PMAT 501/601 L01 Winter 2009

FINAL EXAMINATION

Please use a University of Calgary exam booklet, or LaTeX your solutions. Write on one side of the page only. Due April 23, 2009. Do the problems in the given order and label the number of each problem. Give references from the text or notes for any lemmas, propositions, theorems used. If not explicitly mentioned, problems refer to a measure space $(X, \mathcal{A}, \mu)$.

1. (a) In Lebesgue's Dominated Convergence Theorem, suppose (in addition) that f, f_n are integrable, that (as before) $f = \lim_n f_n$ μ-ae, but drop the hypothesis about the existence of a dominating function g. Give an example showing that the theorem is no longer true.

   (b) (p.60-1) Give an example of two functions $f, g : \mathbb{R} \to \mathbb{R}$ which are Borel measurable, and for which $f = g$ on a dense subset of $\mathbb{R}$ but $f \neq g$ λ-ae.

2. (p. 137-1) Define the positive measure ν on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ by

   $$\nu(A) = \int_A |j| d\lambda \ ,$$

   for any $A \in \mathcal{B}(\mathbb{R})$, where j is the identity function on $\mathbb{R}$. Show that $\nu \ll \lambda$, but that, on the other hand, for any $\varepsilon > 0$ there is no $\delta > 0$ such that $\lambda(A) < \delta$ implies $\nu(A) < \varepsilon$ for all $A \in \mathcal{B}(\mathbb{R})$.

3. (p. 137-3) Let μ, ν be σ-finite positive measures on $(X, \mathcal{A})$, with $\nu \ll \mu$ and let g be a Radon-Nikodym derivative of ν with respect to μ.

   (a) Show that an $\mathcal{A}$-measurable function $f : X \to \mathbb{R}$ is ν-integrable iff fg is μ-integrable.

   (b) In case (a) holds, show $\displaystyle\int f d\nu = \int fg d\mu$.

   [Hint : It may help to first look at the case where f is a characteristic function $f = \chi_A$, then positive simple functions.]

4. If μ is a finite measure and $f : X \to \mathbb{R}$ a measurable function, show that $g := f/(1 + |f|)$ is integrable. [Hint : Prop. 2.3.7]

5. If f is an integrable function, $A \in \mathcal{A}$, and $a, b \in \mathbb{R}$ such that $a \le f(x) \le b$ for all $x \in A$, then

$$a\mu(A) \le \int_A f d\mu \le b\mu(A) \ .$$

6. (p. 137-9) Let μ, ν be σ-finite positive measures on $(X, \mathcal{A})$. Then the following are equivalent.

   (a) $\nu \ll \mu$ and $\mu \ll \nu$,

   (b) μ, ν have the same measure 0 sets,

   (c) there exists an $\mathcal{A}$-measurable function $g : X \to (0, \infty)$ which is a Radon-Nikodym derivative of ν with respect to μ.

   Do one of the final two questions, *not* both.

7. (p. 90-3) Let $f, f_n : X \to \mathbb{R}$ be measurable, and let $g : \mathbb{R} \to \mathbb{R}$ be Borel measurable. Show that if $\{f_n\}$ converges to f μ-ae and if g is continuous at $f(x)$ for almost every x, then $\{g \circ f_n\}$ converges to $g \circ f$ μ-ae.

8. (p. 90-4) Let $f, f_n \in \mathcal{L}^1(X, \mathcal{A}, \mu, \mathbb{R})$. Show that if $\{f_n\}$ converges to f in mean so fast that $\sum_n \int |f_n - f| d\mu < \infty$, then $\{f_n\}$ converges to f ae.