^{n} |f(s_i) - f(t_i)|\Delta x_i < \frac{\epsilon}{2M}.$$

Use the Mean Value Theorem to obtain numbers t_i so that $\Delta\alpha_i = \alpha'(t_i)\Delta x_i$. Then

$$\left|\sum f(s_i)\Delta\alpha_i - \int f\alpha'\right| =$$
$$\left|\sum f(s_i)\alpha'(s_i)\Delta x_i - \int f\alpha' + \sum f(s_i)\alpha'(t_i)\Delta x_i - \sum f(s_i)\alpha'(s_i)\Delta x_i\right| \le$$
$$\left|\sum f(s_i)\alpha'(s_i)\Delta x_i - \int f\alpha'\right| + \left|\sum f(s_i)\left(\alpha'(t_i) - \alpha'(s_i)\right)\Delta x_i\right| <$$
$$\epsilon/2 + \sum |f(s_i)||\alpha'(t_i) - \alpha'(s_i))|\Delta x_i < \epsilon/2 + \sum M|\alpha'(t_i) - \alpha'(s_i))|\Delta x_i <$$
$$\epsilon/2 + M\frac{\epsilon}{2M} = \epsilon.$$

□