# Metric spaces

N. W. Sauer
University of Calgary, Department of Mathematics and Statistics
Calgary, Alberta, Canada T2N 1N4
nsauer@math.ucalgary.ca

**Abstract**

Lecture notes for a course on combinatorial aspects of metric spaces at the University of Calgary in the Fall of 2007.

# 1 Basic notions

A *metric space* on a set $M$ is a function $\mathrm{d} : M^2 \to \mathbb{R}_{\geq 0}$ so that for all $x, y, z \in M$:

1. $\mathrm{d}(x, y) = 0$ if and only if $x = y$.
2. $\mathrm{d}(x, y) = \mathrm{d}(y, x)$.
3. $\mathrm{d}(x, y) + \mathrm{d}(y, z) \geq \mathrm{d}(x, z)$.

It is usual to speak of a metric space $M$ and assume tacitly that the distance function is denoted by d. Actually, a metric space is a function d with domain some set $M^2$ and range $\mathbb{R}_{\geq 0}$ for which the three axioms above hold.

## 1.1 Examples of metric spaces

Any set $M$ of numbers with $\mathrm{d}(x, y) = |x - y|$.

The space $\mathbb{R}^n$ with the Euclidean distance $\mathrm{d}(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$.

The space $\mathbb{R}^n$ with the supremum norm $\mathrm{d}_s(\mathbf{x}, \mathbf{y}) = \max\{|x_i - y_i| : 1 \leq i \leq n\}$.

More generally, if $(\mathrm{d}_i; i \in I)$ is a family of metric spaces all with the same base set $M$, then $\mathbf{d} : M \to \mathbb{R}_{\geq 0}$ given by $\mathbf{d}(x,y) = \{\sup\{\mathrm{d}_i(x,y) : i \in I\}$ for all $x, y \in M$ is a metric on $M$.

Let $(\mathrm{d}_i : (M_i)^2 \to \mathbb{R}_{\geq 0}; i \in I)$ be a family of metric spaces. Let $\mathbf{d} : \left(\prod_{i\in I} M_i\right)^2 \to \mathbb{R}_{\geq 0}$ be given by $\mathbf{d}\big((x_i; i \in I), (y_i; i \in I)\big) = \sup\{\mathrm{d}_i(x_i, y_i) : i \in I\}$ for $x_i, y_i \in M_i$ all $i \in I$. Then $\mathbf{d}$ is a metric on $\prod_{i\in I} M_i$.

Let $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$ be the set of all sequences $s = (s_i; i \in \mathbb{N})$ whose entries are non negative integers. For any two sequences $s, t \in \mathcal{N}$ let $\mathrm{d}(s,t) = \frac{1}{\delta(s,t)}$ where $\delta(s,t)$ is the smallest number $i$ with $s_i \neq t_i$. The set $\mathcal{N}$ with the topology induced by this metric is the *Baire space*. The subspace $\mathcal{C} = 2^{\mathbf{N}}$ is he Cantor space.

Let $M$ be a set and $\mathrm{d} : M^2 \to \mathbb{R}_{\geq 0}$. The function d is an *ultrametric* on $M$ if for all $x, y, z \in M$:

1. $\mathrm{d}(x,y) = 0$ if and only if $x = y$.
2. $\mathrm{d}(x,y) = \mathrm{d}(y,x)$.
3. $\max\{\mathrm{d}(x,y), \mathrm{d}(y,z)\} \geq \mathrm{d}(x,z)$.

Every ultrametric space is a metric space. The metric space $\mathcal{N}$ above is an ultrametric space.

The length of a shortest path between vertices of a graph is a metric distance function on the graph.

If $M$ is a metric space with distance function d and $N \subseteq M$ then the restriction of d to $N^2$ is a metric distance function on $N$. We say $N$ *is a subspace of* $M$.

An *inner product* on a vector space $V$ over $\mathbb{R}$ is a function $\langle\ ,\ \rangle : V \times V \longrightarrow \mathbb{R}$ such that, for all $k_1, k_2 \in \mathbb{R}$ and $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}, \mathbf{w} \in V$, the following properties hold:

1. $\langle k_1\mathbf{v}_1 + k_2\mathbf{v}_2, \mathbf{w}\rangle = k_1\langle \mathbf{v}_1, \mathbf{w}\rangle + k_2\langle \mathbf{v}_2, \mathbf{w}\rangle$.
2. $\langle \mathbf{v}, \mathbf{w}\rangle = \langle \mathbf{w}, \mathbf{v}\rangle$.
3. $\langle \mathbf{v}, \mathbf{v}\rangle \geq 0$, and $\langle \mathbf{v}, \mathbf{v}\rangle = 0$ if and only if $\mathbf{v} = \mathbf{0}$.

The standard example of an inner product is the dot product on $\mathbb{R}^n$:

$$((x_1, \ldots, x_n), (y_1, \ldots, y_n)) := \sum_{i=1}^{n} x_i y_i.$$

An *inner product space* (or *pre-Hilbert space*) is a vector space $\mathbb{R}$ with an inner product $\langle \cdot, \cdot \rangle$.

A norm on a real vector space $V$ is a function $\| \cdot \| : X \to \mathbb{R}$ such that for all $\mathbf{v}, \mathbf{w} \in V$ and $a \in \mathbb{R}$.

1. $\|\mathbf{v}\| \geq 0$ and $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v} = \mathbf{0}$.
2. $\|a\mathbf{v}\| = |a|\|\mathbf{v}\|$.
3. $\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$.

Let $V$ be a vector space with norm $\| \cdot \|$. Then d with $\mathrm{d}(\mathbf{v}, \mathbf{w}) = \|\mathbf{v} - \mathbf{w}\|$ is a metric on $V$.

Every inner product space is also a normed vector space, with the norm defined by $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$. This norm of an inner product space satisfies the parallelogram law, that is $2\|\mathbf{v}\|^2 + 2\|\mathbf{w}\|^2 = \|\mathbf{v} + \mathbf{w}\|^2 + \|\mathbf{v} - \mathbf{w}\|^2$. (Rewrite in terms of inner product and use axioms (i) and (ii) of inner product repeatedly.)

The Cauchy-Schwarz inequality

$$|\langle \mathbf{v}, \mathbf{w} \rangle| \leq \|\mathbf{v}\| \cdot \|\mathbf{w}\|$$

holds in any inner product space.

*Proof.* Note

$$\|x\mathbf{v} + \mathbf{w}\|^2 = \langle x\mathbf{v} + \mathbf{w}, x\mathbf{v} + \mathbf{w} \rangle = x^2\|\mathbf{v}\|^2 + 2x\langle \mathbf{v}, \mathbf{w} \rangle + \|\mathbf{w}\|^2.$$

If $\mathbf{v}$ and $\mathbf{w}$ are linearly independent then $\|x\mathbf{v} + \mathbf{w}\|^2 > 0$ and hence the quadratic on the right has no real roots which in turn implies that its discriminant is negative and hence $\langle \mathbf{v}, \mathbf{w} \rangle^2 < \|\mathbf{v}\|^2\|\mathbf{w}\|^2$.

If $\mathbf{w} = x\mathbf{v}$ then $\langle \mathbf{v}, \mathbf{w} \rangle^2 = \langle \mathbf{v}, x\mathbf{v} \rangle^2 = x^2\langle \mathbf{v}, \mathbf{v} \rangle^2 = x^2\|\mathbf{v}\|^4 = \|\mathbf{v}\|^2\|\mathbf{w}\|^2$. □

Let $M$ be a metric space. The sequence $(x_n)$ of elements of $M$ is a *Cauchy sequence* if for every $\epsilon > 0$ there is $N \in \mathbb{N}$ so that $\mathrm{d}(x_n, x_m) < \epsilon$ for all $n, m \geq N$. A metric space is *complete* if every Cauchy sequence of elements of $M$ converges.

A *Banach space* $(X, \| \cdot \|)$ is a normed vector space such that $X$ is complete under the metric induced by the norm $\| \cdot \|$. A *Hilbert space* is an inner product space for which the metric $\|\mathbf{v} - \mathbf{w}\|$ induced by the norm is complete.

Let $\ell_2$ be the Hilbertspace of all square summable sequences. Then $\mathrm{d} : \ell_2 \to \mathbb{R}_{\geq 0}$ given by $\mathrm{d}(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=1}^{\infty}(x_i - y_i)^2}$ is a metric with domain $\ell_2$.

Remember that in the special case of normed spaces we have the following theorem:

**Theorem 1.1.** *Let $X$ be a normed space. Then $X$ is complete if and only if for every sequence $(\mathbf{v}_n)$ of elements in $X$ with $\sum_{n=1}^{\infty} \|\mathbf{v}_n\| < \infty$ the series $\sum_{n=1}^{\infty} \mathbf{v}_n$ converges.*

*Proof.* Let $X$ be complete and $(\mathbf{v}_n)$ a sequence with elements in $X$ with $\sum_{n=1}^{\infty} \|\mathbf{v}_n\| < \infty$. Let $\epsilon > 0$ and $\mathbf{w}_n = \sum_{i=1}^{n} \mathbf{v}_i$ and $n > m$. Then:

$$\|\mathbf{w}_n - \mathbf{w}_m\| = \left\| \sum_{i=m+1}^{n} \mathbf{v}_i \right\| \leq \sum_{i=m+1}^{n} \|\mathbf{v}_i\| < \epsilon \text{ for all sufficiently large } m.$$

Suppose $\sum_{n=1}^{\infty} \mathbf{v}_n$ converges if $\sum_{n=1}^{\infty} \|\mathbf{v}_n\| < \infty$. Let $(\mathbf{w}_n)$ be Cauchy. The sequence $(\mathbf{w}_n)$ contains a subsequence $(\mathbf{u}_i)$ with $\|\mathbf{u}_{i+1} - \mathbf{u}_i\| < \frac{1}{2^i}$. Let $\mathbf{v}_i = \mathbf{u}_{i+1} - \mathbf{u}_i$. Then

$$\sum_{i=1}^{n} \|\mathbf{v}_i\| = \sum_{i=1}^{n} \|\mathbf{u}_{i+1} - \mathbf{u}_i\| < \sum_{i=1}^{n} \frac{1}{2^i}.$$

It follows that $\sum_{i=1}^{\infty} \|\mathbf{v}_i\| < \infty$ which in turn implies that $\sum_{i=1}^{m} \mathbf{v}_i = \sum_{i=1}^{m}(\mathbf{u}_{i+1} - \mathbf{u}_i) = \mathbf{u}_{m+1} - \mathbf{u}_1$ converges to some element $\mathbf{v} \in X$. Hence $\mathbf{u}_m$ converges to $\mathbf{v} + \mathbf{u}_1$. Hence, the Cauchy sequence $(\mathbf{w}_n)$ converges because it has a convergent subsequence. □

If $Y$ is a Banach space and $X$ is any normed vector space, then the set of continuous linear maps $f : X \to Y$ forms a Banach space, with norm given by the operator norm. In particular, since $\mathbb{R}$ and $\mathbb{C}$ are complete, the continuous linear functionals on a normed vector space $\mathcal{B}$ form a Banach space, known as the *dual space* of $\mathcal{B}$.

Examples:

- Finite-dimensional normed vector spaces.
- $L^p$ spaces are by far the most common examples of Banach spaces.
- $\ell^p$ spaces are $L^p$ spaces for the counting measure on $\mathbb{N}$.
- Continuous functions on a compact set under the supremum norm.

## 1.2 Properties of Metric Spaces

Let $M$ be a metric space. For $a \in M$ and $r > 0$ we denote by $\mathrm{B}_r(a)$ the *ball in $M$ with center $a$ and radius $r$*, that is:

$$\mathrm{B}_r(a) = \{x \in M : \mathrm{d}(a,x) < r\}.$$

A *ball with center $a$* is a set $\mathrm{B}_r(a)$ for some $r > 0$.

The subset $S \subseteq M$ is *open* if for every $a \in S$ there exists $0 < r \in \mathbb{R}$ with $\mathrm{B}_r(a) \subseteq S$. The *closure* $\overline{S}$ of a set $S \subseteq M$ is the set of all $a \in M$ with the poperty that $\mathrm{B}_r(a) \cap S \neq \emptyset$ for all $0 < r \in \mathbb{R}$. The subset $S$ of $M$ is *dense* in $M$ is $\overline{S} = M$. The metric space $M$ is *separable* if it contains a countable dense subset.

The sequence $(p_i; i \in \mathbb{N})$ of elements in $M$ *converges* to the element $p \in M$ if for every $\epsilon > 0$ there exists an $n \in \mathbb{N}$ so that $\{p_i : i > n\} \subseteq \mathrm{B}_\epsilon(p)$. The sequence $(p_i; i \in \mathbb{N})$ is a Cauchy sequence of elements in $M$ and hence has a limite, say $p$. Then $p \in \bigcap_{i\in\mathbb{N}} S_i \cap \mathrm{B}_r(q)$. The sequence $(p_i; i \in \mathbb{N})$ of elements of $M$ is a *Cauchy sequence* if for every $\epsilon > 0$ there is $n \in \mathbb{N}$ so that $\mathrm{d}(p_i, p_j) < \epsilon$ for all $i, j \geq n$.

Every convergent sequence is Cauchy but a Cauchy sequence need not converge. A metric space $M$ is *complete* if every Cauchy sequence of elements of $M$ converges. The *Cauchy completion* of a metric space $M$ is the intersection of all complete metric spaces which contain $M$ as a subspace. Every metric space has an, up to isomorphism, unique Cauchy completion.

**Theorem 1.2** (Baire Category theorem)**.** *The intersection of a countable set of dense open subsets of a complete metric space $M$ is dense in $M$.*

*Proof.* Let $(S_i; i \in \mathbb{N})$ be open and dense. Let $q \in M$ and $r > 0$. We will show that there is an element $p \in \bigcap_{i\in\mathbb{N}} S_i$ with $p \in \mathrm{B}_r(q)$.

There exists a sequence $(\mathrm{B}_{s_i}(p_i); i \in \omega)$ of balls in $M$ with $s_0 = r$ and $0 < s_i \leq \frac{1}{2^i}$ and $p_0 = q$ and $\mathrm{B}_{s_0}(p_0) \supseteq \mathrm{B}_{s_1}(p_1) \supseteq \mathrm{B}_{s_2}(p_2) \supseteq \mathrm{B}_{s_3}(p_3) \supseteq \cdots$. $\square$

Let $M$ be a metric space. The set $S \subseteq M$ is *nowhere dense* if no ball of $M$ is a subset of the closure of $S$. A countable union of nowhere dense sets is a set of the *first category* and all the other subsets of $M$ are of the *second category*. By taking complements in the statement of Theorem 1.2 we obtain the fact that no complete metric space is of the first category, the original result of Baire.

The *spectrum* $\mathrm{Spec}(M, p)$ of the element $p$ in the metric space $M$ is the set $\{d(p, x) : x \in M\}$ and the spectrum $\mathrm{Spec}(M)$ of $M$ is the set $\bigcup_{p \in M} \mathrm{Spec}(M, p)$. The spectrum of the ultrametric space $\mathcal{N}$ of the examples is the set $\{\frac{1}{n} : n \in \mathbb{N}\}$. The *diameter* of the metric space $M$ is the supremum of the spectrum of $M$.

The function $f$ of the metric space $M$ to the metric space $N$ is an *isometry* if $\mathrm{d}_M(x, y) = \mathrm{d}_N(f(x), f(y))$ for all $x, y \in M$. An isometry of a finite subspace of $M$ to a subspace of $N$ is a *local isometry* of $M$ to $N$. A local isometry of $M$ to $M$ will usually just be called an isometry of $M$. An isometry of $M$ onto $M$ is an *automorphism* of $M$. Note that the set of automorphisms of $M$ form a group under function composition. We denote the isomorphism group of $M$ by $\mathrm{Iso}(M)$.

**Definition 1.1.** *The metric space $M$ is* homogeneous *if every local isometry of $M$ can be extended to an automorphism of $M$. The metric space $M$ is $n$-homogeneous if every local isomorphism of an $n$-element subset of $M$ can be extended to an automorphism.*

Caution: In more analysis oriented areas of mathematics metric spaces which we call homogeneous are called *ultrahomogeneous* and 1-homogeneous spaces which we call transitive are called homogeneous. In model theory, homogeneous has still another, but somewhat related, meaning.

For example, the metric space on $\mathbb{N}$ with $\mathrm{d}(n, m) = |n - m|$ is homogeneous. More generally the spaces $\mathbb{R}^n$ are homogeneous. The unit spere $\mathbb{S}^n := \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| = 1 \text{ in } \mathbb{R}^{n+1}\}$ is a homogeneous metric space. The ultrametric space $\mathcal{N}$ of the examples is homogeneous. The Hilbert space $\ell_2$ and the unit sphere $\mathbb{S}^\infty := \{\mathbf{x} \in \ell_2 : \|\mathbf{x}\| = 1\}$, the *Hilbert sphere*, are homogeneous metric spaces.

In contrast to this it is known that: The Hilbert space $\ell_2$ is the only separable infinite dimensional homogeneous Banach space. Every separable infinite dimensional 3-homogeneous Banach space is a Hilbert space. The problem whether the number 3 here can be replaced by 2 is known as the Banach-Mazur rotation problem. Mazur [10] verified it in the finite dimensional case and Pełczynski and Rolewicz [11] showed that the condition to be separable is necessary. The paper [12] is a survey on this problem.

The image of a metric space $M$ by an isometry is a *copy* of $M$. In particular an isometric image of the metric space $M$ in the metric space $M$ is a *copy* of $M$ in $M$. The metric space $M$ is *indivisible* if for every partition

$(A_0, A_1, \ldots, A_{n-1})$ of $M$ into finitely many parts there is $i \in n$ and a copy $M^*$ of $M$ with $M^* \subseteq A_i$. Of the metric spaces listed in the examples only the Baire space is indivisible, [1]. Note that if for every partition $(A_0, A_1)$ of $M$ into two parts there is $i \in 2$ and a copy $M^*$ of $M$ with $M^* \subseteq A_i$, then $M$ is indivisible.

For $N \subseteq M$ and $\epsilon > 0$ let:

$$(N)_\epsilon = \{p \in M : \exists q \in N \ (\mathrm{d}(p, q) \leq \epsilon)\}.$$

The metric space $M$ is *$\epsilon$-indivisible* if for every partition $(A_i; i \in n \in \omega)$ of $M$ there is $i \in n$ and a copy $M^*$ of $M$ with $M^* \subseteq (A_i)_\epsilon$. The metric space $M$ is *approximately indivisible* if it is $\epsilon$-indivisible for every $\epsilon > 0$. Note that $M$ is indivisible if it is 0-indivisible.

Let $f$ be a function of a metric space $M$ to a metric space $N$. Then the *oscillation* of $f$ on $M$ is:

$$\mathrm{osc}(f|M) = \sup\{\mathrm{d}_N(f(x), f(y)) : x, y \in M\}.$$

The function $f$ is L-Lipschitz if

$$\mathrm{d}_N(f(x), f(y)) \leq L\mathrm{d}_M(x, y)$$

for all $x, y \in M$. Note that every L-Lipschitz function is uniformly continuous. (For $\epsilon > 0$ let $\delta = \frac{\epsilon}{L}$. Then for $d(x, y) < \delta$: $d(f(x), f(y)) < Ld(x, y) = L\frac{\epsilon}{L} = \epsilon$.) It follows that if there exists a 1-Lipschitz function $f$ of $M$ to a normed vector space $N$ with $\mathrm{osc}(f|M) > \epsilon$ then there exists a uniformly continuous function $g$ of $M$ to $N$ with $\mathrm{osc}(g|M) > 1$. ($g := \frac{1}{\epsilon}f$. Then $\mathrm{osc}(g|M) = \sup\{\|g(x), g(y)\| : x, y \in M\} = \sup\{\frac{1}{\epsilon}\|f(x), f(y)\| : x, y \in M\}$.)

**Observation 1.1.** *Let $f$ be a uniformly continuous function of the approximately indivisible metric space $M$ to the compact metric space $N$. Then there exists for every $\epsilon > 0$ a copy $M^*$ of $M$ so that $\mathrm{osc}(f|M^*) < \epsilon$.*

*Proof.* For $\epsilon > 0$ let $\delta > 0$ be such that $\mathrm{d}(f(x), f(y)) < \epsilon/3$ for all $x, y \in M$ with $\mathrm{d}(x, y) < \delta$. There is a finite subset $Z \subseteq N$ with $N \subseteq \bigcup_{z \in Z} \mathrm{B}_{\epsilon/3}(z)$. For $z \in Z$ let $A_z = \{x \in M : f(x) \in \mathbf{B}_{\epsilon/3}(z)\}$.

There exists $z \in Z$ and a copy $M^*$ of $M$ with $M^* \subseteq (A_z)_\delta$. Hence for $x, y \in M^*$ and $x_1, y_1 \in A_z$ with $\mathrm{d}(x_1, x) < \delta$ and $\mathrm{d}(y_1, y) < \delta$:

$$\mathrm{d}(f(x), f(y)) \leq \mathrm{d}(f(x), f(x_1)) + \mathrm{d}(f(x_1), f(y_1)) + \mathrm{d}(f(y_1), f(y)) < \epsilon.$$

□

This Oberservation 1.1 has in the case where $N$ is a closed interval of $\mathbb{R}$ a converse. There is a general notion for a topological space $X$ acted upon by a a topological group $G$ continuously to be *oscillation stable*, see [9] chapter 5 where the following result is stated.

**Theorem 1.3** (Pestov)**.** *For a complete homogeneous metric space $M$ the following are equivalent:*

1. *Acted upon by the automorphism group* $\mathrm{Iso}(M)$ *with the point open (pointwise convergent) topology, the metric space $M$ is oscillation stable.*

2. *For every bounded 1-Lipschitz function $f : M \to \mathbb{R}$ and every $\epsilon > 0$, there is a copy $M^*$ of $M$ such that* $\mathrm{osc}(f|M^*) < \epsilon$.

3. *The space $M$ is approximately indivisible.*

A metric space which is not oscillation stable is said to have *distortion*. Negating we obtain:

**Corollary 1.1.** *For a complete homogeneous metric space $M$ the following are equivalent:*

1. *Acted upon by the automorphism group* $\mathrm{Iso}(M)$ *with the point open (pointwise convergent) topology, the metric space $M$ has distortion.*

2. *There is a bounded uniformly continuous function $f : M \to \mathbb{R}$ whose oscillation on every copy $M^*$ of $M$ is at least 1.*

3. *There is a partion $(A, B)$ of $M$ and an $\epsilon > 0$ so that neither $(A)_\epsilon$ nor $(B)_\epsilon$ has a copy of $M$ as a subset.*

We will not prove Theorem 1.3 but use it to state some historical results in terms of approximate indivisibility. The way those results have been stated have undergone a long development, mainly by Milman, who eventually stated them in terms of oscillation of uniformly bounded functions. The story began in 1959 with the following theorem of Dvoretzky [13]:

**Theorem 1.4** (Dvoretzky)**.** *For every $k \in \mathbb{N}$ and every $\epsilon > 0$ there exists a number $N(k, \epsilon)$ so that if $(X, \| \cdot \|)$ is a Banach space of dimension larger than or equal to $N(k, \epsilon)$, then there exists a subspace $E \subseteq X$ of dimension*

*$k$ and a positive quadratic form $Q$ on $E$ so that the corresponding Euclidean norm*

$$| \cdot | = \sqrt{Q(\cdot)}$$

*on $E$ satisfies:*

$$|\mathbf{v}| \leq \|\mathbf{v}\| \leq (1+\epsilon)|\mathbf{v}| \quad \textit{for every } \mathbf{v} \in E.$$

Milman then extended and rerformulated Dvoretzky's theorem and proofed [6]:

**Theorem 1.5** (Milman)**.** *Let $(A_0, A_1, \dots, A_{k-1})$ be a partition of $\mathbb{S}^\infty$. Then for every $\epsilon > 0$ and every $n \in \omega$ there is $i \in k$ and a copy $C$ of $\mathbb{S}^n$ in $\mathbb{S}^\infty$ with $C \subseteq (A_i)_\epsilon$.*

Theorem 1.5 led naturally to the conjecture that $\mathbb{S}^\infty$ is approximately indivisible, but [8]:

**Theorem 1.6** (Odell-Schlumprecht)**.** *The Hilbert sphere $\mathbb{S}^\infty$ is not approximately indivisible.*

We will construct various indivisible and approximately indiviseble metric spaces in the next section.

# 2 Urysohn spaces

## 2.1 The age of homogeneous metric spaces

Given a metric space $M$, a map $f : M \longrightarrow ]0,+\infty[$ is *Katětov over $M$* if

$$\forall x, y \in M, \quad |f(x) - f(y)| \leqslant \mathrm{d}(x, y) \leqslant f(x) + f(y).$$

Equivalently, one can extend the metric d to $M \cup \{f\}$ by defining, for every $x, y$ in X,

$$\mathrm{d}(x, f) = f(x) \text{ and } \mathrm{d}(x, y) = \mathrm{d}(x, y).$$

The corresponding metric space is then written $M \cup \{f\}$. The set of all Katětov maps over $M$ is denoted by $\mathbf{K}(M)$. For a metric subspace $M$ of a metric space $N$ and a Katětov map $f \in \mathbf{K}(M)$ the point $y \in N$ *realizes $f$ over $M$* if

$$\forall x \in M \quad \mathrm{d}(y, x) = f(x).$$

The set of all $y \in N$ realizing $f$ over $M$ is denoted by $\mathrm{T}(f, N)$ called the *typeset of $f$ in $N$*. When $N$ is clear from the context, the set $\mathrm{T}(f, N)$ is simply written $\mathrm{T}(f)$.

The *skeleton* of a metric space $M$ is the set of all finite subspaces of $M$. The *age* of $M$ is the class of all metric spaces which are isometric to an element of the skeleton of $M$. A class $\mathbf{A}$ of finite metric spaces is an *age* if:

1. If $B \in \mathbf{A}$ and $C$ is a metric space isometric to $B$ then $C \in \mathbf{A}$.
2. $\mathbf{A}$ contains a non empty metric space.
3. If $B \in \mathbf{A}$ and $C$ is a subspace of $B$ then $C \in \mathbf{A}$.
4. If $B \in \mathbf{A}$ and $C \in \mathbf{A}$ with $A \cap C = \emptyset$ then there exists a $D \in \mathbf{A}$ which has $B$ and $C$ as subspaces. (*updirected.*)

Note that the age of a metric space is an age. The equivalence relation of being isomorphic partitions every age into equivalence classes. An age $\mathbf{A}$ is called countable if the cardinality of the cardinality of the set of isomorphism equivalence classes of $\mathbf{A}$ is countable. Items (ii) and (iii) imply that the empty metric space $\emptyset$ with distance function the empty function and the one point metric space are elements of every age. The empty function is Katětov over the empty metric space and $\emptyset \cup \{\emptyset\}$ is the one point metric space.

The *spectrum* Spec($\mathbf{A}$) of an age is the set $\bigcup_{A\in\mathbf{A}}\mathrm{Spec}(A)$. Note that if the spectrum of $\mathbf{A}$ is countable then the age $\mathbf{A}$ is countable.

The metric space $M$ has the *mapping extension property* if for every finite subspace $F \subset M$ and every Katětov map $f$ over $F$ with $F \cup \{f\}$ an element of the age of $M$, the set $\mathrm{T}(f, M) \neq \emptyset$; that is, if there exists an isometry $g$ of $F \cup \{f\}$ into $M$ so that $g$ restricted to $F$ is the idendity map on $F$. Of course then $g(f) \in \mathrm{T}(f, M)$.

**Theorem 2.1.** *Let $M$ and $N$ be two countable metric spaces both having the mapping extension property and the same age. Let $g$ be a local isomorphism of $M$ to $N$.*

*Then there exists an isometry of $M$ onto $N$ which extends $g$.*

*Proof.* Let the domain of $g$ be the finite set $F$ with $|F| = k$. Enumerate $M$ into an $\omega$-sequence $m_0, m_1, m_2, m_3, dots$ so that $F = \{m_i : i \in k\}$. Let $n_0, n_1, n_2, n_3, \ldots$ be an $\omega$-enumeration of $N$ so that $n_i = g(m_i)$ for all $i \in k$.

Let $f$ be the Katětov map on $g''(F)$ with $f(n_i) = d(m_i, m_k)$ for all $i \in k$. It follows that every copy of $F \cup m_k$ in $N$ is a copy of $g''(F) \cup f$. Hence $f$ can be realized in $N$ by an element, say $n_t$. We extend $g$ to $g^*$ by $g^* = g \cup \{(m_k, n_t\}$. It follows from the construction that $g^*$ is a local isomorphism. Similarly we find an element, say $m_s \in M$ so that $g^* \cup \{(m_s, n_k)\}$ is a local isomorphism.

Continuing this process in a suitable way will produce an isometry of $M$ onto $N$. □

The type or argument used in the proof of Theorem 2.1 is called a *back and forth argument*. The most common use of Theorem 2.1 is in the case in which $g$ is the empty function. By just using the forth part of the argument we obtain:

**Theorem 2.2.** *Let $M$ and $N$ be countable metric spaces with the age of $M$ a subset of the age of $N$. If $N$ has the mapping extension property then there exists an isometry of $M$ into $N$, that is there exists a copy of $M$ in $N$.*

**Theorem 2.3.** *Let $M$ be a metric space. If $M$ is homogeneous it has the mapping extension property. If $M$ is countable and has the mapping extension property it is homogeneous.*

*Proof.* Let $M$ be homogeneous and $F$ a finite subspace of $M$ and $f$ a Katětov map over $F$. Let $F' \cup f'$ be a copy of $F \cup f$ in $M$. There exists a local isometry of $F'$ to $F$ which has an extension, say $g$, to an isometrie in $\mathrm{iso}(M)$. Then $g(f')$ realizes $f$ over $F$.

Let $M$ be countable with the mapping extension property. Let $F$ be a finite subspace of $M$ and $g$ a local isometry of $F$ into $M$. It follows from Theorem 2.1 that $g$ has an extension to an element of $\mathrm{iso}(M)$. $\square$

**Definition 2.1.** *Let* $\mathbf{A}$ *be a class of metric spaces which is an age. The age* $\mathbf{A}$ *has* amalgamation *if for every* $B \in \mathbf{A}$ *and every subspace* $F$ *of* $B$ *and every Katětov function* $f$ *over* $F$ *with* $F \cup \{f\} \in \mathbf{A}$ *there exists a metric space* $D \in \mathbf{A}$*, with* $D = B \cup \{f\}$ *and so that* $B$ *and* $F \cup \{f\}$ *are subspaces of* $D$*.*

*The triple* $(B, F, f)$ *is an* amalgamation instance *of* $\mathbf{A}$ *and the metric space* $D$ *is the* amalgam *of the amalgamtion instanze* $(B, F, f)$*.*

**Theorem 2.4.** *The age of every homogeneous metric space has amalgamation.*

*Let* $\mathbf{A}$ *be a countable age of metric spaces which has amalgamation. Then there exists a homogeneous metric space* $M_{\mathbf{A}}$ *whose age is* $\mathbf{A}$*. The metric space* $M_{\mathbf{A}}$ *unique up to isomorphism.*

*Proof.* Let $M$ be a homogeneous metric space and $B$ a finite subspace of $M$ and $F$ a subspace of $B$ and $f$ Katětov over $F$ with $F \cup \{f\}$ an element of the age of $M$. Because $M$ has the mapping extension property according to Theorem 2.3 there exists an isometry $g$ of $F \cup \{f\}$ into $M$ which is the idendity on $F$. The subspace $B \cup \{g(f)\}$ is an amalgam of $B$ and $f$.

Let $\mathbf{A}$ be a countable age of metric spaces and let $\mathbf{F}$ be the set of all pairs $(F, f)$ with $F \in \mathbf{A}$ and $f$ Katětov over $F$ and $F \cup \{f\} \in \mathbf{A}$. The pair $(F, f) \in \mathbf{F}$ is *isomorphic* to the pair $(F_1, f_1) \in \mathbf{F}$ if there exists an isometry $g$ of $F$ onto $F_1$ so that $f(x) = f_1(g(x))$ for all $x \in F$. Let $\mathbf{T}$ be the set of all triples $(g, F, f)$ so that $(F, f) \in \mathbf{F}$ and $F \subseteq \omega$ and $g$ is an order preserving map of $F$ into $\omega$. The set $\mathbf{T}$ is countable. Let $(g_i, F_i, f_i)$ be an enumeration of $\mathbf{T}$ for which every element of $\mathbf{T}$ appears infinitely often. (Note that for every $(F, f) \in \mathbf{F}$ there exists an $i \in \omega$ with $(F, f)$ being isomorphic to $(F_i, f_i)$.)

Let $B \in \mathbf{A}$ with $B \subseteq \omega$ and $g_i$ an isometry of $F_i$ into $B$. Then $f$ with $f(g_i(x) = f_i(x)$ for all $x \in F_i$ is Katětov over $g_i''(F_i) = \{g_i(x) : x \in F_i\} \subseteq B$ and $g_i''(F_i) \cup \{f\} \in \mathbf{A}$. Because $\mathbf{A}$ has amalgamation there exists a metric space $D$ so that $g_i'' \cup \{f\}$ and $B$ are subspaces of $D$. Indeed, there exists such a metric space $D$ with $D \subseteq \omega$. Such a metric space $D$ will be called an extension of $B$ via $(g_i, F_i, f_i)$.

We construct a sequence $B_0 \subseteq B_1 \subseteq B_2 \subseteq \cdots$ of elements of $\mathbf{A}$ and a sequence $n_0 < n_1 < n_2 < \cdots$ of numbers so that:

1. $B_0 = \emptyset$ and $n_0 = 0$.

2. $n_{i+1}$ is the smallest number $j$ larger than $i$ so that $g_j$ is an isometry of $F_j$ into $B_i$.

3. $B_{i+1}$ is an extension of $B_i$ via $(g_i, F_i, f_i)$.

It follows that $M_{\mathbf{A}} := \bigcup_{i \in \omega} B_i$ has the mapping extension property. □

## 2.2 Amalgamation

**Definition 2.2.** *The metric spaces $A$ and $B$ are* compatible *if* $\mathrm{d}_A(x,y) = \mathrm{d}_B(x,y)$ *for all $x, y \in A \cap B$. For compatible metric spaces $A$ and $B$ we denote by $A \amalg B$ the set of all metric spaces $D = A \cup B$ with $\mathrm{d}_D(x,y) = \mathrm{d}_A(x,y)$ if $x, y \in A$ and $\mathrm{d}_D(x,y) = \mathrm{d}_B(x,y)$ if $x, y \in B$.*

*An amalgamation instance of the form $(B \cup \{a\}, B, f)$ is an* elementary amalgamation instance*.*

Note that an age $\mathbf{A}$ of metric spaces has amalgamation if $\mathbf{A} \cap A \amalg (B \cup \{f\}) \neq \emptyset$ for every amalgamation instance $(A, B, f)$.

**Theorem 2.5.** *Let $\mathbf{A}$ be an age of metric spaces. The following are equivalent.*

1. *For every elementary amalgamation instance $(B \cup \{a\}, B, f)$ there exist an element of $\mathbf{A}$ in $(B \cup \{a\}) \amalg (B \cup \{f\})$.*

2. *$\mathbf{A}$ has amalgamation.*

3. *The set $A \amalg B \cap \mathbf{A} \neq \emptyset$ for all compatible metric spaces $A, B \in \mathbf{A}$.*

*Proof.* Obviously, condition (iii) implies condition (ii) implies condition (i). We will prove that condition (i) implies (iii).

We prove by induction on the cardinality of $|A \cup B|$ that if $\mathbf{A}$ satisfied condition (i) and $A, B \in \mathbf{A}$ are compatible then $A \amalg B \cap \mathbf{A} \neq \emptyset$. If $A \subseteq B$ then $B \in A \amalg B$. If $B \subseteq A$ then $A \in A \amalg B$. Otherwise let $a \in A \setminus B$ and $b \in B \setminus A$. Then $A \setminus \{a\} \in \mathbf{A}$ is compatible to $B$. Hence there is $D \in (A \setminus \{a\} \amalg B) \cap \mathbf{A}$. Then $A$ and $D \setminus \{b\}$ are compatible. Hence there exists $E \in A \amalg (D \setminus b) \cap \mathbf{A}$.

Let $b$ play the role of the Katětov function on $E \setminus \{a\}$ with $b(x) = \mathrm{d}_E(b, x)$ for all $x \in E \setminus \{a\}$. Then $(E, E \setminus \{a\}, b)$ is an elementary amalgamation instance whose amalgam is an element of $A \amalg B \cap \mathbf{A}$.

□

Let $\mathbf{A}$ be an age of metric spaces. Because every subspace of a metric space in $\mathbf{A}$ is again an element of $\mathbf{A}$ and the elements of $\mathbf{A}$ are finite, there exists a set $\mathbf{B}$ of finite metric spaces so that:

1. There is no isometry of $A$ to $B$ for all $A, B \in \mathbf{B}$.
2. $A \in \mathbf{A}$ if and only if for all $B \in \mathbf{B}$ there is no isometry of $B$ to $A$.

The set $\mathbf{B}$ is the *boundary* of $\mathbf{A}$. For $M$ a metric space, the *boundary of* $M$ is the boundary of the age of $M$. The boundary of an age determines the age. It is often the most convenient way to define an age. That is after specifying the spectrum of the age on lists the boundary. For example, all finite metric spaces whose spectrum is an integer and which do not contain four points with all the six distances equal. The boundary of that age consists of all two element metric spaces with the distance between the two points not an integer and the set $\{\{x_i, y_i, z_i, u_i\} : i \in \omega \text{ and } \mathrm{d}(x, y) = \mathrm{d}(y, z) = \mathrm{d}(z, u) = \mathrm{d}(u, x) = \mathrm{d}(x, z) = \mathrm{d}(y, u)\}$. Of course another convenient way of defining an age is as the age of a metric space.

Given a boundary $\mathbf{B}$, that is a set of finite metric spaces which can not be embedded into each other, so that the age $\mathbf{A}$ determined by that boundary has amalgamation and is countable, there is then a unique up to isomorphism countable homogeneous metric space whose age is $\mathbf{A}$. Hence the problem arises, given a boundary $\mathbf{B}$, does the age determined by the boundary $\mathbf{B}$ have amalgamation. In many cases this is a very difficult problem in finite combinatorics. The following general observations can be made in case that the boundary consists of two element subsets only, that is if $\mathbf{A}$ consists of all finite metric spaces whose spectrum is a subset of the given spectrum for $\mathbf{A}$.

**Definition 2.3.** *Let $S \subseteq \mathbb{R}$. Then $\mathbf{A}_S$ is the class of all finite metric spaces whose spectrum is a subset of $S \cap \mathbb{R}_{\geq 0} \cup \{0\} := S_{\geq 0}$.*

Let $S \subseteq \mathbb{R}$ and $(A, F, f)$ an elementary amalgamation instance of $\mathbf{A}_S$ with $A = F \cup \{a\}$. In order to find an amalgam of this amalgamation instance we have to find a suitable distance $r$ from $a$ to $f$. Such a distance $r$ will be suitable if and only if all triangles of the form $a, x, f$ with $x \in F$ satisfy the triangle inequality. That is, if $|\mathrm{d}_A(a,x) - f(x)| \leq r \leq \mathrm{d}_A(a,x) + f(x)$. Hence $(A, F, f)$ has an amalgam in $\mathbf{A}$ if and only if the *basic amalgamation condition*

$$\max_{x \in F}\{|\mathrm{d}_A(a,x) - f(x)|\} \leq r \leq \min_{x \in F}\{\mathrm{d}_A(a,x) + f(x)\} \tag{1}$$

is saisfied for some $r \in S_{\geq 0}$.

**Theorem 2.6.** *The set $\mathbf{A}_S$ of metric spaces is an age for every set $S$.*

*The age $\mathbf{A}_S$ has amalgamation if every amalgamation instanz of $\mathbf{A}_S$ of the form $(\{a,b,c\},\{b,c\},f)$ has an amalgam in $\mathbf{A}_S$.*

*Proof.* Items (i), (ii) and (iii) of the definition of age are easy to check. Let $B, C \in \mathbf{A}_S$ with $B \cap C = \emptyset$. Let $r = \max\{\text{diameter of } B, \text{ diameter of } C\}$. Then $D \in B \amalg C$ with $\mathrm{d}_D(x,y) = \mathrm{d}_D(y,x) = r$ if $x \in B$ and $y \in C$ is an element of $\mathbf{A}_S$.

The simplest non trivial case of an amalgamation instance is of the form $(\{a,b\},\{b\},f)$. Then $D = \{a,b\} \amalg (\{b\} \cup \{f\})$ with $\mathrm{d}_D(a,f) = \max\{\mathrm{d}(a,b), f(b)\}$ is an amalgam in $\mathbf{A}_S$.

Let $a \in A \in \mathbf{A}_S$ and $F = A \setminus \{a\}$ and $f$ katětov over $F$ with $F \cup \{f\} \in \mathbf{A}_S$. Then $(A, F, f)$ is an amalgamation instance of $\mathbf{A}_S$. We have to find a number $r \in S$ so that $\mathrm{d}_D(a,f) = r$ for some $D \in A \amalg (F \cup \{f\})$. Such a number $r$ has to satisfy the basic amalgamation condition (1).

Let $b \in F$ maximise the left hand side of condition (1) and $c \in F$ minimise the right hand side of condition (1). For $b = c$ we observed above that the amalgamation instance $(\{a,b\},\{b\},f)$ of $\mathbf{A}_S$ has an amalgam in $\mathbf{A}_S$. If $b \neq c$ the amalgamation instance $(\{a,b,c\},\{b,c\},f)$ has an amalgam in $\mathbf{A}_S$ by assumption.

Let $(A, F, f)$ be an amalgamation instanze of $\mathbf{A}_S$. We use induction on $|A \setminus F|$ to find an amalgam of $(A, F, f)$ in $\mathbf{A}_S$. If $|A \setminus F| = 1$ we have the previous case. Let $|A \setminus F| > 1$ and $a \in A \setminus F$. There exists an amalgam $D = (A \setminus \{a\}) \cup F$ in $\mathbf{A}_S$. Extend $f$ from $F$ to $A \setminus \{a\}$ by stipulating that $f(x) = \mathrm{d}_D(x, f)$ for all $x \in A \setminus \{a\}$. Then $(A, A \setminus \{a\}, f)$ is an amalgamation instance of $\mathbf{A}_S$. Let $E$ be an amalgam of $(A, A \setminus \{a\}, f)$ in $\mathbf{A}_S$. Then $E$ is an amalgam of $(A, F, f)$. □

**Corollary 2.1.** *An age $\mathbf{A}_S$ has amalgamation if and only if the set $A \amalg B$ contains an element of $\mathbf{A}_S$ for all $A, B \in \mathbf{A}_S$ with $|A| = |B| = 3$ and $|A \cap B| = 2$.*

**Theorem 2.7.** *Let* $(\{a,b,c\},\{b,c\},f)$ *be an amalgamation instance of* $\mathbf{A}_{\mathbb{R}}$. *Then*

$$|\mathrm{d}(a,b)-f(b)| \leq \mathrm{d}(a,c)+f(c).$$

*Proof.* We may assume without loss of generality that $\mathrm{d}(a,b) \geq f(b)$. Assume for a contradiction that

$$\mathrm{d}(a,b)-f(b) > \mathrm{d}(a,c)+f(c). \tag{2}$$

If $\mathrm{d}(a,b) \geq \mathrm{d}(a,c)$ then $\mathrm{d}(a,b)-\mathrm{d}(a,c) \leq \mathrm{d}(b,c) \leq f(b)+f(c)$ implies $\mathrm{d}(a,b)-f(b) \leq \mathrm{d}(a,c)+f(c)$ in contradiction to inequality (2).

Hence $\mathrm{d}(a,b) < \mathrm{d}(a,c)$ and $\mathrm{d}(a,c)-\mathrm{d}(a,b) \leq \mathrm{d}(b,c) \leq f(b)+f(c)$ implies $\mathrm{d}(a,b)+f(b) \geq \mathrm{d}(a,c)-f(c)$ which added into inequality (2) yields $2\mathrm{d}(a,b) > 2\mathrm{d}(a,c)$.

□

Let $S \subseteq \mathbb{R}$ and $s \in S$ with $s > 0$. Then $s^+ := \min\{x+y : 0 < x \in S \text{ and } 0 < y \in S \text{ and } s \leq x+y\}$ and $s^- := \max\{x-y : 0 < x \in S \text{ and } 0 < y \in S \text{ and } x \geq y \text{ and } s+y \geq x\}$. It follows from Theorem 2.7 that $s^- \leq s^+$ and hence from Theorem 2.6 that $\mathbf{A}_S$ has amalgamation if and only if $S \cap [s^+ - s^-] \neq \emptyset$ for all $0 < s \in S$.

**Corollary 2.2.** *The ages* $\mathbf{A}_{\mathbb{R}}$, $\mathbf{A}_{\mathbb{Q}}$, $\mathbf{A}_n$ *for* $n = \{0,1,2,\ldots,n-1\}$, $\mathbf{A}_{\mathbb{N}}$, $\mathbf{A}_{\mathbb{Q}\leq 1}$, $\mathbf{A}_{\{1/(2^n):n\in\mathbb{N}\}}$ *have amalgamation. (Note that* $\mathbf{A}_{\mathbb{R}}$ *is not countable while all the others are countable.)*

**Definition 2.4.** *A homogeneous metric space with age* $\mathbf{A}_S$ *is denoted by* $\mathbf{U}_S$, *the* Urysohn space with spectrum $S \cap \mathbb{R}_{\geq 0} \cup \{0\}$.

It follows from Theorem 2.4 that a countable Urysohn space $\mathbf{U}_S$ exists if and only if the age $\mathbf{A}_S$ is countable and has amalgamation.

## 2.3 The Urysohn space U

Let $\mathbf{U}$, the *"Urysohn space"* be the completion of the Urysohn space $\mathbf{U}_{\mathbb{Q}}$.

**Lemma 2.1.** *Let* $M$ *be a finite metric space and* $\epsilon > 0$ *and* $w \in M$. *Then there exists a Katětov map* $f$ *over* $M$ *with* $0 < f(w) < \epsilon$ *and* $f(x) \in \mathbb{Q}$ *for all* $x \in M$.

*Proof.* Let $\mu > 0$ be smaller than $\epsilon$ and smaller than $\min\{\mathrm{d}(w, x) : x \in M \setminus \{w\}\}$ and smaller than:
$\min\{|\mathrm{d}(w, x) - \mathrm{d}(w, y)| :$ for all $x, y \in M \setminus \{w\}$ for which $\mathrm{d}(w, x) \neq \mathrm{d}(w, y)\}$. For every $x \in M \setminus \{w\}$ let $0 < \epsilon_x < \mu/2$ be so that $\mathrm{d}(w, x) + \epsilon_x \in \mathbb{Q}$ and $\epsilon_x > \epsilon_y$ if $\mathrm{d}(w, x) > \mathrm{d}(w, y)$ and $\epsilon_x = \epsilon_y$ if $\mathrm{d}(w, x) = \mathrm{d}(w, y)$.

Then $f$ with $f(x) = \mathrm{d}(w) + \epsilon_x$ and $f(w)$ rational with $\mu/2 < f(w) < \mu$ satisfies the conditions of being a Katětov function over $M$ with values in $\mathbb{Q}$ and with $f(w) < \epsilon$. □

**Corollary 2.3.** *For every finite metric space $M = \{x_i : i \in n \in \omega\}$ and $\epsilon > 0$ there exists a metric space $N = M \cup \{x'_i : i \in \omega\}$ so that $\mathrm{d}(x_i, x'_i) < \epsilon$ for all $i \in n$ and $\mathrm{d}(x'_i, x'_j) \in \mathbb{Q}$ for all $i, j \in n$.*

**Lemma 2.2.** *For every $n \in \omega$ and $\epsilon > 0$ there exists a $\delta > 0$ so that if $\{a_0, a_1, a_2, \ldots, a_{n-1}\} \cup \{b_0, b_1, b_2, \ldots, b_{n-1}\}$ is a metric space with distance function* d *so that if $\mathrm{d}(a_i, b_i) < \delta$ for all $i \in n$ and $f$ is a Katětov function over $\{a_0, a_1, a_2, \ldots, a_{n-1}\}$ then there exists a Katětov function $g$ over $\{b_0, b_1, b_2, \ldots, b_{n-1}\}$ so that $g(b_i) \in \mathbb{Q}$ and $|f(a_i) - g(b_i)| < \epsilon$.*

*Proof.* Let $A = \{a_i : i \in n\}$ and $B = \{b_i : i \in n\}$. Assume that $f(a_0) \geq f(a_1) \geq f(a_2) \geq \cdots \geq f(a_{n-1})$ and let $\delta$ be smaller than $\epsilon/(2n + 1)$ and smaller than $|f(x) - f(y)|/(2n + 1)$ for all $x, y \in A$ with $x \neq y$. For $i \in n$ let $0 < \delta_i < \delta$ be such that $f(a_i) + 2i\delta + \delta_i \in \mathbb{Q}$ and such that $\delta_{i-1} < \delta_i$ for all $1 \leq i \leq n - 1$.

For $i \in n$ let $g(b_i) = f(a_i) + 2i\delta + \delta_i$. Noting that $\mathrm{d}(a_i, a_j) - 2\delta \leq \mathrm{d}(b_i, b_j) \leq \mathrm{d}(a_i, a_j) + 2\delta$ for all $i, j \in n$ it is easy to check that $g$ is a Katětov function over $B$. Also $|g(b_i) - f(a_i)| = |f(a_i) + 2i\delta + \delta_i - f(a_i)| = 2i\delta + \delta_i \leq (2n+1)\delta < \epsilon$. □

**Lemma 2.3.** *Let $M = A \cup B$ with $A = \{a_i : i \in n \in \omega\}$ and $B = \{b_i : i \in n\}$ be a metric space with distance function* d *so that $\mathrm{d}(a_i, b_i) < \epsilon$ for $\epsilon > 0$. Let $f$ be a Katětov function over $A$ and $g$ a Katětov function over $B$ so that $|f(a_i) - g(b_i)| < \epsilon$ for all $i \in n$.*

*Then every function $h$ over $B \cup \{f\}$ with $h(x) = g(x)$ for all $x \in B$ and $2\epsilon \leq g(f) \leq \min\{g(x) : x \in B\}$ and $g(f) \leq \min\{f(x) : x \in A\}$ is a Katětov function over $B \cup \{f\}$.*

*Proof.* We have to varify that $|g(x) - g(f)| \leq \mathrm{d}(f, x) \leq g(x) + g(f)$ for all $x \in B$.

$$|g(b_i) - g(f)| = |g(b_i) - f(a_i) + f(a_i) - g(f)| \leq \epsilon + f(a_i) - g(f) \leq$$
$$f(a_i) - \epsilon \leq \mathrm{d}(f, x) \leq f(a_i) + \epsilon \leq f(a_i) + 2\epsilon \leq f(a_i) + g(f).$$

□

**Theorem 2.8.** *The Urysohn space* $\mathbf{U}$ *has the mapping extension property.*

*Proof.* Let $F = \{x_i : i \in n \in \omega\}$ be a finite subset of $\mathbf{U}$ and $f$ a Katětov map over $F$. We will construct a Cauchy sequence $c_0, c_1, c_2, c_3, \ldots$ of elements in $\mathbf{U}_\mathbb{Q}$ so that for all $x \in F$ the sequence $(\mathrm{d}(c_n, x); n \in \omega)$ is a Cauchy sequence converging to $f(x)$.

For a given $\epsilon/6 > 0$ with $3\epsilon < \min\{f(x) : x \in F\}$, let $\{a_i : i \in n\} \subseteq \mathbf{U}_\mathbb{Q}$ with $\mathrm{d}(a_i, x_i) < \delta$ given by Lemma 2.2. There exists a Katětov function $g$ with over $\{a_i : i \in n\}$ with rational values and $|f(x_i) - g(a_i)| < \epsilon/6$ for all $i \in n$. Because $\mathbf{U}_\mathbb{Q}$ has the mapping extension property, the function $g$ has a realisation, say $c$ in $\mathbf{U}_\mathbb{Q}$. Let $\epsilon'/6 < \epsilon/6$ be given and $\{b_i : i \in n\} \subseteq \mathbf{U}_\mathbb{Q}$ with $\mathrm{d}(b_i, x_i) < \delta'$ given by Lemma 2.2 for $\epsilon'/6$. There exists a Katětov function $h$ over $\{b_i : i \in n\}$ with rational values and $|f(x_i) - h(b_i)| < \epsilon/6$ for all $i \in n$. Then $\mathrm{d}(a_i, b_i) < \epsilon/3$.

According to Lemma 2.3 there exists a Katětov function $k$ over $\{b_i : i \in n\} \cup \{g\}$ with $k(b_i) = g(b_i)$ for all $i \in n$ and $k(c) < \epsilon$ and with rational values. Again, because of the mapping exension property of $\mathbf{U}_\mathbb{Q}$ there exists a realisation $c_1$ of $k$ in $\mathbf{U}_\mathbb{Q}$.

Let $(\epsilon_j : j \in \omega$ be a strictly decreasing sequence of positive numbers with $\sum_{j\in\omega} \epsilon_j < \infty$. Let $\epsilon_0$ determin $c_0$ as above and $\epsilon_1$ the point $c_1$ with $\mathrm{d}(c_0, x_1) < \epsilon_0$ and $\mathrm{d}(c_0, x_i) < \epsilon_0$ and $\mathrm{d}(c_1, x_i) < \epsilon_1$ for all $i \in n$ and so on with $\mathrm{d}(c_j, c_{j+1}) < \epsilon_j$ and $\mathrm{d}(c_j, x_i) < \epsilon_j$ for all $j \in \omega$ and $i \in n$.

□

**Theorem 2.9.** *The Urysohn space* $\mathbf{U}$ *is characterized up to isomorphism by the following properties:*

1. $\mathbf{U}$ *embeds every separable metric space isometrically.*
2. $\mathbf{U}$ *is homogeneous.*
3. $\mathbf{U}$ *is separable.*

*Proof.* Let $\mathbf{V}$ be a separable and homogeneous metric space which embeds every separable metric space isometrically. We will show that $\mathbf{U}$ and $\mathbf{V}$ are isomorphic.

Let $W$ be a countable dense subset of $\mathbf{V}$. Enumerate $W$ into an $\omega$-sequence $(w_i : i \in \omega)$ in which every element of $W$ appears infinitely often.

We will construct a sequence $(a_i :\in \omega)$ so that $\mathrm{d}(a_i, a_j) \in \mathbb{Q}$ and $\mathrm{d}(a_i, w_i) < 1/i$ for all $i, j \in \omega$. Let $a_0 \in \mathbf{V}$ with $\mathrm{d}(a_0, w_0) < 1$. If $a_0, a_1, \ldots, a_{n-1}$ are constructed, let $f$ be a Katětov function over $\{a_i : i \in n\} \cup \{w_n\}$ with values in $\mathbb{Q}$ and with $f(w_n) < 1/n$. Such a function $f$ exists according to Lemma 2.1.

The space $\mathbf{V}$ is homogeneous and hence satisfies the mapping extension property according to Theorem 2.3. Let $a_n$ be the realisation of $f$ in $\mathbf{V}$. The set $S = \{a_i : i \in \omega\}$ is dense in $\mathbf{V}$ and any two elements in $S$ have a rational distance.

Using a back and forth argument we construct an isometry $g$ of $\mathbf{U}_\mathbb{Q}$ into $\mathbf{V}$ so that $g^{-1}$ is an isometry of $S$ into $\mathbf{U}_\mathbb{Q}$. Hence $\mathbf{V}$ has an isometric copy of $\mathbf{U}_\mathbb{Q}$ as a subspace which is dense in $\mathbf{V}$. It follows that $\mathbf{V}$ and $\mathbf{U}_\mathbb{Q}$ are isomorphic.

It remains to prove that $\mathbf{U}$ the completion of $\mathbf{U}_\mathbb{Q}$ satisfies the three characterizing properties listed above. Let $M$ be a separable metric space with $W$ a countable dense subset. According to Theorem 2.2 there exists an embedding $f$ of $W$ into $\mathbf{U}_\mathbb{Q}$. The extension of $f$ to limits of sequences of elements in $W$ is an embedding of $M$ into $\mathbf{U}$.

Let $F, H$ be finite subsets of $\mathbf{U}$ and $f$ an isometry of $F$ onto $H$. The Urysohn space $\mathbf{U}$ satisfies the mapping extension property according to Theorem 2.8 and hence we can use the back and forth argument to construct countable sets $F_1 \supseteq \mathbf{U}_\mathbb{Q} \cup F$ and $H_1 \supseteq \mathbf{U}_\mathbb{Q} \cup H$ and an isometry $f_1$ of $F_1$ onto $H_1$ which is an extension of $f$. Extending $f_1$ to the limits of $F_1$ yields the automorphism of $\mathbf{U}$ extending $f$.

□

## 3 Some indivisibility and divisibility theorems

Much of the information in this section and the next is taken from [1].

Let $S$ be a countable subset of $[a,2a]$ for $a>0$. Then $\mathbf{A}_S$ is countable and has the amalgamation property. According to Theorem 2.4 there exists a unique homogenous metric space $\mathbf{U}_S$ with age $\mathbf{A}_S$. The metric space $\mathbf{U}_S$ has the mapping extension property according to Theorem 2.3.

In order to see that $\mathbf{U}_S$ is indivisible consider the metric space $M$ on the set of pairs $(a,b)$ with $a,b\in\mathbf{U}_S$ and $\mathrm{d}_M\big((a,b),(a_1,b_1)\big)=\mathrm{d}_{\mathbf{U}_S}(b,b_1)$ if $b\neq b_1$ and $\mathrm{d}_M\big((a,b),(a_1,b_1)\big)=\mathrm{d}_{\mathbf{U}_S}(a,a_1)$ if $b=b_1$. For $b\in\mathbf{U}_S$ let $M_b:=\{(a,b):a\in\mathbf{U}$. The metric space $M$ contains for every partition $A\cup B=M$ of $M$ a copy of $\mathbf{U}_S$ which is a subset of $A$ or a copy of $\mathbf{U}_S$ which is a copy of $B$. For if each of the subspaces $M_b$ conains an element $a_b\in A$ then $\{a_b:b\in\mathbf{U}_S\}$ is a copy of $\mathbf{U}_S$ and otherwise there is a $b\in\mathbf{U}_S$ with $M_b\subseteq B$. The metric space $M$ imbeds isometrically into $\mathbf{U}_S$ according to Theorem 2.2. It follows that $\mathbf{U}_S$ is indivisible.

In particular, the metric spaces $\mathbf{U}_{\{1\}}$ and $\mathbf{U}_{\{1,2\}}$ are indivisible. That $\mathbf{U}_{\{1,2,3\}}$ is indivisible follows from [14]. It has recently been proven that $\mathbf{U}_{1,2,3,\dots,n\}}$ is indivisible for all $n\in\mathbb{N}$, [15]. More generally, if $S$ is a finite set of positive real numbers for which there exists a number $a$ with $0\neq a\in S$ and $2\min S\leq a$ and $\min S+a\leq\max S$ it follows that $\mathbf{A}_S$ is indivisible and then from [14] that $\mathbf{U}_S$ is indivisible.

The sequence $a_0,a_1,\dots,a_{n-1},a_n$ of elements in a metric space $M$ is an *$\epsilon$-chain joining $a_0$ and $a_n$* if $d(a_i,a_{i+1})\leq\epsilon$ for all $i\in n$. The space $M$ is *Cantor connected* if any two of its elements can be joined by an $\epsilon$-chain for any $\epsilon>0$. The *Cantor connected component of an element $a\in M$* is the largest Cantor connected subset of $M$ containing $a$. The space $M$ is *totally Cantor disconnected* if the Cantor connected component of every $a$ reduces to $a$. See [16] for more details and references.

For $a\in M$ let $\lambda_\epsilon(a)$ be the supremum of all reals $l\leq 1$ for which there exists an $\epsilon$-chain $a_0,a_1,\dots,a_{n-1},a_n$ with $d(a_0,a_n)\geq l$ containing $a$. (The condition $l\leq 1$ saves us from having to consider the special case $\infty$.) Let

$$\lambda(a):=\sup\{l\in\mathbb{R}\mid\forall\epsilon>0\,(\lambda_\epsilon(a)\geq l)\}.$$

A space $(M;d)$ is *restricted* if $\lambda(a)=0$ for all $a\in M$. It follows that every restricted space is totally Cantor disconnected. There are totally Cantor disconnected spaces which are not restricted. Here is an example with a finite diameter :

**Example 3.1.** *Let $(M;d)$ be the metric space so that:*

1. $M = \{(0,0)\} \cup \{(m,n) \in \mathbb{N} \times \mathbb{N} \mid m < n\}$
2. $d((0,0),(m,n)) = \frac{m+1}{n}$
3. $d((m_1,n),(m_2,n)) = \frac{|m_1-m_2|}{n}$
4. $d((m_1,n_1),(m_2,n_2)) = \frac{m_1+1}{n_1} + \frac{m_2+1}{n_2}$ *when* $n_1 \neq n_2$.

This example falls into the following category:

**Definition 3.1.** *A spider is a metric space $(M;d)$ so that*

1. $M = \{(0,0)\} \cup \{(m,n) \in \mathbb{N} \times \mathbb{N} \mid m < n\}$
2. $d((0,0),(n-1,n)) \geq r$ *for some non-negative $r$ and all $n \in \mathbb{N}^*$*
3. $d((m,n),(m+1,n)) \leq r_n$ *for all $n \in \mathbb{N}^*$ where $lim_{n\to\infty} r_n = 0$.*

With this definition, we have easily

**Lemma 3.1.** *A metric space is restricted if it does not isometrically embed a spider.*

**Definition 3.2.** *Let $M$ be a metric space, $a \in M$ and $0 \leq r < s$. Then*

$$\mathcal{R}_a(r,s) := \{x \in M \mid r \leq d(a,x) < s\}.$$

**Lemma 3.2.** *Let $c \in M$ and $0 \leq r_0 < r_1 < r_2 < r_3$ and $a \in \mathcal{R}_c(r_0,r_1)$ and $b \in \mathcal{R}_c(r_2,r_3)$ then:*

1. $d(a,b) > r_2 - r_1$.
2. $d(x,y) < 2r_2$ *for all $x,y \in \mathcal{R}_c(r_1,r_2)$.*
3. *If $0 < \epsilon < \min\{r_1-r_0, r_3-r_2\}$ and $x_0, x_1, x_2, \ldots, x_{n-1}$ is an $\epsilon$-sequence with $x_i \notin \mathcal{R}_c(r_0,r_1) \cup \mathcal{R}_c(r_2,r_3)$ for all $i \in n$ but with $x_i \in \mathcal{R}_c(r_1,r_2)$ for at least one $i \in n$, then $x_i \in \mathcal{R}_c(r_1,r_2)$ for all $i \in n$.*
4. *Let $f$ be an isometry of $M$ with $f[M] \cap (\mathcal{R}_c(r_0,r_1) \cup \mathcal{R}_c(r_2,r_3)) = \emptyset$ and let $z \in M$ with $\lambda(z) > 2r_2$. Then $f(z) \notin \mathcal{R}_c(r_1,r_2)$.*

*Proof.* Items 1 and 2 follow from the triangle inequality. Item 3 follows from item 1 and item 4 follows from items 2 and 3. □

**Definition 3.3.** *Let $c \in M$ and $0 < l$. Then*

$$\mathcal{E}_c(l) := \bigcup_{n \geq 2,\, n \text{ even}} \mathcal{R}_c\Big(\frac{l(n-1)}{n}, \frac{ln}{n+1}\Big)$$

*and*

$$\mathcal{O}_c(l) := \bigcup_{n \text{ odd}} \mathcal{R}_c\Big(\frac{l(n-1)}{n}, \frac{ln}{n+1}\Big).$$

**Theorem 3.1.** *Let $M$ be a countable metric space. If there exists an element $a \in M$ with $\lambda(a) > 0$ then $M$ is divisible.*

*Proof.* Since $M$ is countable, it can be covered by a family of pairwise disjoint open balls with radius less that $\frac{\lambda(a)}{2}$. In fact, there exists a subset $C$ of $M$ and for every $c \in C$ a positive real $l_c$ so that:

1. $l_c \neq d(x, y)$ for every $c \in C$ and $x, y \in M$.

2. $2l_c < \lambda(a)$ for every $c \in C$.

3. For every element $x \in M$ there is one and only one element $c \in C$ with $x \in \mathcal{R}_c(0, l_c)$.

(After enumerating $M$ into an $\omega$-sequence $m_0, m_1, m_2, m_3, \ldots$ such a set $C$ and function $l$ can be constructed step by step exhausting all of the elements of $M$.)

Let

$$\mathcal{E} := \bigcup_{c \in C} \mathcal{E}_c(l_c) \text{ and } \mathcal{O} := \bigcup_{c \in C} \mathcal{O}_c(l_c).$$

Then $\mathcal{E} \cup \mathcal{O} = M$ and $\mathcal{E} \cap \mathcal{O} = \emptyset$.

Assume for a contradiction that there is an isometry $f$ which maps $M$ into $\mathcal{E}$. Then there is a $c \in C$ so that $f(a) \in \mathcal{E}_c(l_c)$. But this is not possible according to Lemma 3.2 item 4. Similarly it is not possible that $f$ maps $M$ into $\mathcal{O}$. □

**Corollary 3.1.** *A countable metric space which is indivisible is restricted and hence totally Cantor disconnected.*

The second part of the conclusion of the corollary above extends to uncountable metric spaces.

**Theorem 3.2.** *Let $M$ be a metric space and $r$ be a positive real, then there is a partition into two parts $A_0$ and $A_1$ which contains no Cantor connected subspace of diameter larger than $r$.*

**Lemma 3.3.** *Le $M$ be a metric space and $r$ be a positive real number. Then there is a sequence $(E_\mu)_{\mu<\lambda}$ such that:*

1. *$E_0 = \emptyset$ and each $E_\mu$ is open in $M$*
2. *the sequence is strictly increasing and continuous, that is $E_\mu$ is the union of $E_\nu$ for $\nu < \mu$ if $\mu$ is a limit ordinal,*
3. *the union covers $M$*
4. *$F_\mu := E_{\mu+1} \setminus E_\mu$ has diameter at most $r$ and decomposes into two sets $A_{\mu,0}$ and $A_{\mu,1}$ such that each Cantor connected subspace $Y$ of $A_{\mu,i}$ is contained into some subset $\mathcal{B}_Y$ of $F_\mu$ such that $d(y, A_{\mu,i} \cap \mathcal{B}_Y) \geq \epsilon_Y$ for some $\epsilon_Y > 0$ and every $y \in M \setminus (\mathcal{B}_Y \cup E_\mu)$.*

*Proof.* Suppose the sequence defined for all $\nu$, $\nu < \mu$. If $\mu$ is a limit ordinal, set $E_\mu := \bigcup\{E_\nu : \nu < \mu\}$. If $\mu$ is a successor, say $\mu := \nu + 1$, pick $x \in E' := M \setminus E_\nu$, set $\mathcal{R}'_x(0, r/2) := \{y \in E' : d(x,y) < r/2\}$ and set $E_\mu := E_\nu \cup R'_x(0, r/2)$. Decompose $\mathcal{R}'_x(0, r/2)$ into countably many crowns $\mathcal{R}'_x\big(\frac{r(n-1)/2}{n}, \frac{rn/2}{n+1}\big)$ as in the proof of Theorem 3.1 , the union of the even ones gives $A_{\nu,0}$, the rest gives $A_{\nu,1}$.
Finally any Cantor connected subspace $Y$ of $A_{\mu,i}$ must be included into $\mathcal{R}'_x\big(\frac{r(n-1)/2}{n}, \frac{rn/2}{n+1}\big)$ for some $n$, and therefore the required $\mathcal{B}_Y$ may be taken to be $\mathcal{R}'_x\big(0, s_Y\big)$ with $\frac{rn/2}{n+1} < s_Y < \frac{r(n+1)/2}{n+2}$ with $\epsilon_Y := s_Y - \frac{rn/2}{n+1}$.

□

**Proof of Theorem 3.2** Let $A_i := \bigcup\{A_{\mu,i} : \mu < \lambda\}$. Then $A_i$ contains no Cantor connected subspace $X$ of diameter larger than $r$.

Indeed, suppose the contrary. Let $\mu$ be minimum such that $E_\mu$ meets $X$. Clearly $\mu$ is a successor, say $\mu = \nu + 1$. Let $x \in X_\nu := X \cap F_\nu$. Let $Y$ be the Cantor connected component of $x$ in $A_{\nu,i}$ and let $\mathcal{B}_Y$ given by the above lemma. **Claim** $X \subseteq \mathcal{B}_Y$. Indeed suppose not, let $y \in X \setminus \mathcal{B}_Y$, let $\epsilon$, $0 < \epsilon < \epsilon_Y$ and $x_0 := x, \ldots, x_k, \ldots, x_n = y$ be an $\epsilon$ path contained in

$X$. Let $\ell$ be least index such that $x_\ell \notin \mathcal{B}_Y$. From $x_{\ell-1} \in A_{\mu,i} \cap \mathcal{B}_Y$, we get $d(x_\ell, A_{\mu,i} \cap \mathcal{B}_Y) < \epsilon_Y$. A contradiction.

Since $X \subseteq B_Y \subseteq F_\mu$, the diameter of $X$ is at most $r$. The proof is complete.

**Definition 3.4.** *Let $M$ be totally Cantor disconnected. Then*

$$d^*(x,y) := \inf\{\epsilon > 0 \mid \text{ there exists an } \epsilon\text{-sequence containing } x \text{ and } y\}.$$

**Lemma 3.4.** *Let $M$ be totally Cantor disconnected. Then $\mathbb{M}^* := (M; d^*)$ is an ultrametric space.*

*Proof.* Let $x, y, z \in M$ with $d^*(x,y) \geq d^*(x,z) \geq d^*(y,z)$. Then for every $\epsilon > d^*(x,z)$ there are $\epsilon$-sequences joining $x$ to $z$ and $z$ to $y$, then one joining $x$ to $y$, hence $d^*(x,y) \leq \epsilon$. Thus $d^*(x,y) \leq d^*(x,z)$. □

(See [16], Theorem 1 and Lemma 8.)

**Theorem 3.3.** *Let $M$ be a countable homogeneous indivisible metric space then $\mathbb{M}^*$ is an homogeneous indivisible ultrametric space.*

*Proof.* Since $M$ is indivisible it is totally Cantor disconnected, hence $d^*$ is well defined. Since $M$ is homogeneous then $d(x,y) = d(x',y')$ implies $d^*(x,y) = d^*(x',y')$ for all $x, y, x', y' \in M$. From this property, every local isometry of $M$ is a local isometry of $M^*$. Hence, since $M$ is indivisible, $M^*$ is indivisible. Since every automorphism of $M$ is an automorphism of $M^*$, $M^*$ is point-homogeneous. According to Theorem 4.2, $M^*$ is homogeneous. □

**Theorem 3.4.** *Let $M$ be a homogeneous metric space and $V := Spec(M)$. If $M$ is totally Cantor disconnected and every three element metric space $\mathbb{T}$ with $Spec(\mathbb{T}) \subseteq V$ embeds into $M$ then the set $V \setminus \{0\}$ is either contained into an interval of the form $[a \to +\infty)$ for some $a \in \mathbb{R}_+ \setminus \{0\}$ or into an union of intervals of the form $\cup\{[a_{2(n+1)}, a_{2n+1}] : n < \omega\} \cup [a_0 \to +\infty)$ where $\{a_n : n < \omega\}$ is a sequence such that $a_{2n+1} \leq \frac{a_{2n}}{2}$.*

*Proof.* **Claim** For every $w \in V^* := Spec(\mathbb{M}^*)$, $]\frac{w}{2}, w[\cap V = \emptyset$.

Suppose the contrary. Pick $r \in ]\frac{w}{2}, w[\cap V = \emptyset$. Since $w \in V^*$, we may find $x, y$ such that $d^*(x,y) = w$. Let $n < \omega$ and $\epsilon := 2r$, then there is an $\epsilon$-sequence $x_0, \dots, x_n$ containing $x, y$. For $i < n$, let $\mathbb{T}_i := (\{x_i, x_{i+1}, z_i\}, d_i)$ where $d_i(x_i, x_{i+1}) := d(x_i, x_{i+1})$, $d_i(x_i, z_i) = d_i(x_{i+1}) := r$. Each $\mathbb{T}_i$ is a metric space whith spectrum included into $V$, hence can be isometrically embedded into $M$. Since $M$ is homegenous, we may suppose that $z_i \in M$ and that the embedding is the inclusion. By adding the $z_i'$'s to the $x_i's$ we get a $r$-sequence containing $x$ and $y$. Since $r < w$ this gives a contradiction.

Since every element of $V^*$ is the infimum of elements of $V$ it also follows that $]\frac{w}{2}, w[\cap V^* = \emptyset$.

Let $\alpha := Inf(V \setminus \{0\})$. If $\alpha \neq 0$ set $a := \alpha$; in this case $V \setminus \{0\} \subseteq [a \to +\infty)$. If $\alpha = 0$ then, since every element de $V \setminus \{0\}$ majorizes some element of $V^* \setminus \{0\}$ it follows that $Inf(V^* \setminus \{0\}) = 0$ too. Let $\{a_{2n} : n < \omega\}$ be a strictly decreasing sequence of elements of $V^*$ which converges to 0. Set $a_{2n+1} := a_{2n}$. From the Claim $]\frac{a_{2n}}{2}\ a_{2n}[\cap V = \emptyset$, hence $a_{2n+2} \leq a_{2n+1}$. The rest follows.

□

**Theorem 3.5.** *Every unbounded metric space is divisible.*

*Proof.* Let $M$ be an unbounded metric space. Construct a sequence of reals $r_0, r_1, r_2, \ldots$ and a sequence $a_0, a_1, a_2, \ldots$ of elements of $M$ so that for every integer $i \in \mathbb{N}$

1. $d(a_0, a_{i+1}) > 2r_i$.
2. $d(a_0, a_{i+1}) + r_i < r_{i+1}$.

Let $r_0 := 0$ and $a_0 \in M$ be arbitrary. Suppose that $(r_i : i \leq n)$ and $(a_i : i \leq n)$ have already been constructed. From the fact that $M$ is unbounded, we can find $a_{n+1} \in M$ such that $d(a_0, a_{n+1}) > 2r_n$. Next, choose $r_{n+1} > d(a_0, a_{n+1}) + r_n$. Note that the set $\{r_i : i \in \mathbb{N}\}$ such constructed is unbounded.

Let, given any $c \in M$,

$$\mathcal{E} := \bigcup_{i \in \mathbb{N}} \mathcal{R}_c(r_{2i}, r_{2i+1}) \text{ and } \mathcal{O} := \bigcup_{i \in \mathbb{N}} \mathcal{R}_c(r_{2i+1}, r_{2i+2}).$$

We prove that there is no isometric embedding of $M$ into $\mathcal{E}$ or into $\mathcal{O}$.

Let $f$ be an isometric embedding of $M$ into $M$. Let $i$ be minimal so that $d(c, f(a_0)) < r_i$; notice that $i > 0$ and $f(a_0) \in \mathcal{R}_c(r_{i-1}, r_i)$. We have:

$$\begin{aligned} &d(c, f(a_{i+1})) \geq d(f(a_0), f(a_{i+1})) - d(c, f(a_0)) = \\ &= d(a_0, a_{i+1}) - d(c, f(a_0)) > 2r_i - r_i = r_i. \end{aligned}$$

Also:

$$d(c, f(a_{i+1})) \leq d(c, f(a_0)) + d(f(a_0), f(a_{i+1})) \leq r_i + d(a_0, a_{i+1}) < r_{i+1}.$$

It follows that $f(a_{i+1}) \in \mathcal{R}_c(r_i, r_{i+1})$. Therefore $f[M]$ intersects both $\mathcal{E}$ and $\mathcal{O}$.

$\square$

## 4 Ultrametric spaces

A metric space is an *ultrametric space* if it satisfies the strong triangle inequality $d(x,z) \leq \max\{d(x,y), d(y,z)\}$. See [16] for example. Note that a space is an ultrametric space if and only if $d(x,y) \geq d(y,z) \geq d(x,z)$ implies $d(x,y) = d(y,z)$. What we did in the previous section for general metric spaces work for ultrametric spaces.

Let $V$ be a set such that $0 \in V \subseteq \mathbb{R}_+$. Let $\mathcal{M}ult_V$ (resp. $\mathcal{M}ult_{V,<\omega}$) be the collection of ultrametric metric spaces (resp. finite ultrametric spaces) $M$ whose spectrum is included into $V$. Then $\mathcal{M}ult_{V,<\omega}$ is the age of a metric space whose spectrum is $V$; it is closed under embeddability and has the amalgamation property. If $V$ is countable then there is a countable homogeneous ultrametric space $\mathbb{U}ult_V$ whose age is $\mathcal{M}ult_{V,<\omega}$ and has spectrum $V$; we call it the *Urysohn ultrametric space with spectrum* $V$. We give a description of this space in Proposition 4.1.

For a given set $V$, $\mathbf{U}_V$ and $\mathbb{U}ult_V$ are in general different, except if $V = \{a_n : n \in D\}$ where $D$ is an interval of the set $\mathbb{I}$ of integers and $2a_{i+1} < a_i$ for all $i, i+1 \in D$.

Homogeneous ultrametric are easy to describe. In fact ultrametric spaces can be described by means of real-valued trees. An ordered set $P$ is a *forest* if for every $x \in P$ the set $\downarrow x := \{y \in P : y \leq x\}$ is a chain; this is a *tree* if in addition every pair of elements of $P$ has a lower bound. If every pair $x, y \in P$ has an infimum, denoted $x \wedge y$, we will say that $P$ is a *meet-tree*. We say that $P$ is *ramified* if for every $x, y \in P$ such that $x < y$ there is some $y' \in P$ such that $x < y'$ and $y'$ incomparable to $y$. In the sequel, we consider ramified meet-trees such that every element is below some maximal element. These posets are meet-semilattices generated by their coatoms. We will need the following property

**Lemma 4.1.** *Let $P$ be a ramified meet-tree such that every element is below some maximal element. For every $x \in P \setminus \max(P)$ there is a subset $X \subseteq \max(P)$ of maximum cardinality such that $x = a \wedge b$ for every pair of distinct elements $a, b$ of $X$*

*Proof.* For two elements $a$ and $b$ above an element $x$, set $a \equiv b$ if $x \neq a \wedge b$. Observe that this is an equivalence relation. A set $X$ which meets each equivalence classe has maximum size. □

The cardinality of $X$, denoted $d_P(x)$, is the *degree* of $x$. For $x \in \max(P)$ we set $d_P(x) := 0$. If $P$ is finite or well-founded, this is the number of upper-covers of $x$, which is the ordinary notion of out-degree in the poset $P$. Two meet-trees $P$, $P'$ are *isomorphic* if they are isomorphic as posets; in particular, an isomorphism $f$ from $P$ to $P'$ preserves meets, that is $f(x \wedge y) = f(x) \wedge f(y)$ for all $x, y \in P$. A *positive real-valued meet-tree* , valued meet-tree for short, is a pair $(P, v)$ where $P$ is a meet-tree and $v$ a map from $P$ to $\mathbb{R}^*$. Two valued meet-trees $(P, v)$, $(P, v')$ are *isomorphic* if there is an isomorphism $f$ from $P$ onto $P'$ such that $v' \circ f = v$. A *subtree* of a meet-tree $P$ is a subset $P'$ of $P$ such that the meet of two arbitrary elements of $P'$ belongs to $P'$; a *valued subtree* of a valued meet-tree $(P, v)$ is a pair $(P', v')$ where $P'$ is a subtree and $v' := v_{\restriction P'}$. The *age* of a valued meet-tree $(P, v)$ is the collection of finite valued meet-trees which are isomorphic to some valued subtree of $P$.

Let $M$ be a metric space, $r \in R_+$ and $a \in M$, the *closed ball of center $a$, radius $s$* is the set $\mathcal{B}_a(s) := \{x \in M \mid d(a, x) \leq s\}$. The *diameter* of a subset $B$ of $E$ is $\delta(B) := sup\{d(x, y) : x, y \in B\}$. We denote by $\mathcal{B}all(M)$ be the collection of closed balls of $M$ and by $Nerv(M) := \{\mathcal{B}_a(s) : a \in M, s \in Spec(M, a)\}$.
Notice that $\delta(B_a(s)) = s$ whenever $s \in Spec(M, a)$, but more genarally let us recall the following fact.

**Lemma 4.2.** *If $M$ is an ultrametric space then for every $B \in \mathcal{B}all(M)$ and $a \in B$, $B = \mathcal{B}_a(s)$ where $s := \delta(B)$,*

We give below a description of ultrametric spaces in terms of valued trees. A very close description is given by Lemin [17] (who instead of $Nerv(M)$ considered $Ball(M)$).

**Theorem 4.1.** *1. Let $M := (M, d)$ be an ultrametric space, then the pair $(P, v)$, where $P := (Nerv(M), \supseteq)$, $v$ is the diameter function, is a valued ramified meet-tree such that every element is below some maximal element and the map $\delta : Nerv(M) \to Spec(M)$ is strictly decreasing, $\delta(X) = 0$ for every $X \in M' := \max(P)$ and $d(x, y) = \delta(\{x\} \wedge \{y\})$ for every $x, y \in M$.*

2. *Conversely, let $(P, v)$a valued ramified meet-tree such that every element is below some maximal element of $P$ and the map $v : P \to \mathbb{R}_+$ is strictly decreasing with $v(x) := 0$ for each maximal element of $P$. Then the map $d$ defined on $M' := \max(P)$ by $d(x, y) := v(x \wedge y)$ is an ultrametric distance and $Nerv(M') = up(P)_{\restriction M'}$ where $up(P)_{\restriction M'} := \{M' \cap \uparrow x : x \in P\}$.*

*3. The two correspondences are inverse of each other.*

*Proof.* 1) According to Lemma 4.2, balls are disjoint or comparable w.r.t. inclusion, hence $P$ is a tree. Since $\{x\} \in P$ for every $x \in M$, $P$ is ramified and every element is below some maximal element. Let $B, B' \in P$. Pick $a \in B$, $a' \in B'$ and set $r := d(a, a')$. It is easy to se that $\mathcal{B}_a(r) = B \wedge B'$, hence $P$ is a meet-tree. The properties of $\delta$ follow from Lemma 4.2.

2) a) $d$ is an ultrametric distance : Let $x \in M'$. We have $d(x, x) := v(x \wedge x) = v(x) = 0$. If $x \neq y$ then, since $v$ is strictly decreasing, $d(x, y) := v(x \wedge y) > v(x) = 0$. Clearly $d(x, y) = d(y, x)$. Let $x, y, z \in M'$. Since $P$ is a tree, $x \wedge z$ and $y \wedge z$ are comparable. Suppose $x \wedge z \leq y \wedge z$. Then $x \wedge z \leq x \wedge y$. Since $v$ is decreasing, we have $d(x, y) \leq d(x, z) \leq \max\{d(x, z), d(y, z)\}$.

b) $Nerv(M') = up(P)_{\restriction M'}$ :

Let $B := M' \cap \uparrow x \in up(P)_{\restriction M'}$, $r := v(x)$ and $y \in B$.

**Claim 1.** $B = \mathcal{B}_y(r)$ and $r \in Spec(M', y)$. Thus $B \in Nerv(M')$.

Indeed, let $z \in B(y, r)$, that is $v(y \wedge z) \leq r$. Since $x \leq y$ and $y \wedge z \leq y$, $x$ and $y \wedge z$ are comparable, since $v$ is strictly decreasing $x \leq y \wedge z$ hence $z \in B$. Conversely, if $z \in B$ then $x \leq y \wedge z$ thus, since $v$ is strictly decreasing, $d(y, z) := v(y \wedge z) \leq v(x) = r$ proving $z \in \mathcal{B}_y(r)$. Thus $B = \mathcal{B}_y(r)$ as claimed. Since $P$ is ramified and every element of $P$ is below some element of $M'$, there is some $z \in M'$ such that $x = y \wedge z$. Clearly, $z \in B$ and $r = d(y, z)$ thus $r \in Spec(M', y)$.

Let $B := B(y, r) \in Nerv(M')$ with $r \in Spec(M', y)$

**Claim 2.** $B \in up(P)_{\restriction M'}$.

Indeed, since $r \in Spec(M', y)$ there is some $z \in M'$ such that $d(y, z) = r$. Let $x := y \wedge z$. Since $v(x) = r$ we get $B = \uparrow x \cap M' \in up(P)_{\restriction M'}$ from the previous claim.

3) We simply note that if $P := (Nerv(M), \supseteq)$ then, for $M' := \max(P)$, $P$ is isomorphic to $(up(P)_{\restriction M'}, \supseteq)$; moreover, if $v : P \to \mathbb{R}_+$ is the diameter function associated to $M$, then $v(x) = \delta'(M' \cap \uparrow x)$ where $\delta$ is the diameter function associated to the metric defined on $M'$ in part 2. □

**Lemma 4.3.** *Two ultrametric spaces have the same age if and only if the corresponding valued trees have the same age.*

The verification is immediate.

The *reduced valued tree associated to* an ultrametric space $M$ is the pair $(P', v')$ where $P' := P \setminus \max(P)$ and $v' := v_{\restriction P'}$. The age of the reduced valued tree does not determine the age of the tree, because the information about the degree, in $P$, of terminal nodes in $P'$ is missing. With this information added, we have easily:

**Lemma 4.4.** *If two reduced valued trees are isomorphic via a map which preserves the degree of the original trees then the ultrametric spaces have the same age.*

Let $\lambda$ be a chain and let $\overline{a} := (a_\mu)_{\mu\in\lambda}$ such that $2 \leq a_\mu \leq \omega$. Set $\omega^{[\overline{a}]} := \{\overline{b} := (b_\mu)_{\mu\in\lambda} : \mu \in \lambda \Rightarrow b_\mu < a_\mu$ and $supp(\overline{b}) := \{\mu < \omega : b_\mu \neq 0\}$ is finite $\}$. If $a_\mu = \omega$ for every $\mu \in \lambda$, the set $\omega^{[\overline{a}]}$ is usually denoted $\omega^{[\lambda]}$. Add a largest element, denoted $\infty$ to $\lambda$. Given $\overline{b}, \overline{c} \in \omega^{[\overline{a}]}$, set $\Delta(\overline{b}, \overline{c}) := \infty$ if $\overline{b} = \overline{c}$, otherwise $\Delta(\overline{b}, \overline{c}) := \mu$ where $\mu$ is the least member of $\lambda$ such that $b_\mu \neq c_\mu$.

Suppose $\lambda$ be countable. Let $w : \lambda \cup \{\infty\} \to \mathbb{R}_+$ be a strictly decreasing map such that $w(\infty) = 0$, let $d_w := w \circ \Delta$ and let $V$ be the image of $w$. For $\mu \in \lambda \cup \{\infty\}$ set $\downarrow^* \mu :=\downarrow \mu \setminus \{\mu\}$. Let $P' := \{f_{\restriction\downarrow^*\mu} : f \in \omega^{[\overline{a}]}, \mu \in \lambda \cup \{\infty\}\}$ ordered by extension and let $v'(f_{\restriction\downarrow^*\mu}) := w(\mu)$.

We have the following property, which is easy to check.

**Lemma 4.5.** *The pair $M := (\omega^{[\overline{a}]}, d_w)$ is an ultrametric space, $Spec(M) = V$ and the valued tree associated to $M$ is isomorphic to $(P', v')$.*

We say that $M$ is *point-homogeneous* if the automorphism group of $M$ acts transitively on $M$.

**Theorem 4.2.** *Let $M$ be a countable ultrametric space, $P := (Nerv(M), \supseteq)$, $v : P \to \mathbb{R}_+$ where $v(B) := \delta(B)$, $M' := \max(P)$. The following properties are equivalent:*

*(i)* *$M$ is isometric to some $(\omega^{[\overline{a}]}, d_w)$ .*

*(ii)* *$M$ is homogeneous;*

*(iii)* *$M$ is point-homogeneous;*

*(iv)* *(a) $v(x) = v(y) \Rightarrow d_P(x) = d_P(y)$ for every $x, y \in P$;*
*(b) $v[\downarrow x] = v[\downarrow y]$ for every $x, y \in \max(P)$.*

*Proof.* $(i) \Rightarrow (iv)$ Let $M := (\omega^{[\overline{a}]}, d_w)$. According to Lemma 4.5, the valued tree associated to $M$ is isomorphic to $(P', v')$. Condition $(b)(iv)$ immediately follows. Let $x := f_{\restriction\downarrow^*\mu} \in P'$; if $\mu = \infty$ then $d_{P'}(x) = 0$, otherwise $d_{P'}(x) = a(\mu)$. Thus Condition $(a)(iv)$ holds too.

$(ii) \Rightarrow (iii)$ Trivial

$(iii) \Rightarrow (iv)$ Suppose $M$ point homogeneous. First, Condition $(b)(iv)$ holds. Indeed, let $x, y \in M' := \max(P)$. Then $x := \{x'\}$ and $y := \{y'\}$, with $x', y' \in M$. Let $f$ be an isometry from $M$ onto itself such that $f(x') = y'$. Then $Spec(x', M) = Spec(y', M)$ and the result follows. Next, Condition $(a)(iv)$ holds. Let $x := B \in P, y := C \in P$ and $r := v(x) = v(y)$. Pick $x' \in B$, $y' \in C$. Let $f$ be an isometry from $M$ onto itself such that $f(x') = y'$. Then $f(B) = C$. For two elements $x', y'$ of $B$, set $x' \equiv y'$ if $d(x', y') < r$. This relation is an equivalence relation whose number of classes is the degree of $x := B$ in the poset $P := Nerv(M)$. The desired conclusion follows.

$(iv) \Rightarrow (ii)$ Let $f$ be an isometry from a finite subset $A$ of $M$ onto a subset $B$ of $M$. Let $x \in M \setminus A$. We prove that $f$ extends to an isometry defined on $A \cup \{x\}$. If $A$ is empty, we may send $x$ onto any element $b$ of $M$. If $A$ is non-empty, set $r := \min(\{d(x, y) : y \in A\})$. In order to extend $f$ we only need to send $x$ onto some $b \in M$ such that $f(B(x, r)) = B(b, r) \cap f(A)$. There is some $u \in P$ such that $x \wedge x' = u$ for all $x' \in B(x, r) \cap A$ and moreover $v(u) = r$. Select $y \in f(B(x, r))$. Since $v[\downarrow x] = v[\downarrow y]$ there is some $u' \in \downarrow y$ such that $v(u') = r$. Since $d_P(u) = d_P(u')$, there is $b \in M$ such that $y' \wedge b = u'$ for all $y' \in f(A)$. Such an element will do.

$(ii) \Rightarrow (i)$. Let $\lambda := Spec(M) \setminus \{0\}$ ordered with the dual of the order induced by the natural order on $\mathbb{R}$, let $w : \lambda \cup \{\infty\} \to \mathbb{R}_+$ with $w(x) := x$ for $x \in \lambda$ and $w(\infty) := 0$ and let $\overline{a} : \lambda \to \omega + 1$ such that $\overline{a} \circ w = d_P$ (such a map exists because of (iv) Condition 1).

**Claim** $M$ is isometric to $(\omega^{[\overline{a}]}, d_w)$. According to the implications $(i) \Rightarrow (iv) \Rightarrow (ii)$ proved above, $(\omega^{[\overline{a}]}, d_w)$ is homogeneous. Since $M$ is homogeneous, it suffices to prove that $(\omega^{[\overline{a}]}, d_w)$ and $M$ have the same age to get the desired conclusion. From the implication $(iii) \Rightarrow (iv)$, the reduced valued trees associated to $(\omega^{[\overline{a}]}, d_w)$ and $M$ are isomorphic by an isomorphism which preserves the degree. From Lemma 4.4, $(\omega^{[\overline{a}]}, d_w)$ and $M$ have the same age. □

**Proposition 4.1.** *The space $(\omega^{[\lambda]}, d_w)$ is the countable homogeneous ultrametric space $\mathbb{U}lt_V$ associated with $V$.*

*Proof.* We only neeed to prove that every finite ultrametric space $\mathbb{M} := (M, d)$ with spectrum included into $V$ embeds isometrically into $(\omega^{[\lambda]}, d_w)$. We argue by induction on the number $n$ of elements of $M$. If $n \leq 1$, the result is obvious. Suppose $n \geq 2$. Let $x \in M$. We may suppose that there is an isometric embedding $f$ of $\mathbb{M}_{-x} := \mathbb{M}_{\restriction M \setminus \{x\}}$ into $(\omega^{[\lambda]}, d_w)$. We prove that

$f$ extends to $\mathbb{M}$. Set $r := \min(\{d(x,y) : y \in M \setminus \{x\}\})$ and $\mu \in \lambda$ such that $w(\mu) = r$. In order to extend $f$ we only need to find some element $b \in \omega^{[\lambda]}$ such that $f(B(x,r)) = B(b,r) \cap f(M \setminus \{x\})$. For every $\overline{b}', \overline{b}'' \in f(B(x,r))$ we have $b'_{\mu'} = b''_{\mu'}$ for all $\mu' < \mu$. Select $b \in \omega^{[\lambda]}$ such that $b_{\mu'} = b'_{\mu'}$ for all $\mu' < \mu$ and $b_\mu \in \omega \setminus \{b'_\mu : \overline{b}' \in f(B(x,r)\}$. □

## 4.1 Indivisible ultrametric spaces

**Definition 4.1.** *Let $M$ be a metric space, $a \in M$ and $0 \leq r < s$. Then*

$$\mathcal{R}_a(r,s) := \{x \in M \mid r \leq d(a,x) < s\}.$$

**Lemma 4.6.** *Let $M$ be an indivisible ultrametric space. Then the spectrum of every element of $M$ is dually well founded.*

*Proof.* Let $a \in M$. Suppose for a contradiction that $r_0 = 0 < r_1 < r_2 < r_3 < \ldots$ is an infinite sequence of reals in the spectrum of $a$. Let $s$ be its supremum. Cover $M$ by a family $\mathcal{B} := \{\mathcal{R}_{a_\alpha}(0,s) : \alpha < \kappa\}$ of open balls of radius $s$ such that $a_\alpha \notin M_\alpha := \cup\{\mathcal{R}_{a_\beta}(0,s) : \beta < \alpha\}$ (with the convention that if $s = \infty$ then $\mathcal{B}$ consists of $M$). Since $d$ is an ultrametric distance, these balls are pairwise disjoint and therefore, the rings $\mathcal{R}_{a_\alpha}(r_i, r_{i+1})$ make-up a partition of $M$. Let:

$$\mathcal{E} := \bigcup_{\alpha<\kappa, i\in\omega} \mathcal{R}_{a_\alpha}(r_{2i}, r_{2i+1}) \text{ and } \mathcal{O} := \bigcup_{\alpha<\kappa, i\in\omega} \mathcal{R}_{a_\alpha}(r_{2i+1}, r_{2i+2})$$

and let $f$ be an isometry of $M$ into $M$. Let $\alpha < \kappa$ and $i \in \omega$ so that $f(a) \in \mathcal{R}_{a_\alpha}(r_i, r_{i+1})$. Let $b \in M$ with $d(a,b) = r_{i+1}$.

Then $d(f(a), f(b)) = r_{i+1}$ and because $d(f(a), a_\alpha) < r_{i+1}$ it follows that that $d(a_\alpha, f(b)) = r_{i+1} < s$. Thus $f(b) \in \mathcal{R}_{a_\alpha}(r_{i+1}, r_{i+2})$. □

**Corollary 4.1.** *If an ultrametric space is indivisible then the collection of balls, once ordered by inclusion, is dually well-founded and the diameter is attained.*

*Proof.* Let $(B_n)_{n<\omega}$ be an increasing sequence of balls of an ultrametric space $M := (M,d)$. Pick $a \in \cap\{B_n : n \in\}$. Since $M$ is ultrametric, $a$ is the center of each $B_n$ thus their radii belong to the spectrum of $a$. If $M$ is indivisible, then from Lemma 4.6 above *Spec*$(M,a)$ is dually well-founded, thus the sequence is eventually constant. Let $s$ be the maximum of *Spec*$(M,a)$. Let $x, y \in M$. We have $d(x,y) \leq \max(\{d(x,a), d(y,a)\}) \leq s$, hence $s$ is the maximum of the spectrum of $M$, that is the diameter of $M$. □

**Theorem 4.3.** *Let $M$ be a denumerable ultrametric space. The following properties are equivalent:*

*(i) $M$ is isometric to some $\mathbb{U}lt_V$, where $V$ is dually well-ordered;*

*(ii) $M$ is point-homogeneous, $P := (Nerv(M), \supseteq)$ is well founded and the degree of every non maximal element is infinite;*

*(iii) $M$ is homogeneous and indivisible;*

*Proof.* $(i) \Rightarrow (ii)$ By definition, $\mathbb{U}lt_V$ is homogeneous, hence point-homogeneous. In fact, according to Proposition 4.1, $M$ is isometric to some $(\omega^{[\lambda]}, d_w)$ where $\lambda$ is a well-ordered chain. Thus, from Lemma 4.5, $P := (Nerv(M), \supseteq)$ is well-founded and the degree of every non maximal element is infinite.

$(iii) \Rightarrow (i)$ Suppose that $(iii)$ holds. Theorem 4.2 asserts that $M$ is isometric to some $(\omega^{[\overline{a}]}, d_w)$. Since $M$ is indivisible, it follows from Lemma 4.6 that $V := Spec(M)$ is well-founded, hence we may suppose that $\lambda$ is an ordinal. To conclude it suffices to prove that $a_\mu = \omega$ for every $\mu < \lambda$. Let $\mu < \lambda$; set $r := w(\mu)$. First, observe that $M = \cup\mathcal{B}$ where $\mathcal{B}$ is a collection of pairwise disjoint balls, all of diameter $r$. Next, each member $B$ of $\mathcal{B}$ is the union of $a_\mu$ balls $B_i$ each of smaller diameter than $r$. Indeed, since $M$ is point-homogeneous, all balls having the same radius are isometric spaces, thus it suffices to prove this property for the ball $B := \mathcal{B}_0(r)$, where 0 is the ordinal sequence which only takes value 0. This is easy: set $\overline{x}_i := (b_\nu)_{\nu<\lambda}$ where $i < a_n$, $b_\nu = 0$ if $\nu \neq \mu$ and $b_\mu := i$ otherwise, set $r^+ := w(\mu^+)$ where $\mu^+ := \mu + 1$ if $\mu + 1 < \lambda$ and $\mu^+ := \infty$ otherwise, then $B := \cup\{B(\overline{x}_i, r^+) : i < a_\mu\}$. With these two observations we have $M = \cup\{M_i : i < a_\mu\}$ where $M_i := \cup\{B_i : B \in \mathcal{B}\}$. Clearly, there is no isometry from $M$ into an $M_i$ hence if $a_\mu < \omega$, $M$ cannot be indivisible.

$(ii) \Rightarrow (iii)$ According to Theorem 4.2, $M$ is homogeneous. Let us show that it is indivisible. Let $f : M \to 2$ be a partition of $M$ into two parts. Set $\mathcal{F}_0$ be the set of balls $B \in Nerv(M)$ such that there is some isometry $\varphi_B$ from $B$ into$B \cap f^{-1}(0)$ and let $M_0 := \cup\mathcal{F}_0$.

**Claim 1** There is an isometry from $M_0$ to $M_0 \cap f^{-1}(0)$.

Indeed, let $\mathcal{F}'_0$ be the subset of $\mathcal{F}_0$ made of its maximal members (w.r.t. inclusion). Let $\varphi := \cup\{\varphi_B : B \in \mathcal{F}'_0\}$. Since balls are either disjoint or comparable, $\varphi$ is a map and, since $P := (Nerv(M), \supseteq)$ is well-founded, $M_0 = \cup\mathcal{F}'_0$, hence the domain of $\varphi$ is $M_0$.

For $B$ in $Nerv(M)$, set $Pred(B) := \max(\{B' : B' \subset B, B' \in Nerv(M)\})$.

**Claim 2** If $B \notin \mathcal{F}_0$ then $Pred(B) \cap \mathcal{F}_0$ is finite.

Indeed, suppose not. Then, since the space is point-homogeneous, all members of $Pred(B)$ have the same radius and there is an isometry $\psi$ from $B$ into $B$ which transforms each member of $Pred(B)$ to a member of $Pred(B) \cap \mathcal{F}_0$. Let $\varphi := \cup\{\varphi_{B'} : B' \in Pred(B) \cap \mathcal{F}_0\}$. Then $\varphi$ is an isometry from $\cup(Pred(B) \cap \mathcal{F}_0)$ into $B \cap f^{-1}(0)$. Consequently, $\varphi \circ \psi$ is an isometry from $B$ into $B \cap f^{-1}(0)$, thus $B \in \mathcal{F}_0$, a contradiction.

Suppose that $M \notin \mathcal{F}_0$. We construct an isometry $h$ from $M$ into $f^{-1}(1) \setminus M_0$ as follows. We start with an enumeration $(x_n)_{n<\omega}$ of the elements of $M$. According to Claim 1, $M \setminus M_0 \neq \emptyset$. We may also suppose that it contains an element of $f^{-1}(1)$ (otherwise the union of the identity map on $M \setminus M_0$ and an isometry as constructed in Claim 1, is an isometry from $M$ into $f^{-1}(0)$). Let $y_0$ such an element. We set $h(x_0) := y_0$.

Suppose $h$ be defined for all $m$, $m < n$. Let $p := \min(\{d(x_m, x_n) : m < n\})$. Let $I := \{i, i < n : d(x_i, x_n) := p\}$ . Let $B := \mathcal{B}_{h(i)}(p)$ for $i \in I$. This set does not depend upon the choice of $i$. Since $h(i) \in f^{-1}(1) \setminus M_0$, $B \notin \mathcal{F}_0$. For each $i \in I$ let $B'_i$ such that $h(i) \in B'_i \in Pred(B)$. According to Claim 2, there is some $B'' \in Pred(B) \setminus \mathcal{F}_0$ which is distinct from all the $B'_i s$. As in our first step, $B'' \setminus M_0$ is nonempty and in fact contains an element, say $y_n$ of $f^{-1}(1)$. We set $h(x_n) := y_n$.

$\square$

## References

[1] C. Delhommé, C. Laflamme, M. Pouzet, N. Sauer, Divisibility of countable metric spaces, em European Journal of Combinatorics. Volume 28, Issue 6 (August 2007) Pages: 1746-1769, 2007, ISSN: 0195-6698

[2] M. El-Zahar, N. W. Sauer, The indivisibility of the homogeneous $K_n$-free graphs, *J. Combin. Theory Ser. B*, 47, 162-170, 1989.

[3] R. Fraïssé, Theory of relations, Studies in Logic and the Foundations of Mathematics, 145, North-Holland Publishing Co., Amsterdam, 2000.

[4] A. S. Kechris, V. Pestov and S. Todorcevic, Fraïssé limits, Ramsey theory, and topological dynamics of automorphism groups, *Geom. Funct. Anal.*, 15, 106-189, 2005.

[5] J. Lopez-Abad and L. Nguyen Van Thé, The oscillation stability problem for the Urysohn sphere: A combinatorial approach, *Topology Appl.*, *submitted*, 2006.

[6] V. D. Milman, A new proof of A. Dvoretzky's theorem on cross-sections of convex bodies, *Funkcional. Anal. i Priložen.*, 5 (4), 28-37, 1971 (in Russian).

[7] L. Nguyen Van Thé, Théorie de Ramsey structurale des espaces métriques et dynamique topologique des groupes d'isométries, Ph.D. Thesis, Université Paris 7, 2006.

[8] E. Odell, T. Schlumprecht, The distortion problem, *Acta Mathematica*, 173, 259-281, 1994.

[9] V. Pestov, Dynamics of Infinite-dimensional Groups and Ramsey-type Phenomena, *Publicações matematicas (IMPA, Rio de Janeiro)*, 2005.

[10] S. Mazur, Quelques propriétés des espaces euclidiens, *C.R. Acad. Sc. Paris*, **207**, 761-764, 1938.

[11] A. Pełczinski and S. Rolewicz, Best norms with respect to isometry groups in normed linear spaces, *Short Communication in International Math. Congress in Stockholm*, **104**, 1962.

[12] F. Cabello Sánchez, Regards sur le probléme des rotations de Mazur, *Extracta Math.,* **12**, 97-116, 1997.

[13] A. Dvoretzky, A theorem on convex bodies and applications to Banach spaces, *Proc. Nat. Acad. Sci. USA* **45**, (1959), 223-226.

[14] N.W. Sauer, Canonical vertex partitions, *Combinatorics Probability and Computing* **12** (2003) no. 5-6, 671-704.

[15] L. Nguyen Van The, N Sauer, *The Urysohn Sphere is oscillation stable*, submitted.

*Divisibility of countable metric spaces*, European Journal of Combinatorics. Volume 28, Issue 6 (August 2007) Pages: 1746-1769, 2007, ISSN: 0195-6698

[16] A. J. Lemin, On ultrametrization of general metric spaces, *Proceedings of the American Mathematical Society* **131** (2002) no. 3, 979-989.

[17] A. Lemin,The category of ultrametric spaces is isomorphic to the category of complete, atomic, tree-like, and real graduated lattices LAT*, *Algebra universalis* **50** (2003) 35-49.

[18] L.Nguyen VanThé, Big Ramsey degrees and indivisibility in classes of ultrametric spaces, preprint, July 5, 2005, 13pp.