Dynamics of infinite-dimensional groups and Ramsey-type phenomena

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Contents

Preface

This set of lecture notes is an attempt to present an account of basic ideas of a small new theory located on the crossroads of geometric functional analysis, topological groups of transformations, and combinatorics. The presentation is certainly incomplete.

The lecture notes are organized on a "need-to-know" basis, and in particular I try to resist the temptation to achieve extra generality for its own sake, or else introduce concepts or results that will not be used more or less immediately.

The notes have grown out of mini-courses given by me at the Nipissing University, North Bay, Canada (26–27 May 2004), at the 8ième atélier sur la théorie des ensembles, CIRM, Luminy, France (20–24 september, 2004), and in preparation for a mini-course to be given at IMPA, Rio de Janeiro, Brazil (25–29 July 2005).

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As always, the author bears full responsibility for the remaining errors and omissions, which are certainly there; the hope is that none of them is too serious. The reader is warmly welcomed to send me comments at <vpest283@uottawa.ca>.

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Chapter 0

Introduction

The "infinite-dimensional groups" in the title of this set of lecture notes refer to a vast collection of concrete groups of automorphisms of various mathematical objects. Examples include unitary groups of Hilbert spaces and, more generally, operator algebras, the infinite symmetric group, groups of homeomorphisms, groups of automorphisms of measure spaces, isometry groups of various metric spaces, etc. Typically such groups are supporting additional structures, such as a natural topology, or sometimes the structure of an infinite-dimensional Lie group. In fact, it appears that those additional structures are often encoded in the algebraic structure of the groups in question. The full extent of this phenomenon is still unknown, but, for instance, a recent astonishing result by Kechris and Rosendal [100] states that there is only one non-trivial separable group topology on the infinite symmetric group S_∞. It could also well be that there are relevant structures on those groups that have not yet been formulated explicitely.

Those infinite-dimensional groups are being studied from a number of different viewpoints in various parts of mathematics: most notably, representation theory [135, 128], logic and descriptive set theory [11, 91], infinite-dimensional Lie theory [85, 136], theory of topological groups of transformations [5, 186] and abstract harmonic analysis [17, 37], as well as from a purely group theoretic viewpoint [9, 18] and from that of set-theoretic topology [164]. Together, those studies mark a new chapter in the theory of groups of transformations, a post Gleason–Yamabe–Montgomery–Zippin development [126].

The present set of lecture notes outlines an approach to the study of infinite-dimensional groups based on the ideas originating in geometric functional analysis, and exploring an interplay between the dynamical properties of those groups, combinatorial Ramsey-type theorems, and geometry of high-dimensional structures (asymptotic geometric analysis).

The theory in question is a result of several developments, some of which have happened independently of, and in parallel to, each other, so it is a "partially ordered set." On the other hand, each time those developments have been absorbed into the theory and put into connection to each other, so the set is, hopefully, "directed."

It is not clear how far back in time one needs to go towards the beginnings. For example, one of the three components of the theory — the phenomenon of concentration of measure, the subject of study of the modern-day asymptotic geometric analysis — was explicitely described and studied in the 1922 book by Paul Lévy [103] for spheres. On the other hand, another major manifestation of the phenomenon – the Law of Large Numbers – was stated by Émile Borel in the modern form back in 1904, see a discussion in [120]. And according to Gromov [76], the concentration of measure was first extensively used by Maxwell in his studies of statistical physics (Maxwell–Boltzmann distribution law, 1866).

We will take as the point of departure for interactions between geometry of high-dimensional structures, topological transformation groups, and combinatoric the *Dvoretzky theorem* [38, 39], regarded in light of later results by Vitali Milman.

As Aryeh Dvoretzky remarks in his 1959 Proc. Nat. Acad. Sci. USA note [38],

> "this result, or various weaker variations, have often been conjectured. However, the only explicit statement of this conjecture in print known to the author is in a recent paper by Alexandre Grothendieck" [80].

Interestingly, Grothendieck's paper was published (in the *Bol. Soc. Mat. São Paulo*) during the period when he was a visiting professor at the Universidade de São Paulo, Brazil (1953–54).

Theorem 0.0.1 (Dvoretzky theorem, 1959). *For every* $\varepsilon > 0$*, each* n*-dimensional normed space* X *contains a subspace* E *of dimension* $\dim E = k \geq c\varepsilon^2 \log n$*, which is Euclidean to within* ε*:*

$$d_{BM}(E, \ell^2(k)) \leq 1 + \varepsilon.$$

Here d_{BM} is the *multiplicative Banach–Mazur distance* between two isomorphic normed spaces:

$$d_{BM}(E, F) = \inf\{\|T\| \cdot \left\|T^{-1}\right\| : \ T \text{ is an isomorphism}\}.$$

(In geometric functional analysis, *isomorphic* normed space are those isomorphic as topological vector spaces, that is, topologically isomorphic.)

The geometric meaning of the Banach-Mazur distance is clear from the following figure.

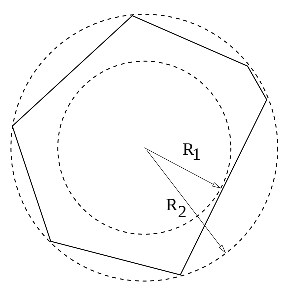

Figure 0.1: Banach-Mazur distance: $R_2 = (1 + \varepsilon)R_1$.

The logarithmic dependence of k on n is best possible for all sufficiently small $\varepsilon > 0$. With k fixed, the smallest Banach-Mazur distance between a k-dimensional subspace of an n-dimensional normed space goes to zero only as the logarithm of n, and therefore, generally speaking, the dimension of the space X should be very high in order to guarantee the existence of "almost elliptical" sections. For instance, Fig. 0.2 shows sections of the unit cube $\mathbb{I}^n$ (that is, the unit ball of $\ell^\infty(n)$) by random 2-planes. Those sections are normalized to the same size.

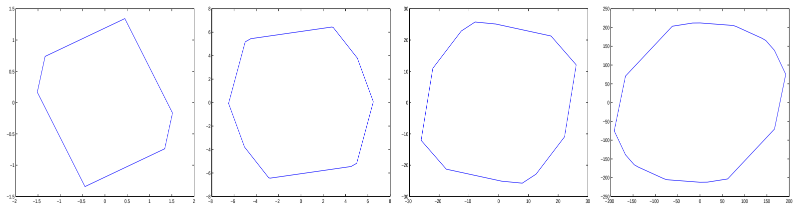

Figure 0.2: Normalized sections of $\mathbb{I}^n$ by random planes, $n = 3, 300, 10000, 1000000$.

Even in the dimension million (the highest the author could manage on his standard desktop computer) the section remains rather rough. This is what theory predicts, as the unit cube (that is, the unit ball of $\ell^\infty(n)$) is the worst case presently known ([56], Sect. 4.1).

Things are very different if one considers the unit ball of the space $\ell^1(n)$, the so-called *crosspolytope* $\lozenge^n$. At the first sight, there is little difference between the “ruggedness” of the crosspolytope and of the cube, cf. Fig. 0.3.

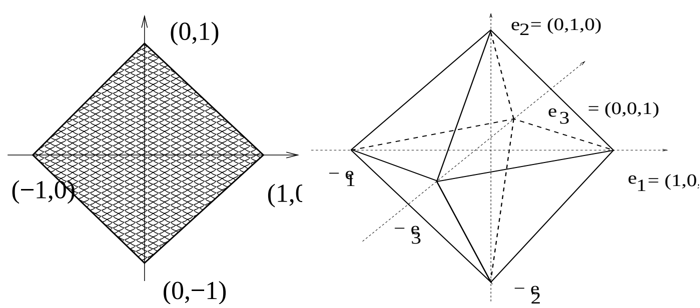

Figure 0.3: 2- and 3-crosspolytopes: diamond $\lozenge^2$ and octahedron $\lozenge^3$.

However, the behaviour of the random sections is markedly different, cf. Fig. 0.4.

In the late 60’s and early 70’s Vitali Milman in a series of papers [112]–[117] has isolated a geometric property of the unit sphere of

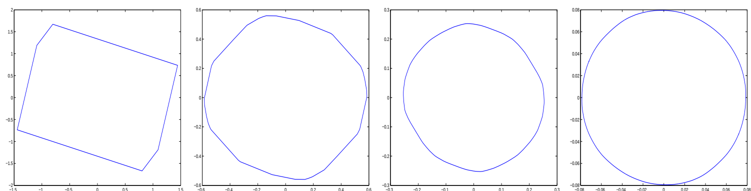

Figure 0.4: Normalized sections of crosspolytopes by random planes, $n = 3, 20, 100, 1000$.

the Hilbert space which, as a particular application, has led to a new proof of the Dvoretzky theorem [117]. If one forgets about the exact bounds on the dimension k, Milman's property is equivalent, by a routine ultraproduct argument, to the following property of the unit sphere in the infinite-dimensional Hilbert space ℓ^2.

Theorem 0.0.2 (Milman). *The unit sphere* $\mathbb{S}^\infty$ *in* ℓ^2 *is* finitely oscillation stable *in the following sense. Let* $f\colon \mathbb{S}^\infty \to \mathbb{R}$ *be a uniformly continuous function. Then for every* $\varepsilon > 0$ *and every natural* N *there exists a linear subspace* V *with* $\dim V = N$ *such that the restriction of the function* f *to the unit sphere of* V *is constant to within* ε*:*

$$\operatorname{osc}(f|V \cap \mathbb{S}^\infty) < \varepsilon.$$

Here the oscillation, $\operatorname{osc}(f|X)$, of a function f on a set X is the difference between the supremum and the infimum. One should also mention here that the original concept of Milman is that of *finite spectrum*, while oscillation stability is a later terminology. A real number a is in the finite spectrum, $\gamma(f)$, of a function $f\colon \mathbb{S}^\infty \to \mathbb{R}$ if for every $\varepsilon > 0$ the set $f^{-1}(a-\varepsilon, a+\varepsilon)$ contains linear subspaces of an arbitrary high finite dimension. Milman's theorem was stated in this form: the finite spectrum of every function f as above is non-empty.

As we will see shortly, the above property has a strong combinatorial flavour, it is a Ramsey-type statement but in the continuous setting, the first of a kind.

Intuitively, a Ramsey-type statement says that if a very large, highly symmetric structure is smashed into finitely many pieces, then

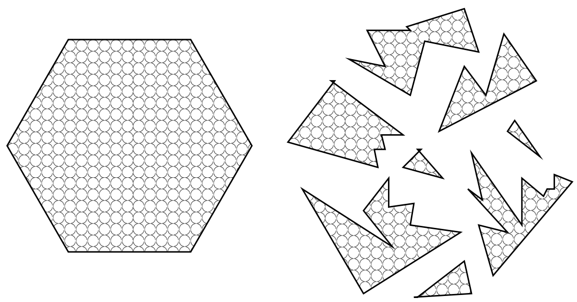

Figure 0.5: A Ramsey-type theorem.

at least one of them contains a large symmetric chunk that survived intact. (Cf. Fig. 0.5.)

For instance, here is the classical Ramsey theorem, in its infinite version.

Theorem 0.0.3 (Infinite Ramsey Theorem). *Let X be an infinite set, and let k be a natural number. For every finite colouring of the set $[X]^k$ of k-subsets of X there exists an infinite subset $A \subseteq X$ such that the set $[A]^k$ is monochromatic.*

To appreciate the Ramsey-theoretic spirit of the finite oscillation stability of the sphere $\mathbb{S}^\infty$, here is an equivalent reformulation of the latter property.

Theorem 0.0.4. *Let γ be a finite colouring of the sphere of $\mathbb{S}^\infty$ (that is, a finite partition of it). Then for every $\varepsilon > 0$ there exists a sphere $\mathbb{S}^N \subset \mathbb{S}^\infty$ of an arbitrarily high finite dimension N that is monochromatic to within ε. That is, $\mathbb{S}^N$ is contained within the ε-neighbourhood of one of the elements of the partition γ.*

Milman has established his theorem for wide classes of concrete infinite-dimensional manifolds coming from geometric functional analysis (including the Stiefel manifolds of ordered orthonormal frames, the Grassmann manifolds of subspaces of a given finite dimension, etc.) These discoveries have led Milman to the formulation of the following principle. (Quoted from [121], Principle 1.5.)

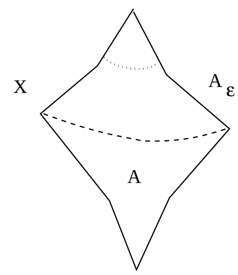

Figure 0.6: An illustration to the concentration of measure.

> *Uniformly continuous functions on high-dimensional (or infinitely-dimensional) structures have the property of being almost constant on smaller substructures. In other words, small local oscillations imply small global oscillations on smaller substructures.*

The above principle was dubbed by Gromov in 1983 [75] the *Ramsey–Dvoretzky–Milman phenomenon.* By that time, the known manifestations of this phenomenon were numerous, and the underlying philosophy has considerably influenced developments in the discrete Ramsey theory as well: combinatorialists were led to search for Ramsey-type statements for other discrete structures. And indeed, the period of active development of discrete Ramsey theory, marked by such papers as [67, 66], has historically followed the emergence of continuous, Milman-type combinatorial theorems.

Finite oscillation stability has been established by Milman using the *phenomenon of concentration of measure on high-dimensional structures.* Here is a heuristic way to describe the phenomenon:

> *for a typical "high-dimensional" structure X, if A is a subset containing at least half of all points, then the measure of the ε-neighbourhood A_ε of A is overwhelmingly close to* 1 *already for small values of $\varepsilon > 0$.*

Within the commonly used setting for concentration proposed by Gromov and Milman [78], the basic geometric object is an *mm-*

space, that is, a triple (X, d, μ), consisting of a metric space (X, d) and a probability Borel measure μ on it. (Alternative settings for concentration are discussed in [77] and [56].) The phenomenon is best captured in the following asymptotic way: a family (X_n, d_n, μ_n) of mm-spaces is called a *Lévy family* if, whenever $A_n \subseteq X_n$ are measurable subsets whose measures $\mu_n(A_n)$ are uniformly separated away from zero, one has $\mu_n((A_n)_\varepsilon) \to 1$ for every $\varepsilon > 0$.

Examples of Lévy families are numerous: Euclidean spheres with the normalized Haar measure and geodesic distance (Paul Lévy, 1922, [103]), Hamming cubes with the normalized counting measure and the Hamming distance normalized to one (the Law of Large Numbers is a particular consequence of this fact, hence the name Geometric Law of Large Numbers sometimes used in place of Concentration Phenomenon), unitary groups of finite rank, etc. Various aspects of the phenomenon of concentration of measure on high-dimensional structures, including all the above examples and many more, are discussed in monographs [123], [102], [76], Ch. $3\frac{1}{2}{}_+$, [107], [16], and articles [162], [77], [120], [122].

Another way to reformulate the concentration phenomenon is this: on a high-dimensional structure, every uniformly continuous function is close to being constant everywhere except on a set of a vanishingly small measure. This is the formulation that is being used to establish the finite oscillation stability of the sphere: on every subsphere $\mathbb{S}^N$ of sufficiently high dimension N the set where the function f is constant to within ε has measure so close to one that one can find inside of that set a sphere $\mathbb{S}^n$ of a smaller dimension n, moreover even a randomly chosen n-sphere will have the desired property with high probability.

Now it is time for topological groups of transformations to enter into the picture. The unit sphere $\mathbb{S}^\infty$ is a highly symmetric object, and its symmetries (isometries) form a group, the unitary group $U(\ell^2)$ of the separable Hilbert space. The unitary group is acting on the sphere by isometries, and this action determines one of the two most important topologies on $U(\ell^2)$: the *strong operator topology,* which is simply the topology of simple convergence on the sphere. The unitary group with the strong topology is a Polish (separable completely metrizable) group, arguably one of the most important infinite-dimensional groups ever. The dynamical properties of the sphere, equipped with the action of the unitary group, and of the unitary group acting on itself by left multiplications, are very similar: the unitary group is also finitely oscillation stable. The proof of this fact is based on the phenomenon of concentration of measure applied to the family of unitary groups of growing finite rank.

For a topological group G acting on itself by left multiplication, finite oscillation stability is equivalent to a dynamical property called *extreme amenability.* A topological group G is *extremely amenable* if every continuous action of G on a compact space X has a fixed point: there is an $x \in X$ with the property $g \cdot x = x$ for all $g \in G$. The proof is very straightforward and intuitively clear: finite oscillation stability says that bounded left (or right) right uniformly continuous functions on G become ever closer to being constant on suitable one-sided left (resp., right) translates of ever larger finite subsets of G. There is a direct correspondence between bounded right uniformly continuous functions on a topological group G and continuous functions on compact G-spaces, and the standard compactness argument leads to the existence of a fixed point in every compact G-space.

The concept of an extremely amenable (semi)group has first appeared in mid-60's in the papers by Mitchell [124] and Granirer [69]. Since it was soon realized that no locally compact group can be extremely amenable [71] (for discrete groups it had already been known for a while [42]), it was asked by Mitchell in 1970 ([125], footnote 2) if extremely amenable topological groups existed at all.

The first example of a non-trivial extremely amenable group was published in 1975 by Herer and Christensen [90]. They were motivated by Maharam's Control Measure Problem (which remains unsolved to the day and whose links with extreme amenability are certainly not yet fully understood). Nevertheless, the construction of Herer and Christensen was certainly a "counter-example" rather than a naturally occuring example of a topological group.

Things have changed in late 70's when Gromov and Milman have proved (independently of the above mentioned investigations, and answering a question of H. Furstenberg) that the unitary group $U(\ell^2)$ with the strong topology has the fixed point on compacta property. (Their paper [78] was submitted for publication in 1978 but only published in 1983.) Indeed, the standing of the unitary group $U(\ell^2)$ with the strong topology in modern mathematics is such that that

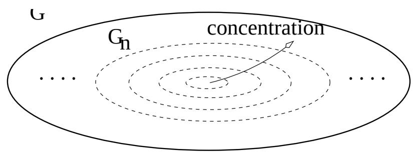

Figure 0.7: To the concept of a Lévy group.

every property of this group has to be taken seriously.

The technique used in the proof can be captured in the general concept of a *Lévy group* introduced in the same paper [78]. So is called a topological group G admitting an increasing sequence of compact subgroups K_n with everywhere dense union and which forms a Lévy family with regard to the normalized Haar measures and the restrictions of some compatible left-invariant metric on G. (Cf. Fig. 0.7.)

Every Lévy group is extremely amenable, and many concrete infinite-dimensional groups turned out to be Lévy groups. Glasner [59] and (independently, unpublished) Furstenberg and Weiss, have shown that the group $L^0(\mathbb{I}, U(1))$ of measurable maps from the standard Lebesgue measure space to the circle rotation group $U(1)$, equipped with the pointwise operations and the topology of convergence in measure, is Lévy. Other known Lévy groups include unitary groups of some von Neumann algebras, groups of measure and measure-class preserving automorphisms of the standard Lebesgue measure space, full groups of amenable equivalence relations [57, 58], the isometry group of the universal Urysohn metric space [142, 146].

There are however extremely amenable groups which are not Lévy, coming from discrete combinatorics as groups of automorphisms of various ultrahomogeneous structures. A direct link between extreme amenability and Ramsey theory has been established by the present author, who proved in [138] the extreme amenability of the group $\mathrm{Aut}\,(\mathbb{Q}, \leq)$ of order-preserving bijections of the rationals, equipped with the topology of pointwise convergence on $\mathbb{Q}$ viewed as discrete. This statement is just a reformulation of the Finite Ramsey Theorem. Clearly, the group $\mathrm{Aut}\,(\mathbb{Q}, \leq)$ is not Lévy simply because it contains no non-trivial compact subgroups.

The recent paper by Kechris, Pestov and Todorcevic [99] explores this trend extensively, by establishing extreme amenability of groups of automorphisms of numerous countable Fraïssé structures. In particular, all extremely amenable topological subgroups of the infinite symmetric group S_∞ have been described in combinatorial terms related to their orbit structure.

Applications of extreme amenability in the theory of infinite-dimensional groups are not yet numerous, but already sufficiently interesting. They are essentially of two kinds.

One of them concerns measurable dynamics of Lévy groups. The recent work by Glasner, Tsirelson and Weiss [63] and Glasner and Weiss [62] links Lévy groups to the problem of (non)existence of spatial models for near actions. Specifically, they have shown that a non-trivial near-action of a Polish Lévy group on a measure space (that is, a weakly continuous action where every motion $X \ni x \mapsto gx \in X$ is only defined μ-almost everywhere) never admits a Borel model: if a Polish Lévy group G acts in a Borel way on a Polish space X equipped with an invariant probability measure, and the action is ergodic, then the measure is a Dirac point mass and so the action is trivial. This is in sharp contrast to the classical result by Mackey, Varadarajan and Ramsay stating that a measure-preserving near-action of a locally compact second countable group always admits a Borel model. As a consequence of results by Glasner, Tsirelson and Weiss, one can describe all left-invariant means on a Lévy group as elements of the closed convex hull of the set of multiplicative invariant means.

Another kind of applications concerns the universal minimal flows of infinite-dimensional groups. To every topological group G one can associate the *universal minimal flow,* that is, a compact G-space $\mathscr{M}(G)$ that is *minimal* (the orbit of every point is everywhere dense) and *universal* (every other minimal G-space is an image of $\mathscr{M}(G)$ under a continuous G-equivariant surjection). Minimal flows form the major object of study of abstract topological dynamics [5, 43, 186], where the universal minimal flows feature prominently. At the same time, the universal minimal flow $\mathscr{M}(G)$ of a locally compact and non-compact group G is a highly non-constructive object (for instance, always non-metrizable, [99], Appendix 2). On the other hand, if G is an extremely amenable group, then $M(G) = \{*\}$ is a singleton.

Rather surprisingly, it turns out that there is a class of examples "in between" these two extremes, where the flow $\mathscr{M}(G)$ is both non-trivial and still of reasonable size (a metrizable compactum that lends itself to an explicit description). This is essentially an infinite-dimensional phenomenon.

The first instance where the universal minimal flow $\mathscr{M}(G)$, different from a point, was described explicitely, was the case where $G = \mathrm{Homeo}_+(\mathbb{S}^1)$ is the group of orientation-preserving homeomorphisms of the circle with the compact-open topology. In this situation, $\mathscr{M}(G)$ is the circle $\mathbb{S}^1$ itself, equipped with the canonical action of G [138]. The proof is based on the fact that G contains a large extremely amenable subgroup, namely $\mathrm{Homeo}_+(\mathbb{I})$, and so every action of G on a compact space factors through the homogeneous space $\mathbb{S}^1 \cong G/\mathrm{Homeo}_+(\mathbb{I})$.

The question of whether a similar result holds for groups of homeomorphisms of closed manifolds X in dimension > 1 was answered in the negative by V.V. Uspenskij [174] who has made the following observation: the universal minimal flow $\mathscr{M}(G)$ of a topological group G is never 3-transitive. Not only the result is most interesting, but the method of proof provides a source of examples of minimal flows for many concrete topological groups of transformations: the space $\Phi(X)$ of maximal chains of closed subsets of a compact G-space.

Glasner and Weiss have subsequently described in [60] the universal minimal flow of the infinite symmetric group S_∞ with its (unique) Polish topology as the zero-dimensional compact metric space LO of all linear orders on the natural numbers. In [61] the same authors have computed the universal minimal flow of the group of homeomorphisms of the Cantor set, C, as the space $\Phi(C)$ of maximal chains of closed subsets of C (Uspenskij's construction). Numerous other examples of explicit computations of universal minimal flows of groups of automorphisms of Fraïssé structures can be found in the paper by Kechris, Pestov and Todorcevic [99].

Thus, it is hoped that extremely amenable groups can help to understand other large groups as well. In addition, there are several promising directions for future research.

Vitali Milman's work in geometric functional analysis and his studies of finite oscillation stability have led him to formulate a stronger concept, which is now mostly known under the name of *oscillation stability.* A function f on the sphere $\mathbb{S}^\infty$ is oscillation stable if for every $\varepsilon > 0$ there is an infinite-dimensional subspace $\mathscr{K}$ of ℓ^2 such that

$$\mathrm{osc}\,(f|\mathscr{K} \cap \mathbb{S}^\infty) < \varepsilon.$$

Again, Milman's original notion was that of the *infinite spectrum,* $\gamma_\infty(f)$, of a function f as above: a number $a \in \gamma_\infty(f)$ if and only if for every ε the inverse image $f^{-1}(a - \varepsilon, a + \varepsilon)$ contains an infinite-dimensional subspace.

The question that has become known as the famous *distortion problem* and was the main engine driving the development of geometric functional analysis for three decades, is, in one of its equivalent forms, this: is every uniformly continuous function on the unit sphere of the Hilbert space oscillation stable, that is, is $\gamma_\infty(f) \neq \emptyset$?

A negative answer was obtained in 1994 by Odell and Schlumprecht [134]. To the day, no direct proof based on the intrinsic geometry of the unit sphere $\mathbb{S}^\infty$ is known.

It turns out that the concept of oscillation stability makes sense for every uniform space X equipped with a continuous action of a topological group G by uniform isomorphisms, see [99]. In particular, the question of oscillation stability can be asked about many concrete highly homogeneous structures. For example, the *random graph* R is well known to be oscillation stable in this sense (a property known as *indecomposability* of R: if the set of vertices of R is partitioned into finitely many classes, the induced subgraph on at least one of them is isomorphic to R). However, for many known examples the question of oscillation stability remains open — for instance, for the universal Urysohn metric space of diameter one, the most intriguing object of them all. Understanding the situation here would unquestionably help to achieve a deeper understanding of the distortion phenomenon of the unit sphere in the Hilbert space. In this connection, let us mention a recent remarkable advance by Greg Hjorth [92]: every Polish topological group G, considered as a left G-space, has distortion (that is, is not oscillation stable).

Another interesting topic for investigations is concentration to a nontrivial space in a dynamic framework. One can put a metric on the set of all isomorphism classes of separable mm-spaces in such a way that the convergence of a sequence X_n to the trivial one-point space is equivalent to the family (X_n) being a Lévy family. This was done by Gromov [76]. Thus, a Lévy group is extremely amenable, in a sense, because there is a growing sequence of approximating subgroups that at the same time converges to a point! What happens in a more general situation, where such a family of approximating compact subgroups inside a topological group G concentrates to a non-trivial mm-space? For instance, if this space is compact, does it necessarily have to be the universal minimal flow $\mathscr{M}(G)$? One should mention here that alternative approaches to concentration to a nontrivial space have been proposed by V. Milman (cf. e.g. remarks in [115]) and recently also by A.M. Vershik, and the relationship between the three concepts, as well as their links to dynamics, remain unexplored.

We hope that the present set of lecture notes will provide an accessible entry into the theory outlined on the preceding pages. The bibliography, though still not comprehensive, is relatively extensive, and hope that the reader will find a discussion of open questions at the end of the book stimulating.

These notes are by no means comprehensive or definitive. The main aim is to give the reader a taste of the new theory rather than to offer an encyclopaedic source. We work out in detail a number of selected basic examples and results, while subsequently our exposition becomes cursory and we often refer to the original research papers for details. Some results and trends are not even mentioned at all.

Here are some of the most notable omissions of this kind.

The notion of an extremely amenable topological group can be defined and studied in terms of finitely additive measures on a topological group. Such an approach forms the subject of Chapter 493 in Volume 2 of David Fremlin's fundamental multi-volume treatise on Measure Theory [51], and beyond any doubt, it sheds new light on the concept. We don't even touch it here.

There are constructions of extremely amenable groups by means other than either measure concentration or Ramsey theory. We have already mentioned the first ever example of an extremely amenable group by Herer and Christensen [90] in terms of pathological submeasures. This group has a stronger property: it is a so-called *exotic group,* that is, admits no non-trivial strongly continuous unitary representations. (Every amenable exotic group is extremely amenable.) Even if on the surface the construction looks very similar to the construction of Glasner and Furstenberg and Weiss, presented in Section 3.2 of these notes, it is not at all clear if concentration of measure on subgroups of simple functions plays any role at all in establishing extreme amenability of the Herer–Christensen group. A deeper understanding of this example is called for by scholars of Maharam's problem such as Ilijas Farah and Slawek Solecki.

Banaszczyk [7, 8] has developed a method of constructing abelian exotic (therefore, extremely amenable) groups as topological factor-groups of various locally convex spaces (including the Hilbert space ℓ^2, other Banach spaces and some nuclear spaces) by discrete additive subgroups. His construction certainly deserves a greater attention.

Some examples of extremely amenable groups such as the group of affine isometries of the Hilbert space ℓ^2 [142] are constructed as semidirect products of an extremely amenable group (the unitary group $U(\ell^2)$) and a topological group that is not extremely amenable (the additive group of ℓ^2). The general conditions of extreme amenability of semidirect products, and other constructions of group theory, are not yet understood and used on any scale.

The concept of a Lévy group can be sensibly relaxed. Such generalized Lévy groups include, for instance, groups of measurable maps $L^0(X, \mu; G)$ from the standard Lebesgue measure space to an amenable locally compact group G [142]. It is possible that in this way one can use concentration of measure to deduce Ramsey-type statements.

Last but not least, the very notion of extreme amenability may yet evolve. Recent results by Rosendal and Solecki [154] allow one to deduce from extreme amenability of the Polish group $\mathrm{Aut}\,(\mathbb{Q}, \leq)$ of order-preserving bijections of the rational numbers the following *fixed point on metric compacta property:* every action of $\mathrm{Aut}\,(\mathbb{Q}, \leq)$ by homeomorphisms on a metrizable compact space is automatically continuous and therefore has a fixed point. Thus, the topology is entirely expunged from the definition, everything is now just about the algebraic structure of a group in question. The result is the more astonishing in view of the old observation by Ellis [42] that every discrete group admits a free action on a compact space (in fact, on the Stone-Čech compactification βG). Who knows, at the end it may turn out that namely the above is the "right" version of the fixed point on compacta property of infinite-dimensional groups!

All of the above topics will be no doubt addressed in greater detail should these notes ever come to get re-edited, corrected and expanded.

Chapter 1

The Ramsey–Dvoretzky–Milman phenomenon

1.1 Finite oscillation stability

Uniform — rather than metric — spaces provide the most natural setting for studying the Ramsey–Dvoretzky–Milman phenomenon.

Definition 1.1.1. Recall that a *uniform space* is a pair $(X, \mathcal{U})$, consisting of a set X an a *uniform structure*, $\mathcal{U}$, on X, that is, a collection of subsets of $X \times X$ (binary relations on X), called *entourages of the diagonal*, satisfying the following properties.

1. $\mathcal{U}$ is a family closed under finite intersections and supersets (if $V \in \mathcal{U}$ and $V \subseteq U \subseteq X \times X$, then $U \in \mathcal{U}$).

2. Every $V \in \mathcal{U}$ contains the diagonal $\Delta = \{(x,x) \colon x \in X\}$.

3. If $V \in \mathcal{U}$, then $V^{-1} \equiv \{(x,y) \colon (y,x) \in V\}$ is in $\mathcal{U}$.

4. If $V \in \mathcal{U}$, there exists a $U \in \mathcal{U}$ such that $U \circ U \equiv \{(x,z) \colon \exists y \in X, \ \ (x,y) \in U, \ \ (y,z) \in U\}$ is a subset of V.

Definition 1.1.2. A subfamily $\mathcal{B} \subseteq \mathcal{U}$ is called a *basis* of the uniformity $\mathcal{U}$ if for every $U, V \in \mathcal{B}$ there is a $C \in \mathcal{B}$ with $C \subseteq A \cap B$, and every entourage $V \in \mathcal{U}$ contains, as a subset, an element $U \in \mathcal{B}$.

A prefilter $\mathcal{B}$ on a set X serves as a basis for a uniform structure if and only if it satisfies the following conditions:

1. Every $V \in \mathcal{U}$ contains the diagonal Δ.
2. If $V \in \mathcal{U}$, then for some $U \in \mathcal{B}$, $U \subseteq V^{-1}$.
3. If $V \in \mathcal{U}$, there exists a $U \in \mathcal{U}$ such that $U \circ U \subseteq V$.

For every $V \in \mathcal{U}$ and $x \in X$, denote $V[x] = \{y \in X : (x, y) \in V\}$. This is the V*-neighbourhood* of x. The sets $V[x]$, where $V \in \mathcal{U}$, form a neighbourhood basis for x with regard to a topology on X, called the topology *determined* by, or *associated* to, $\mathcal{U}$. If $\mathcal{U}$ is a uniform structure determining the topology of a given topological space X, then $\mathcal{U}$ is said to be *compatible*.

If $\mathcal{U}$ is *separated* (as we will usually assume), that is, $\cap\mathcal{U} = \Delta$, the associated topology is Tychonoff. Every Tychonoff topological space admits a compatible uniform structure.

To every uniform space $(X, \mathcal{U})$ one can associate a *separated replica*, that is, the quotient set of X by the equivalence relation $\cap\mathcal{U}$, equipped with the image of the uniformity $\mathcal{U}$ under the quotient map. Here we will usually assume that all our uniform spaces are separated.

Exercise 1.1.3. Show that every element of a compatible uniformity $\mathcal{U}$ on a topological space X is a neighbourhood of the diagonal. Deduce that every compact space X admits a *unique* compatible uniformity, consisting of all neighbourhoods of the diagonal in the product topology on $X \times X$.

Example 1.1.4. The *additive uniform structure* on a topological vector space E has, as a basis, the set of all entourages of the diagonal of the form
$$\{(x, y) \in E \times E : x - y \in V\},$$
where V runs over the neighbourhood basis of zero.

Example 1.1.5. The *left uniform structure*, $\mathcal{U}_L(G)$, on a topological group G has as basic entourages of the diagonal the sets

$$V_L = \{(x,y)\colon x^{-1}y \in V\},$$

where V is a neighbourhood of identity.

Example 1.1.6. Every metric space (X,d) supports a natural uniform structure, whose basic entourages are of the form

$$\{(x,y)\colon d(x,y) < \varepsilon\}, \quad \varepsilon > 0.$$

For instance, if d is a left-invariant metric generating the topology of a topological group G, then the corresponding uniform structure is the left uniform structure on G.

Let A be a subset of a uniform space $(X,\mathcal{U})$, and let $V \in \mathcal{U}$ be an entourage of the diagonal. Say that A is *V-small* if $A \times A \subseteq V$, that is, for each $a,b \in A$ one has $(a,b) \in V$.

A *Cauchy filter* on a uniform space X is a filter $\mathscr{F}$ containing a V-small set for every $V \in \mathcal{U}$. A uniform space is complete if all Cauchy filters converge. Two Cauchy filters are *equivalent* if their intersection is again a Cauchy filter. It is clear how to define the completion of a uniform space, along the same lines as it is being done for metric spaces. The uniform structure $\mathcal{U}$ admits a unique extension to the completion $\hat{X}$ of a uniform space $X = (X,\mathcal{U})$.

A uniform structure $\mathcal{U}$ is *totally bounded* if for every $V \in \mathcal{U}$ there is a finite $F \subseteq X$ with $V[F] = X$. Here $V[F]$ is the *V-neighbourhood* of F, given by

$$V[F] = \cup_{x\in F} V[x] = \{y \in X\colon \exists x \in F, \quad (x,y) \in V\}.$$

Totally bounded uniform spaces are exactly those whose completions are compact.

A pseudometric d on a uniform space $(X,\mathcal{U})$ is *uniformly continuous* if for every $\varepsilon > 0$ there is a $V \in \mathcal{U}$ with the property $d(x,y) < \varepsilon$ whenever $(x,y) \in V$. Every uniform space admits a family of uniformly continuous bounded pseudometrics that determine the uniform structure, in the following sense: for every $V \in \mathcal{U}$ there is a bounded uniformly continuous pseudometric d such that

$$\{(x,y) \in X \times X\colon d(x,y) < 1\} \subseteq V.$$

We will only consider such families of pseudometrics that are *directed,* that is, for every two pseudometric d_1, d_2 there is a third, d_3, that is greater than any of the two.

A mapping f between uniform spaces $(X, \mathcal{U}_X)$ and $(Y, \mathcal{U}_Y)$ is *uniformly continuous* if for every $V \in \mathcal{U}_Y$ there is a $U \in \mathcal{U}_X$ with $(f \times f)(U) \subseteq V$.

For a more detailed review of uniform spaces, the reader may consult, for instance, [45] (or its 1989 edition, revised and completed: Berlin, Heldermann Verlag), and [23].

Now comes one of the main concepts of the present theory.

Definition 1.1.7. Let a group G act on a set X. A function f on X, taking values in a uniform space $(Y, \mathcal{U}_Y)$, is called *finitely oscillation stable* if for every finite subset $F \subset X$ and every entourage $V \in \mathcal{U}_Y$ there is a transformation $g \in G$ such that the image $f(gF)$ is V-small:

$$f(gF) \times f(gF) \subseteq V.$$

Exercise 1.1.8. Assume a group G acts on a uniform space X uniformly equicontinuously. (That is, for all $V \in \mathcal{U}_X$ there is a $U \in \mathcal{U}_X$ such that if $(x, y) \in U$, then for all $g \in G$ one has $(gx, gy) \in V$.) Let $f\colon X \to Y$ be uniformly continuous. Show that one can replace in the above definition 1.1.7 finite subsets $F \subset X$ with compact ones.

If (Y, d) is a metric space (equipped with the corresponding uniformity), the definition can be conveniently restated. For a function f on a set A with values in a metric space (Y, d_Y), define *oscillation* of f on A as the value

$$\operatorname{osc}(f|A) = \sup_{x,y \in A} d_Y(f(x), f(y)).$$

Proposition 1.1.9. *Let a group G act on a set X. A function f on X, taking values in a metric space (Y, d_Y), is finitely oscillation stable if and only if for every finite subset $F \subset X$ and every $\varepsilon > 0$ there is a transformation $g \in G$ such that*

$$\operatorname{osc}(f|gF) < \varepsilon.$$

In other words, for every $\varepsilon > 0$ the function f is constant to within ε on a suitable translate of F by an element of G. □

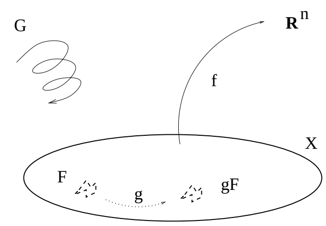

Figure 1.1: A finitely oscillation stable function.

For the case where $X = \mathbb{R}^n$, the concept is illustrated in Fig. 1.1.

Now let G be a group of (not necessarily all) uniform isomorphisms of a uniform space $(X, \mathcal{U}_X)$. We will say that (G, X) is a *uniform G-space.* At this stage, we do not equip G with topology; however, whenever G is a topological group, we will assume that the action of G on X is continuous.

Definition 1.1.10. Say that a uniform G-space X has the *Ramsey–Dvoretzky–Milman property*, or is *finitely oscillation stable,* if every bounded uniformly continuous real-valued function f on X is finitely oscillation stable.

Notice that the above property is dynamical, that is, it depends not only on the uniform space X in question, but also on the selected group G of uniform isomorphisms of X. In most concrete situations, X will be a metric space, and G will be a group of isometries of X.

Remark 1.1.11. The above concept was introduced (in the case where X is the unit sphere in the Hilbert space and G is the group of unitary operators) by V.Milman [112, 113], cf. also [121], though the terminology we use is of later origin (cf. [134]). According to Milman, the *spectrum* of a function f as above is the set $\gamma(f)$ of all $a \in Y$ such that for every $\varepsilon > 0$ and every finite $F \subset X$ there is a $g \in G$ with $d(f(x), a) < \varepsilon$ whenever $x \in gF$. If $f(X)$ is relatively compact in Y (which is the case that matters), then clearly $\gamma(f) \neq \emptyset$ if and only if f is finitely oscillation stable.

It will be useful, to reformulate the concept in several equivalent forms using the language of uniform topology. Let $(X, \mathcal{U})$ be a (separated) uniform space. By $\mathcal{C}^*\mathcal{U}$ we will denote the *totally bounded* *replica* of $\mathcal{U}$, that is, the finest totally bounded uniformity contained in $\mathcal{U}$. Equivalently, $\mathcal{C}^*\mathcal{U}$ is the coarsest uniformity on X with regard to which every bounded uniformly continuous function on X remains uniformly continuous. If $\mathcal{U}$ is separated, then $\mathcal{C}^*\mathcal{U}$ is separated as well.

The completion of the uniform space $(X, \mathcal{C}^*\mathcal{U})$ is a compactification of X, called the *Samuel compactification* or else the *universal uniform compactification* of X. It is denoted by σX. This is the maximal ideal space of the commutative C^*-algebra $\mathrm{UCB}\,(X)$ of all bounded uniformly continuous complex-valued functions on X. In other words, elements of σX are multiplicative functionals on $\mathrm{UCB}\,(X)$ (morphisms of unital involution algebras), which are automatically continuous, and the set σX is equipped with the weak*-topology.

Exercise 1.1.12. Prove that X is everywhere dense in the maximal ideal space of $\mathrm{UCB}\,(X)$.

Example 1.1.13. If X is a Tychonoff topological space equipped with the finest compatible uniformity, then $\sigma X = \beta X$ is the Stone–Čech compactification of X. In particular, for $X = \mathbb{N}$ equipped with the discrete uniformity (containing every entourage of the diagonal $\Delta_{\mathbb{N}}$), one has $\sigma\mathbb{N} = \beta\mathbb{N}$.

We are going to give a convenient description of the totally bounded replica of a given uniformity $\mathcal{U}$ on a set X. If γ is a cover of X, denote by $\tilde{\gamma}$ the corresponding entourage of the diagonal in $X \times X$, that is,

$$\tilde{\gamma} = \cup_{A\in\gamma} A \times A.$$

For an entourage of the diagonal $V \in \mathcal{U}$, denote by γ_V the V-*thickening* of γ,

$$\gamma_V = \{V[A] \colon A \in \gamma\}.$$

Here, as usual,

$$V[A] = \{x \in X \colon \text{ for some } a \in A,\ (a, x) \in V\}$$

is the V-neighbourhood of A.

It is easy to verify that $\widetilde{\gamma_V} = V \circ \tilde{\gamma} \circ V^{-1}$.

Lemma 1.1.14. *Let $(X,\mathcal{U})$ be a uniform space. A basis for the totally bounded replica of $\mathcal{U}$ is given by all entourages of the diagonal of the form $\widetilde{\gamma_V}$, where γ is a finite cover of X and $V \in \mathcal{U}$.*

Proof. Given a finite cover γ and an $V \in \mathcal{U}$, choose a bounded uniformly continuous pseudometric d on X with the property

$$(d(x,y) < 1) \implies (x,y) \in V.$$

For each $A \in \gamma$ and every $x \in X$, set $d_A(x) := \inf\{d(x,a)\colon a \in A\}$. The functions d_A are uniformly continuous bounded. Let $x,y \in X$ be such that $|d_A(x) - d_A(y)| < 1$ for all $A \in \gamma$. For some $A' \in \gamma$ one has $x \in A'$, therefore $d_{A'}(x) = 0$, $d_{A'}(y) < 1$ and $y \in V[A']$. Consequently, $(x,y) \in \widetilde{\gamma_V}$. We conclude: $\widetilde{\gamma_V} \in \mathcal{C}^*\mathcal{U}$.

It remains to prove that every $W \in \mathcal{C}^*\mathcal{U}$ contains an entourage of the form $\widetilde{\gamma_V}$. By definition of the totally bounded replica, there exist $n \in \mathbb{N}$, $f \in \mathrm{UCB}\,(X;\ell^\infty(n))$ and $\varepsilon > 0$ such that $(x,y) \in W$ whenever $\|f(x) - f(y)\|_\infty < \varepsilon$. Find a $V \in \mathcal{U}$ with $(x,y) \in V$ implying $\|f(x) - f(y)\|_\infty < \varepsilon/3$, and cover the set $f(X) \subset \ell^\infty(n)$ with a finite family α of sets of diameter $< \varepsilon/3$ each. Let $\gamma = \{f^{-1}(A)\colon A \in \alpha\}$. If $(x,y) \in \widetilde{\gamma_V}$, there are x', y' and an $A \in \alpha$ with $(x,x') \in V$, $f(x'), f(y') \in A$, and $(y,y') \in V$. By the triangle inequality, $\|f(x) - f(y)\|_\infty < \varepsilon$, meaning that $\widetilde{\gamma_V} \subseteq W$. □

Corollary 1.1.15. *If X is a metric space, then a basis for the totally bounded replica of the metric uniformity on X is given by entourages $\widetilde{\gamma_\varepsilon}$, $\varepsilon > 0$, where γ_ε is the ε-thickening of a finite cover γ.* □

Example 1.1.16. Let $\mathbb{S}^\infty$ be the unit sphere of the separable Hilbert space ℓ^2. The *norm uniformity*, $\mathcal{U}_{\|\cdot\|}$, on $\mathbb{S}^\infty$ is generated by the usual norm distance, $d_{\|\cdot\|}$. The *weak uniformity*, $\mathcal{U}_w$, is the coarsest uniformity making the restriction to $\mathbb{S}^\infty$ of each continuous linear functional on ℓ^2 uniformly continuous. Equivalently, $\mathcal{U}_w$ is the restriction to the sphere of the additive uniformity on the space ℓ^2 with the weak topology. As is easy to see, $\mathcal{U}_w$ is totally bounded and contained in the norm uniformity. It is well known and easily seen that the completion of the sphere with the weak uniformity is the unit ball equipped with the compact weak topology and the corresponding unique compatible uniformity.

The totally bounded replica $\mathcal{C}^*\mathcal{U}_{\|\cdot\|}$ of the norm uniformity is strictly finer than the weak uniformity $\mathcal{U}_w$. To see this, let X be any infinite uniformly discrete subset of $\mathbb{S}^\infty$, for example, the set $X = \{e_1, e_2, \dots\}$ of all standard basic vectors. Every $\{0,1\}$-valued function f on X extends to a bounded uniformly continuous, indeed 1-Lipschitz, function $\tilde{f}$ on $(\mathbb{S}^\infty, \mathcal{U}_{\|\cdot\|})$, e.g. through the Katětov construction:

$$\mathbb{S}^\infty \ni \xi \mapsto \tilde{f}(\xi) = \inf\{\|\xi - x\| + f(x) \colon x \in X\}.$$

(More generally, every uniformly continuous bounded function always admits an extension from a subspace to the uniform space, cf. Exercise 8.5.6 in [45].) Consequently, the Stone-Čech compactification $\beta\mathbb{N}$ embeds into $\sigma\mathbb{S}^\infty$ as a topological subspace. At the same time, the weak completion of $\mathbb{S}^\infty$ is the unit ball $\mathbb{B}^\infty$ of ℓ^2 with the weak topology, which is separable metrizable.

For a natural number k, we denote, following a convention usual in combinatorics, $[k] = \{1, 2, \dots, k\}$.

The following result, although purely technical in nature, provides a useful insight because it shows why the Ramsey-Dvoretzky-Milman property is closely linked both to Ramsey theory and to the existence of fixed points.

Theorem 1.1.17. *For a uniform G-space $(X, \mathcal{U})$ the following are equivalent.*

1. *X has the Ramsey–Milman–Dvoretzky property, that is, every uniformly continuous bounded $f \colon X \to \mathbb{R}$ is finitely oscillation stable.*

2. *Let $\mathcal{R}$ be a directed collection of bounded uniformly continuous pseudometrics generating the uniformity $\mathcal{U}$. Then for every $d \in \mathcal{R}$, every real-valued bounded 1-Lipschitz function on (X, d) is finitely oscillation stable.*

3. *Every bounded uniformly continuous function f from X to each finite-dimensional Euclidean space $\mathbb{R}^n$, $n \in \mathbb{N}$ is finitely oscillation stable.*

4. *Every uniformly continuous mapping from X to a compact space is finitely oscillation stable.*

5. *The canonical mapping from X to the Samuel compactification σX is finitely oscillation stable.*

6. *For every entourage $W \in \mathcal{C}^*\mathcal{U}$ and every finite $F \subseteq X$, there is a $g \in G$ such that gF is W-small: $(gF \times gF) \subseteq W$.*

7. *For every finite cover γ of X ("colouring of X with finitely many colours"), every $V \in \mathcal{U}$, and every finite $F \subseteq X$ there are a $g \in G$ and an $A \in \gamma$ such that $gF \subseteq V[A]$ (that is, gF is "monochromatic to within V").*

8. *For every binary cover $\{A, B\}$ of X, every $V \in \mathcal{U}$ and every finite $F \subseteq X$, there is a $g \in G$ such that gF is contained either in $V[A]$ or in $V[B]$.*

9. *For every entourage $V \in \mathcal{U}$, every $k \in \mathbb{N}$ and every finite $F \subseteq X$, there is a finite $K \subseteq X$ with the property that for every colouring $c\colon K \to [k]$ with k colours, there are $g \in G$ and $i \in [k]$ such that $gF \subseteq K$ and $gF \subseteq V[c^{-1}(i)]$, that is, gF is monochromatic to within V.*

10. *For every $V \in \mathcal{U}$ and every finite $F \subseteq X$, there is a finite $K \subseteq X$ with the property that for every binary cover $\gamma = \{A, B\}$ of K there is a $g \in G$ such that $gF \subseteq K$ and gF is contained either in $V[A]$ or in $V[B]$.*

If in addition X is a metric space and $\mathcal{U}$ is the metric uniformity, then each of the above is equivalent to any of the following:

11. *Every bounded 1-Lipschitz function $f\colon X \to \mathbb{R}$ is finitely oscillation stable.*

12. *For every $n \in \mathbb{N}$, each bounded 1-Lipschitz function from X to $\ell^\infty(n)$ is finitely oscillation stable.*

Proof. (1) $\Longrightarrow$ (2): trivial.

(2) $\Longrightarrow$ (8): for a $V \in \mathcal{U}$, choose a uniformly continuous bounded

pseudometric $d \in \mathcal{R}$ such that $d(x,y) < 1 \implies (x,y) \in V$. Let $\{A,B\}$ be a binary cover of X. The distance function d_A from A is a real-valued, bounded, and 1-Lipschitz function on (X,d), and by assumption, for every finite $F \subseteq X$ there is $g \in G$ such that $\operatorname{osc}(d_A|gF) < 1$. Unless $gF \subseteq B$ (in which case we are of course done), the function d_A assumes the zero value at some point of gF, and consequently $gF \subseteq V[A]$.

(8) $\implies$ (10): if X is finite, there is nothing to prove. Suppose X is infinite, and assume that (10) does not hold, that is, there exist a $V \in \mathcal{U}$ and a finite $F \subseteq X$ such that every finite $K \subseteq X$ admits a binary cover $\{A_K, B_K\}$ with the property that, whenever $g \in G$ and $gF \subseteq K$, one has $gF \not\subseteq V[A_K]$ and $gF \not\subseteq V[B_K]$. Choose an ultrafilter ξ on the collection $\mathcal{P}_{\text{fin}}(X)$ of all finite subsets of X with the property that for every $F' \in \mathcal{P}_{\text{fin}}(X)$, the set $\{K \in \mathcal{P}_{\text{fin}}(X) \colon K \supseteq F'\}$ is in ξ. Define $A = \lim_{K\in\xi} A_K$, that is, $x \in A$ if and only if $\{K \in \mathcal{P}_{\text{fin}}(X) \colon x \in A_K\} \in \xi$. Similarly, set $B = \lim_{K\in\xi} B_K$. Then $\{A,B\}$ is a binary cover of X. Let $g \in G$ be arbitrary. For every finite $K \supseteq gF$, there are elements $a_K, b_K \in gF$ such that for every $x \in A_K$, $y \in B_K$ one has $(a_K, y) \notin V$ and $(b_K, x) \notin V$. Since the set of all finite supersets of gF is in ξ, one can define $a = \lim_{K\to\xi} a_K \in gF$ and $b = \lim_{K\to\xi} b_K \in gF$. It follows that for every $x \in A$, $y \in B$ one has $(a,y) \notin V$ and $(b,x) \notin V$. In other words, $gF \not\subseteq V[A]$ and $gF \not\subseteq V[B]$, meaning that (8) does not hold either.

(10) $\implies$ (9): finite induction in the number of colours, k. The case $k = 2$ forms the base of induction. Suppose the statement is proved for $k = \ell$ (for all finite F and all entourages $V \in \mathcal{U}$). Let F' be chosen so that for every $c \colon F' \to [\ell]$ there is a $g \in G$ such that $gF \subseteq F'$ and gF is monochromatic to within V. Choose a finite $K \subseteq X$ in such a way that for every binary cover $\gamma = \{A,B\}$ of K there is an $h \in G$ such that $hF' \subseteq K$ and hF' is contained either in $V[A]$ or in $V[B]$. Let now $c \colon K \to [\ell+1]$. By property (10), applied to the binary cover of K with $A = c^{-1}[\ell]$ and $B = c^{-1}\{\ell+1\}$, there exists $h \in G$ such that $hF' \subseteq K$ and either $hF' \subseteq V[c^{-1}\{\ell+1\}]$ (in which case we are done, as for some $g \in G$ one has $hgF \subseteq hF'$), or else $hF' \subseteq V[c^{-1}[\ell]]$. In the latter case, choose a $W \in \mathcal{U}$ so that $(x,y) \in W$ implies $(hx, hy) \in V$ (uniform continuity of transla-

tions). Apply the induction hypothesis to this W and the colouring $c_1 = c \circ h \colon F' \to [\ell]$ (that is, $c_1(x) = c(hx)$) to find a $g \in G$ such that $gF \subseteq F'$ and gF is contained in $W[c_1^{-1}(i)]$ for some $i \in [\ell]$. Then $hgF \subseteq K$ and $hgF \subseteq V[c^{-1}(i)]$, completing the step.

(9) $\Longrightarrow$ (7): obvious, if one applies (9) to V, F, and $k = |\gamma|$.

(7) $\Longrightarrow$ (6): if $W \in \mathcal{C}^*\mathcal{U}$, then, by Lemma 1.1.14, there exist a finite cover γ of X and a $V \in \mathcal{U}$ such that the entourage of diagonal $\widetilde{\gamma_V} \subseteq W$. By assumption, for every finite $F \subseteq X$ there are $g \in G$ and $A \in \gamma$ with $gF \subseteq V[A]$. This means exactly that gF is $\widetilde{\gamma_V}$-small.

(6) $\Longrightarrow$ (5): let W be an element of the unique compatible uniformity $\mathcal{U}_{\sigma X}$ on σX, and let $F \subseteq X$ be finite. The restriction W' of W to X is an element of $\mathcal{C}^*\mathcal{U}$, and by assumption there is $g \in G$ such that gF is W'-small, and consequently W-small, meaning that the map $i \colon X \to \sigma X$ is oscillation stable.

(5) $\Longrightarrow$ (4): follows from the fact that every uniformly continuous function f from X to a compact space C factors through the canonical map $i \colon X \to \sigma X$.

(4) $\Longrightarrow$ (3): apply (4) to the closure of the image $f(X)$ in $\mathbb{R}^n$.

(3) $\Longrightarrow$ (1): trivial.

If in addition (X, d) is a metric space, then, in a trivial way, (3) $\Longrightarrow$ (12) $\Longrightarrow$ (11) $\Longrightarrow$ (2) (with $\mathcal{R} = \{d\}$). $\square$

Next we will discuss two examples of oscillation stable G-spaces of fundamental importance.

1.2 First example: the sphere $\mathbb{S}^\infty$

1.2.1 Finite oscillation stability of $\mathbb{S}^\infty$

The first example is the unit sphere $\mathbb{S}^\infty$ of the infinite dimensional separable Hilbert space ℓ^2 with the usual norm distance, equipped with the standard action of the unitary group $U(\ell^2)$ by rotations. We will prove that the pair $(\mathbb{S}^\infty, U(\ell^2))$ is finitely oscillation stable.

The proof of this fact, belonging to V. Milman, is based on three ingredients:

- every finite subset F of the sphere $\mathbb{S}^\infty$ is contained in a sphere $\mathbb{S}^N$ of an arbitrarily high finite dimension N;
- there is a compact subgroup of $U(\ell^2)$, acting transitively on $\mathbb{S}^N$;
- concentration of measure on the spheres $\mathbb{S}^N$, $N \to \infty$.

The first two observations are obvious, while the third one requires explanation.

The *phenomenon of concentration of measure on high-dimensional structures* says, intuitively speaking, that the geometric structures – such as the Euclidean spheres – of high finite dimension typically have the property that an overwhelming proportion of points are very close to every set containing at least half of the points. Technically, the phenomenon is dealt with in the following framework.

Definition 1.2.1 (Gromov and Milman [78]). A *space with metric and measure,* or an *mm-space*, is a triple, (X, d, μ), consisting of a set X, a metric d on X, and a probability Borel measure on the metric space (X, d).

For a subset A of a metric space X and an $\varepsilon > 0$, denote by A_ε the ε-neighbourhood of A in X.

Definition 1.2.2 (*ibid.*). A family $\mathcal{X} = (X_n, d_n, \mu_n)_{n\in\mathbb{N}}$ of *mm*-spaces is a *Lévy family* if, whenever Borel subsets $A_n \subseteq X_n$ satisfy

$$\liminf_{n\to\infty} \mu_n(A_n) > 0,$$

one has for every $\varepsilon > 0$

$$\lim_{n\to\infty} \mu_n((A_n)_\varepsilon) = 1.$$

Remark 1.2.3. The concept of a Lévy family can be reformulated in many equivalent ways. For example, a family as above is Lévy if and only if for every $\varepsilon > 0$, whenever A_n, B_n are Borel subsets of X_n satisfying

$$\mu_n(A_n) \geq \varepsilon, \quad \mu_n(B_n) \geq \varepsilon,$$

then $d(A_n, B_n) \to 0$ as $n \to \infty$. (Exercise.)

This is formalized using the notion of *separation distance,* proposed by Gromov ([76], Section $3\frac{1}{2}.30$). Given numbers $\kappa_0, \kappa_1, \dots, \kappa_N > 0$, one defines the invariant

$$\mathrm{Sep}\,(X; \kappa_0, \kappa_1, \dots, \kappa_N)$$

as the supremum of all δ such that X contains Borel subsets X_i, $i = 0, 1, \dots, N$ with $\mu(X_i) \geq \kappa_i$, every two of which are at a distance $\geq \delta$ from each other. Now a family $\mathcal{X} = (X_n, d_n, \mu_n)_{n\in\mathbb{N}}$ is a Lévy family if and only if for every $0 < \varepsilon < \frac{1}{2}$, one has

$$\mathrm{Sep}\,(X; \varepsilon, \varepsilon) \to 0 \text{ as } n \to \infty.$$

The reader should consult Ch. $3\frac{1}{2}$ in [76] for numerous other characterisations of Lévy familes of mm-spaces, among which the most important could be the one in terms of the *observable diameter.*

Example 1.2.4. The Euclidean spheres $\mathbb{S}^n$, $n \in \mathbb{N}_+$ of unit radius, equipped with the Haar measure (translation-invariant probability measure) and Euclidean (or geodesic) distance, form a Lévy family.

Before proving this fact in Theorem 1.2.27 below, we will deduce from it the oscillation stability of the sphere $\mathbb{S}^\infty$.

Theorem 1.2.5. *The $U(\ell^2)$-space $\mathbb{S}^\infty$ is finitely oscillation stable.*

Proof. Let $F \subset \mathbb{S}^\infty$ be finite, let γ be a finite cover of $\mathbb{S}^\infty$, and let $\varepsilon > 0$. Denote $n = |F|$ and $m = |\gamma|$. Choose a natural N so large that $N \geq n$ and whenever $A \subset \mathbb{S}^N$ has the property $\mu_N(A) \geq 1/m$, one has $\mu_N(A_\varepsilon) > 1 - 1/n$. Now fix a Euclidean sphere $\mathbb{S}^N \subset \mathbb{S}^\infty$ of dimension N containing the set F. The group of isometries of $\mathbb{S}^N$ is isomorphic to $U(N)$ and naturally embeds into $U(\ell^2)$: if an orthonormal basis of ℓ^2 is so chosen that $\mathbb{S}^N$ is the unit sphere of the linear span of the first $N+1$ basic vectors, then the embedding is given by

$$U(N) \ni u \mapsto \begin{pmatrix} u & 0 \\ 0 & \mathbb{I} \end{pmatrix} \in U(\ell^2).$$

Equipped with the compact-open topology with regard to its action on $\mathbb{S}^N$, the group $U(N)$ is compact and acts on the sphere continuously.

For some $A \in \gamma$, one has $\mu_N(A \cap \mathbb{S}^N) \geq 1/m$, and therefore $\mu_N(A_\varepsilon \cap \mathbb{S}^N) \geq 1 - 1/n$. Denote by ν the Haar measure on $U(N)$ normalised to one. Using the uniqueness of a rotation-invariant Borel probability measure on $\mathbb{S}^N$, one has for every $\xi \in F$

$$\nu\{u \in U(N) \colon u\xi \in A_\varepsilon\} = \mu_N(A_\varepsilon) > 1 - \frac{1}{n},$$

and consequently

$$\bigcap_{\xi \in F} \{u \in U(N) \colon u\xi \in A_\varepsilon\} \neq \emptyset.$$

Any element u belonging to the intersection on the left hand side has the property $uF \subset A_\varepsilon$. Since u also belongs to $U(\ell^2)$, we are done by Theorem 1.1.17(7). □

The same argument *verbatim* gives us, at no extra cost, a more general result (Th. 1.2.9 and 1.2.12). It is convenient to slightly generalize the concept of a Lévy family by replacing a metric with a uniformity.

Definition 1.2.6. We say that a net (μ_α) of probability measures on a uniform space $(X,\mathcal{U})$ has the *Lévy concentration property,* or simply *concentrates*, if for every family of Borel subsets $A_\alpha \subseteq X$ satisfying $\liminf_\alpha \mu_\alpha(A_\alpha) > 0$ and every entourage $V \in \mathcal{U}_X$ one has $\mu_\alpha(V[A_\alpha]) \to_\alpha 1$.

Exercise 1.2.7. Suppose a net of measures (μ_α) on a uniform space X concentrates, and consider each μ_α as a measure on the Samuel compactification σX. Show that all cluster points of the net (μ_α) in the compact space $P(\sigma X)$ of probability measures on σX are Dirac measures corresponding to singletons.

Example 1.2.8. The converse implication is false. Let $X = \mathbb{S}^\infty$ be the sphere with the norm distance. For every finite collection $\mathcal{F}$ of bounded uniformly continuous functions on $\mathbb{S}^\infty$ and each $\varepsilon > 0$ there is a pair of points $a_{\mathcal{F},\varepsilon}, b_{\mathcal{F},\varepsilon} \in \mathbb{S}^\infty$ at a norm distance $\sqrt{2}/2$ from each other such that the oscillation of every $f \in \mathcal{F}$ on this pair is $< \varepsilon$

(finite oscillation stability of the sphere). As a consequence, the net of probability measures

$$\mu_{a_{\mathcal{F},\varepsilon}} = \frac{1}{2}(\delta_{a_{\mathcal{F},\varepsilon}} + \delta_{b_{\mathcal{F},\varepsilon}}),$$

indexed by pairs $(\mathcal{F}, \varepsilon)$ as above, concentrates with regard to the totally bounded replica of the norm uniformity on $\mathbb{S}^\infty$. By Exercise 1.2.7, the measures $(\mu_{a_{\mathcal{F},\varepsilon}})$ converge to a point mass on the Samuel compactification $\sigma\mathbb{S}^\infty$. At the same time, the net $(\mu_{a_{\mathcal{F},\varepsilon}})$ does not concentrate with regard to the norm uniformity: it is enough to consider the family of Borel subsets $A_{\mathcal{F},\varepsilon} = \{a_{\mathcal{F},\varepsilon}\}$ and the element of the norm uniformity $V = \{(x,y) \in \mathbb{S}^\infty \times \mathbb{S}^\infty \colon \|x-y\| < \sqrt{2}/2\}$.

Now a family $\mathcal{X} = (X_n, d_n, \mu_n)_{n\in\mathbb{N}}$ of mm-spaces is a Lévy family if and only if the measures (μ_n) concentrate in the sense of Def. 1.2.6 if considered as probability measures on the disjoint union $\oplus_{n=1}^\infty X_n$. This disjoint union is equipped with a metric d inducing the metrics d_n on each X_n and making X_n into an open and closed subset.

Theorem 1.2.9. *Let a topological group G act continuously by isometries on a metric space (X,d). Assume there is a family of compact subgroups (G_α), directed by inclusion and such that for some $\xi \in X$ the orbits $G_\alpha \cdot \xi$*

1. *have an everywhere dense union in X, and*
2. *form a Lévy family with regard to the restriction of the metric d and the probability measures $\mu_\alpha \cdot \xi$, where μ_α is the normalized Haar measure on G_n.*

Then the G-space X is finitely oscillation stable. □

Remark 1.2.10. Here $\mu_\alpha \cdot \xi$ stands for the push-forward of the normalized Haar measure μ_α on the compact group G_α under the orbit map $G_\alpha \ni g \mapsto g\cdot\xi \in X$.

Remark 1.2.11. In fact, one can expunge the compact subgroups from the definition altogether, using two facts: (i) the group of isometries, $\mathrm{Iso}\,(K)$, of a compact metric space K, equipped with the topology of simple convergence, is compact, and (ii) the orbit map $\mathrm{Iso}\,(K) \to K$ is open. (As a continuous map between compact spaces, it is closed, hence a quotient map onto a homogeneous space, hence open; the statement remains true for Polish groups acting on Polish spaces, this is the Effros theorem [41], but the proof becomes much less trivial.) Thus, K is a homogeneous factor-space of the compact group and supports a unique probability measure invariant under isometries. The groups $\mathrm{Iso}\,(K_\alpha)$ will no longer be topological subgroups of G, in fact they will be continuous homomorphic images of suitable subgroups of G, but since their role is auxiliary, like that of the unitary groups $U(N)$ in the proof of 1.2.5, this is enough. We arrive at the following result.

Theorem 1.2.12. *Let a group G act on a metric space X by isometries. Suppose there exists a family (K_α) of compact subspaces of X such that*

- *(K_α) are directed by inclusion,*
- *the union of (K_α) is everywhere dense in X,*
- *for every α, the isometries of X, stabilizing K_α setwise, act on K_α transitively, and*
- *the (unique) normalized probability measures on K_α, invariant under isometries of K_α, concentrate.*

Then the G-space X is finitely oscillation stable. □

1.2.2 Concentration of measure on spheres

There are at least three different known proofs of concentration of measure in the Euclidean spheres. Below, following Gromov and Milman [79], we present the proof based on the Brunn-Minkowski inequality. (The original proof by Lévy [103], based on the isoperimetric inequality, was made completely rigorous by Gromov in the general setting of Riemannian manifolds, cf. his apprendix in [123], and another well-known proof uses the spectral theory of the Laplacian, cf. [78]. Cf. also [123, 16, 102].)

Exercise 1.2.13. In the Definition 1.2.2 above, it is enough to assume that the values $\mu_n(A_n)$ are bounded away from zero by $1/2$ (or by any other constant strictly between zero and one). In other words, prove that a family $\mathcal{X}$ is a Lévy family if and only if, whenever Borel subsets $A_n \subseteq X_n$ satisfy $\mu_n(A_n) \geq 1/2$, one has for every $\varepsilon > 0$

$$\lim_{n\to\infty} \mu_n(A_n)_\varepsilon = 1.$$

(The author had included this exercise in the final exam of a graduate course *Elements of Asymptotic Geometric Analysis* taught at the Victoria University of Wellington in 2002, so it should not be too hard!)

This equivalence leads to the following concept [118, 123], providing convenient quantitative bounds on the rate of convergence of $\mu_n(A_n)_\varepsilon$ to one.

Definition 1.2.14. Let (X, d, μ) be a space with metric and measure. The *concentration function* of X, denoted by $\alpha_X(\varepsilon)$, is a real-valued function on the positive axis $\mathbb{R}_+ = [0, \infty)$, defined by letting

$$\alpha_X(\varepsilon) = \begin{cases} \frac{1}{2}, & \text{if } \varepsilon = 0, \\ 1 - \inf\left\{\mu\left(B_\varepsilon\right) : B \subseteq X, \quad \mu(B) \geq \frac{1}{2}\right\}, & \text{if } \varepsilon > 0. \end{cases}$$

Remark 1.2.15. Thus, a family $\mathcal{X} = (X_n, d_n, \mu_n)_{n\in\mathbb{N}}$ of mm-spaces is a Lévy family if and only if

$$\alpha_{X_n} \to 0 \text{ pointwise on } (0, +\infty) \text{ as } n \to \infty.$$

A Lévy family as above is called *normal* if for suitable constants $C_1, C_2 > 0$,

$$\alpha_{X_n}(\varepsilon) \leq C_1 e^{-C_2\varepsilon^2 n}.$$

As we will see shorlty, the spheres $(\mathbb{S}^n)$ form a normal Lévy family. Fig. 1.2 shows the concentration functions of spheres $\mathbb{S}^n$ in dimensions $n = 3, 10, 100, 500$.

Remark 1.2.16. The concept of a Lévy family was introduced in [78], and that of a normal Lévy family appears in [3].

Lemma 1.2.17. *If Π_1 and Π_2 are two disjoint boxes in $\mathbb{R}^n$, then there are an $i = 1, 2, \ldots, n$ and $c \in \mathbb{R}$, such that for all $x \in \Pi_1$, $x_i \leq c$, while for all $x \in \Pi_2$, $x_i \geq c$.* □

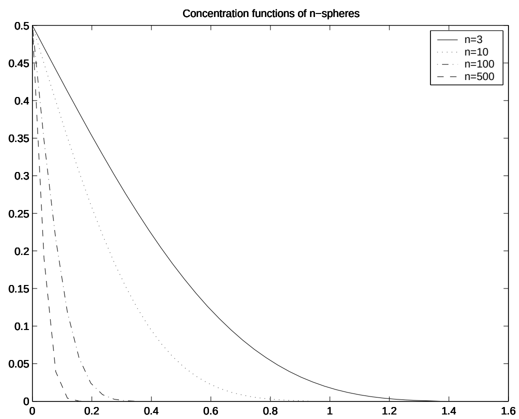

Figure 1.2: The concentration functions of spheres in various dimensions.

Geometrically, the result means that any two boxes are separated by a coordinate hyperplane, and immediately follows from the obvious one-dimensional case.

Lemma 1.2.18. *The geometric mean of a finite collection of positive reals never exceeds the arithmetic mean:*

$$(a_1 a_2 \dots a_n)^{1/n} \leq \frac{1}{n}(a_1 + a_2 + \cdots + a_n), \quad a_1, a_2, \dots, a_n \geq 0.$$

$\square$

Theorem 1.2.19 (Brunn–Minkowski inequality — the additive form). *Let A and B be non-empty subsets of $\mathbb{R}^n$ such that A, B and $A+B$ are all measurable sets. Then*

$$\mu(A+B)^{1/n} \geq \mu(A)^{1/n} + \mu(B)^{1/n}.$$

Proof. Without loss in generality, we can assume that A and B are unions of finite families of disjoint open boxes. (Approximate A and

B by compact sets from the inside and then by unions of open boxes from the outside.) The proof is by induction in k, the total number of such boxes in the two sets.

Basis of induction. Let A and B be open boxes with side lengths $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ respectively. The Minkowski sum of A and B is again an open box, and one has

$$\mu(A) = a_1 a_2 \ldots a_n, \quad \mu(B) = b_1 b_2 \ldots b_n,$$

and

$$\mu(A+B) = (a_1+b_1)(a_2+b_2)\ldots(a_n+b_n).$$

Applying Lemma 1.2.18 twice, first to the non-negative numbers $a_i/(a_i+b_i)$ and then to $b_i/(a_i+b_i)$, $i = 1, 2, \ldots, n$, and adding up the resulting inequalities, one gets

$$\begin{aligned} \prod_{i=1}^n \left(\frac{a_i}{a_i+b_i}\right)^{\frac{1}{n}} + \prod_{i=1}^n \left(\frac{b_i}{a_i+b_i}\right)^{\frac{1}{n}} &\leq \frac{1}{n}\sum_{i=1}^n \frac{a_i}{a_i+b_i} + \frac{1}{n}\sum_{i=1}^n \frac{b_i}{a_i+b_i} \\ &= 1, \end{aligned} \tag{1.1}$$

whence it follows that

$$\begin{aligned} \mu(A)^{\frac{1}{n}} + \mu(B)^{\frac{1}{n}} &= \prod_{i=1}^n (a_i)^{\frac{1}{n}} + \prod_{i=1}^n (b_i)^{\frac{1}{n}} \\ &\leq \prod_{i=1}^n (a_i+b_i)^{\frac{1}{n}} \\ &= \mu(A+B)^{\frac{1}{n}}. \end{aligned} \tag{1.2}$$

Step of induction. Suppose that the inequality has been proved for all pairs of sets A, B representable each as a finite union of disjoint open boxes, $\leq k-1$ in total. Let A and B be finite unions of disjoint open boxes, $k \geq 3$ in total. At least one set, say A, must be made up of at least 2 boxes. (Then B consists of at most $k-2$ boxes.)

Using Lemma 1.2.17, find an $x \in \mathbb{R}^n$ such that the translates by x of at least two open boxes making up A are on the opposite sides of the coordinate hyperplane $x_i = 0$. By cutting boxes up with the hyperplane $x_i = 0$ and throwing away the negligible set

$A \cap \{x \colon x_i = 0\}$, we can assume that in fact each of the newly-diminished boxes is entirely on one side of the hyperplane.

Denote by A' the union of all boxes to the 'left' from the hyperplane (that is, for each $a \in A'$ one has $a_i < x_i$), and by A'' the union of all boxes to the 'right' of the hyperplane. Notice that under such arrangement, both A' and A'' are made up of fewer than k boxes.

Now denote $\lambda = \mu(A')/\mu(A)$. For every $x \in \mathbb{R}$, denote

$$f(x) = \mu\{b \in B \colon b_i < x\}.$$

The function $f \colon \mathbb{R} \to [0, \mu(B)]$ is piecewise-linear and assumes at suitable points both values 0 and $\mu(B)$. By the Intermediate Value Theorem, there is a $\xi \in \mathbb{R}$ with $f(\xi) = \lambda\mu(B)$.

Choose now an arbitrary vector $y \in \mathbb{R}^n$ whose i-th coordinate is $-\xi$, and observe that the translate $B + y$ has the following property. Denote by $B' = \{b \in B \colon b < y\}$ the part of $B + y$ that is to the 'left' from the same coordinate hyperplane $x_i = 0$, and by $B'' = \{b \in B \colon b > y\}$ the part to the 'right' of it. Then

$$\frac{\mu(B')}{\mu(B)} = \lambda \quad \text{and} \quad \frac{\mu(B'')}{\mu(B)} = 1 - \lambda = \frac{\mu(A'')}{\mu(A)}.$$

The total number of boxes in each of the pairs A', B' and A'', B'' is strictly less than k, and the inductive hypothesis applies. Since the sets $A' + B'$ and $A'' + B''$ are on the opposite sides of the hyperplane $x_i = 0$ and so disjoint,

$$\mu[(A' + B') \cup (A'' + B'')] = \mu(A' + B') + \mu(A'' + B'').$$

Clearly, $(A' + B') \cup (A'' + B'') \subseteq A + B + (x + y)$, and therefore

$$\begin{aligned}
\mu(A+B) &\geq \mu(A' + B') + \mu(A'' + B'') \\
&\geq \left[\mu(A')^{\frac{1}{n}} + \mu(B')^{\frac{1}{n}}\right]^n + \left[\mu(A'')^{\frac{1}{n}} + \mu(B'')^{\frac{1}{n}}\right]^n \\
&= \lambda\left[\mu(A)^{\frac{1}{n}} + \mu(B)^{\frac{1}{n}}\right]^n + (1-\lambda)\left[\mu(A)^{\frac{1}{n}} + \mu(B)^{\frac{1}{n}}\right]^n \\
&= \left[\mu(A)^{\frac{1}{n}} + \mu(B)^{\frac{1}{n}}\right]^n, \qquad (1.3)
\end{aligned}$$

finishing the step of induction. □

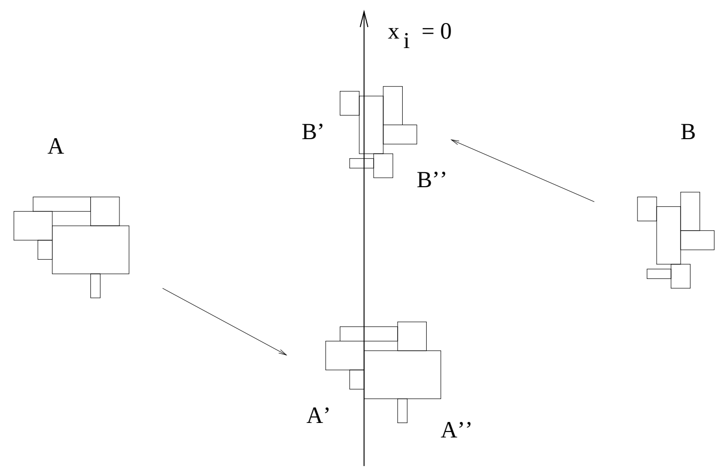

Figure 1.3: To the proof of the Brunn–Minkowski inequality.

Theorem 1.2.20 (Brunn–Minkowski inequality — the multiplicative form). *Let A and B be non-empty Borel subsets of $\mathbb{R}^n$. Then*

$$\mu\left(\frac{A+B}{2}\right) \geq (\mu(A)\mu(B))^{1/2}.$$

Proof. Applying the obvious inequality $a^2+b^2 \geq 2ab$ to $a = \mu(A)^{1/2n}$ and $b = \mu(B)^{1/2n}$, one derives from Theorem 1.2.19:

$$\begin{aligned} \mu\left(\frac{A+B}{2}\right)^{\frac{1}{n}} &\geq \mu\left(\frac{A}{2}\right)^{\frac{1}{n}} + \mu\left(\frac{B}{2}\right)^{\frac{1}{n}} \\ &= \frac{1}{2}\cdot\left(\mu(A)^{1/n} + \mu(B)^{1/n}\right) \\ &\geq (\mu(A)\mu(B))^{1/2n}, \end{aligned}$$

now it is enough to raise to the power of n on both sides. ☐

We need the following geometric property.

Proposition 1.2.21. *The unit ball* $\mathbb{B}$ *in the Hilbert space is* uniformly convex *in the sense that for every* $x, y \in \mathbb{B}$ *at a distance* $\varepsilon = d(x,y)$ *from each other, the distance from the mid-point* $(x+y)/2$ *to the complement of* $\mathbb{B}$ *is at least*

$$\delta(\varepsilon) = 1 - \sqrt{1 - \varepsilon^2/4},$$

which number is called the modulus of convexity *of* $\mathbb{B}$.

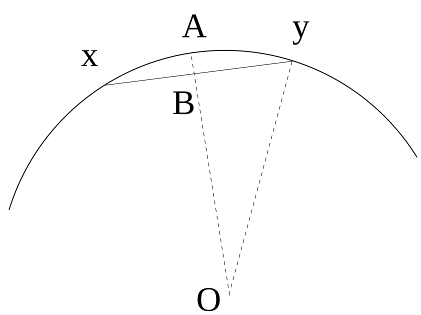

Figure 1.4: Uniform convexity of the Euclidean ball.

Proof. It is rather clear that the minimal value of the distance between $(x+y)/2$ and the boundary of $\mathbb{B}$ is achieved when x and y are themselves elements of the sphere. Consider the section of the ball by the plane spanned by x and y (Fig. 1.4). Since $|By| = \varepsilon/2$ and OBy forms a right triangle, $|OB| = \sqrt{1-\varepsilon^2/4}$ and finally

$$|BA| = 1 - \sqrt{1 - \varepsilon^2/4}.$$

□

Exercise 1.2.22. Give an algebraic proof, using the parallelogram identity.

Lemma 1.2.23. $\delta(\varepsilon) \geq \frac{\varepsilon^2}{8}$.

Proof. The desired inequality

$$1-\left(1-\frac{\varepsilon^2}{4}\right)^{\frac{1}{2}} \geq \frac{\varepsilon^2}{8}$$

is equivalent to

$$1-\frac{\varepsilon^2}{4} \leq \left(1-\frac{\varepsilon^2}{8}\right)^2,$$

and is clearly correct once the brackets on the r.h.s. are opened. □

Remark 1.2.24. Let A be a subset of $\mathbb{S}^n$. Denote by

$$\tilde{A} = \{ta \colon a \in A,\ t \in [0,1]\}$$

the *cone* over A. If $A \subseteq \mathbb{S}^n$ is a Borel subset, then $\tilde{A}$ is a Borel subset of $\mathbb{B}^{n+1}$. Besides,

$$\nu_n(A) \equiv \nu(A) = \frac{\mu(\tilde{A})}{\mu(\mathbb{B}^{n+1})},$$

where ν_n denotes the Haar measure on the sphere and $\mu = \mu_{n+1}$ stands for the Lebesgue measure in $\mathbb{R}^{n+1}$.

Lemma 1.2.25. *Let A be a Borel subset of $\mathbb{S}^{n-1}$, let $\varepsilon > 0$, let $B = \mathbb{S}^{n-1} \setminus A_\varepsilon$, and let $a \in \tilde{A}$ and $b \in \tilde{B}$ belong to the respective cones. Then*

$$\left\|\frac{a+b}{2}\right\| \leq 1-\delta(\varepsilon).$$

□

Remark 1.2.26. Since for every $x, y \in \mathbb{S}^n$,

$$d_{eucl}(x,y) \leq d_{geo}(x,y) \leq \frac{\pi}{2} d_{eucl}(x,y),$$

it makes no difference (apart from an extra constant in a formula), which of the two distances we choose. In the next result it will be more convenient for us to use the Euclidean distance.

Theorem 1.2.27. *The unit spheres $\mathbb{S}^n$, equipped with the rotation-invariant probability Borel measures and the Euclidean distance, form a normal Lévy family:*

$$\alpha_{\mathbb{S}^{n-1}}(\varepsilon) \leq 2e^{-n\varepsilon^2/4}.$$

Proof. Let A be a Borel subset of the $(n-1)$-dimensional Euclidean sphere $\mathbb{S}^{n-1}$, containing at least half of all points: $\nu(A) \geq \frac{1}{2}$. We will show that for every $\varepsilon > 0$ one has

$$\begin{aligned} \nu(A_\varepsilon) &\geq 1 - 2e^{-2n\delta(\varepsilon)} \\ &\geq 1 - 2e^{-n\varepsilon^2/4}, \end{aligned} \tag{1.4}$$

where the second inequality follows easily from Lemma 1.2.23.

Denote $B = \mathbb{S}^{n-1} \setminus A_\varepsilon$. Lemma 1.2.25 implies that the set $(\tilde{A} + \tilde{B})/2$ is contained inside the closed ball of radius $1-\delta(\varepsilon)$ and therefore, modulo the Brunn–Minkowski inequality in the multiplicative form (Theorem 1.2.20),

$$\begin{aligned} (1-\delta(\varepsilon))^n &\geq \tilde{\mu}\left(\frac{\tilde{A}+\tilde{B}}{2}\right) \\ &\geq \left(\tilde{\mu}(\tilde{A})\tilde{\mu}(\tilde{B})\right)^{1/2} \\ &\geq \left(\frac{1}{2}\tilde{\mu}(\tilde{B})\right)^{1/2}, \end{aligned} \tag{1.5}$$

where $\tilde{\mu}$ denotes the Lebesgue measure normalized so that the volume of the unit ball is 1. This formula yields

$$\nu(B) \equiv \tilde{\mu}(\tilde{B}) \leq 2(1-\delta(\varepsilon))^{2n}.$$

It remains to notice that

$$1 - \delta(\varepsilon) \leq e^{-\delta(\varepsilon)} = 1 - \delta(\varepsilon) + \frac{\delta(\varepsilon)^2}{2} - \ldots,$$

from which one concludes

$$\nu(B) \leq 2e^{-2n\delta(\varepsilon)},$$

which is the desired result in a slight disguise. □

Remark 1.2.28. The upper bound in Theorem 1.2.27 is not the best possible and can be improved, using the technique of isoperimetric inequalities, to at least this:

$$\alpha_{\mathbb{S}^{n-1}}(\varepsilon) \leq \sqrt{\frac{\pi}{8}} e^{-n\varepsilon^2/2}.$$

However, even in this case the Gaussian estimate falls short of closely approximating the true concentration function, as can be seen from Fig. 1.5.

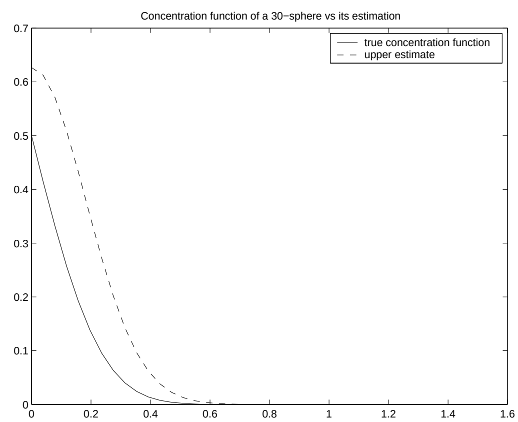

Figure 1.5: The concentration function of 30-sphere versus a Gaussian upper bound.

1.3 Second example: finite Ramsey theorem

Denote by S_∞ the infinite symmetric group, that is, the group of all self-bijections of a countably infinite set ω. For every $n \in \mathbb{N}_+$, let $[\omega]^n$ denote the set of all n-subsets of ω, that is, the collection of all subsets $A \subset \omega$ of cardinality exactly n. The group S_∞ acts on $[\omega]^n$

by permutations in a natural fashion: for each $\tau \in S_\infty$ and $A \in [\omega]^n$, $\tau(A) = \{\tau(a) \colon a \in A\}$.

Notice that the action of S_∞ on ω is *ultratransitive,* that is, every bijection between two finite subsets $A, B \subset \omega$ of the same cardinality extends to a global bijection $\tau \in S_\infty$.

Equip the set $[\omega]^n$ with a discrete uniform structure, that is, the uniform structure $\mathcal{U}_\omega$ containing all entourages of the diagonal. This uniform structure is generated by a discrete metric:

$$d(x,y) = \begin{cases} 1, & \text{if } x = y, \\ 0, & \text{otherwise.} \end{cases}$$

The pair $([\omega]^n, S_\infty)$ forms a uniform G-space.

Theorem 1.3.1. *For every $n \in \mathbb{N}_+$, the pair $([\omega]^n, S_\infty)$ is finitely oscillation stable.*

The proof is based on, and is in fact equivalent to, the following classical result. Recall that a *colouring* of a set A with k colours is a map $c \colon A \to [k]$, where $[k] = \{1, 2, \ldots, k\}$ is a k-element set.

Theorem 1.3.2 (Finite Ramsey Theorem). *Let n, m and k be natural numbers. There exists a natural number $N = N(n, m, k)$ such that for every set A of cardinality N and each colouring c of $[A]^n$ with k colours there is a subset $B \subseteq A$ of cardinality m such that $[B]^n$ is monochromatic: $c|_{[B]^n} = const$.*

Finite Ramsey Theorem (1.3.2) $\implies$ Theorem 1.3.1. We will use finite oscillation stability in the form of the equivalent condition (9) of Theorem 1.1.17. Since the uniform structure on $[\omega]^n$ is discrete, it is enough to set $V = \Delta$. Let $k \in \mathbb{N}$, and let $F \subseteq [\omega]^n$ be finite. Denote $\operatorname{supp} F = \cup_{x \in F} x \subset \omega$ and let $m = |\operatorname{supp} F|$. Choose $N = N(n, m, k)$ as in Finite Ramsey Theorem. Let $A \subseteq \omega$ be any subset of cardinality N, and let $K = [A]^n$. Now consider an arbitrary colouring c of K with k colours. According to Th. 1.3.2, there is a subset $B \subseteq A$ of cardinality m and such that $[B]^n$ is monochromatic. Since the action of S_∞ on ω is ultratransivite, there is a $\tau \in S_\infty$ such that $\tau(\operatorname{supp} F) = B$, and consequently $\tau(F) \subseteq [B]^n$ is monochromatic. In particular, $\tau(F)$ is monochromatic to within Δ, verifying condition (9). $\square$

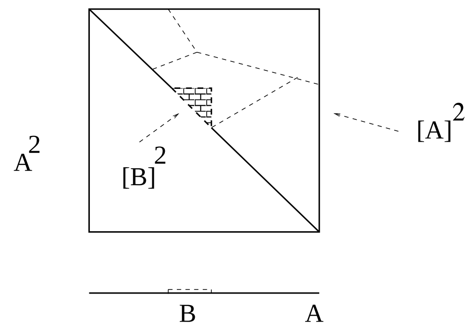

Figure 1.6: To the Finite Ramsey Theorem ($n = 2$, $k = 4$).

Remark 1.3.3. The only property of the symmetric group S_∞ that we have used in the above proof is the transitivity of the action on $[\omega]^n$. Therefore, Theorem 1.3.1 can be extended as follows.

Theorem 1.3.4. *Let n be a natural number, and let G be a subgroup of the symmetric group S_∞ with the property that for every natural n, the action of G on $[\omega]^n$ is transitive. Then the pair $([\omega]^n, G)$ is finitely oscillation stable.* □

Example 1.3.5. Here is one important example of such an acting group. Fix on ω a linear order of type η, that is, a dense linear order without the first and the last elements. Equipped with this order, ω is order-isomorphic to the rational numbers $\mathbb{Q}$ equipped with the usual order. Denote by $\mathrm{Aut}\,(\mathbb{Q}, \leq)$ the group of all order-preserving bijections of the rationals. Clearly, the action of this group on the rationals is ultratransitive.

Corollary 1.3.6. *For every $n \in \mathbb{N}_+$, the pair $([\mathbb{Q}]^n, \mathrm{Aut}\,(\mathbb{Q}, \leq))$ is finitely oscillation stable.* □

In the remaining part of this Section we will show that oscillation stability of the pair $([\omega]^n, S_\infty)$ indeed implies the Ramsey theorem, as well as sketch a proof of the latter.

Theorem 1.3.1 $\implies$ Finite Ramsey Theorem. Assume finite oscillation stability of the pair $([\omega]^n, S_\infty)$, and let m, k be natural numbers.

Let $C \subset \omega$ be any subset with m elements. Apply the condition (9) of Theorem 1.1.17 to k as above, $X = [\omega]^n$, $F = [C]^n$, and $V = \Delta$ being the diagonal of $[\omega]^n$. There exists a finite subset $K \subseteq [\omega]^n$ such that for every colouring of K with k colours there is a $\tau \in S_\infty$ with the properties that $\tau(F) \subseteq K$ and $\tau(F)$ is "monochromatic to within Δ," that is, monochromatic. Let $N = |\operatorname{supp} K|$. We will show that this N has the required property.

Let $A \subset \omega$ be any subset of cardinality N, and let $c\colon A \to [k]$ be a colouring. Because of ultratransitivity of the action of S_∞ on ω, there is a $\sigma \in S_\infty$ with the property $\sigma(\operatorname{supp} K) = A$. Denote by the same letter σ the induced mapping $K \to [A]^n$. By the choice of K, there is a $\tau \in S_\infty$ with the properties that $\tau([C]^n) \subseteq K$ and $\tau(F)$ is monochromatic with respect to the colouring $c \circ \sigma$ of K. Equivalently, if one denotes $B = \sigma \circ \tau(C)$, one has $B \subseteq A$, $|B| = m$, and $[B]^n$ is c-monochromatic, as required. □

Now we will outline, following the book [65] (which is a highly recommended introduction to Ramsey theory), a proof of Ramsey Theorem. We will establish a stronger result, the infinite version.

Theorem 1.3.7 (Infinite Ramsey Theorem). *Let k be a natural number. For every finite colouring of $(\omega)^k$ there exists an infinite subset $A \subseteq X$ such that the set $(A)^k$ is monochromatic.*

Example 1.3.8. For $k = 1$ the statement is simply the familiar pigeonhole principle: if one partitions an infinite set into finitely many subsets, then one of them ought to be infinite.

Example 1.3.9. Here is a popular reformulation of the result in the case $k = 2$. Among infinitely many people, either there exists an infinite subset such that everyone in this subset knows each other, or there exists an infinite subset such that no two persons in this subset know each other.

Remark 1.3.10. One deduces the finite version of the Ramsey theorem 1.3.2 from the infinite version via the same routine ultrafilter construction used by us in the proof of Theorem 1.1.17 (implication (8) $\Longrightarrow$ (10)). To avoid repeating the same construction all over again, just notice that the Infinite Ramsey Theorem implies the oscillation stability of the pair $([\omega]^n, S_\infty)$ (and thus Theorem 1.3.1) through an argument exactly similar to our deduction 1.3.2 $\implies$ 1.3.1 above, but using condition (7) of Theorem 1.1.17, rather than (9).

Proof of the Infinite Ramsey Theorem 1.3.7. We will only give the proof in case $k = 2$, letting the reader to figure out how to handle the further induction over k.

Now let χ be an r-colouring of $(X)^2$. We will think of the latter set as the set of all edges of the complete graph $K_{\mathbb{N}}$ on natural numbers as vertices.

Infinitely many edges incident to 0 have the same colour, say c_1 (an application of the pigeonhole principle); denote

$$X = \{n \in \mathbb{N} \colon n > 0, \chi(\{0, n\}) = c_1\} = \{x_1, x_2, \dots\},$$

where $x_1 < x_2 < \dots$.

Infinitely many of the edges of the form $\{x_1, x_k\}$, where $k > 1$, have the same colour, say c_2. Denote

$$Y = \{x \in X \colon x > x_1, \chi(\{x_1, x\}) = c_2\} = \{y_1, y_2, \dots\},$$

where $y_1 < y_2 < \dots$.

Now, infinitely many edges incident to y_1 and a vertex from Y have the same colour, say c_3, and we let

$$Z = \{y \in Y \colon y > y_1, \chi(\{y_1, y\}) = c_3\} = \{z_1, z_2, \dots\},$$

where $z_1 < z_2 < \dots$.

Denote now $T = \{0, x_0, y_0, z_0, \dots\}$. This is an infinite set. Notice that for a pair of elements $a, b \in T$, the colour of the edge $\{a, b\}$ only depends on $\min\{a, b\}$. Indeed, by the choice of our sets $X, Y, Z, \dots$, the colour of each edge joining a minimal element of the form $0, x_0, y_0, \dots$ with any other element in T is completely determined by the minimal element: e.g., for any $t \in T$, $\chi(\{0, t\}) = c_1$, $\chi(\{x_1, t\}) = c_2$, and so forth.

Thus, one can define a new colouring of T as follows: $\chi^*(s)$ is the colour of any edge of the form $\{s, t\}$, where $s < t$. This colour is independent of t. By the pigeonhole principle, there is an infinite χ^*-monochromatic subset $A \subseteq T$. Now it is clear that for every $a, b \in A$, the colour of the edge $\{a, b\}$ is going to be the same. $\square$

Remark 1.3.11. Essentially the same proof, with mathematical induction being replaced by transfinite induction, gives the following: if τ is an infinite cardinal and X is a set of cardinality $> 2^\tau$, then for any τ-colouring of $[X]^2$ there exists a monochromatic $Y \subset X$ of cardinality $> \tau$. This result has numerous consequences in set-theoretic topology. For instance:

Exercise 1.3.12. Deduce the following result by Hajnal and Juhasz: every first countable Hausdorff topological space satisfying the countable chain condition has cardinality $\leq 2^{\aleph_0}$.

Notice that in the result in Remark 1.3.11 one cannot replace the cardinality $> 2^\tau$ with 2^τ.

Theorem 1.3.13 (W. Sierpiński [158]). *Let X be a set of cardinality $2^{\aleph_0}$. There exists a colouring $c\colon [X]^2 \to [2]$ without uncountable homogeneous subsets $A \subseteq X$ (that is, such A that the set $[A]^2$ is monochromatic).*

If one is allowed to use countably infinitely many colours, the proof is immedaite.

Exercise 1.3.14. Prove a weaker version of Sierpiński's theorem (a colouring with countably many colours) by identifying X with the set of all 0-1 sequences, and for each pair of distinct sequences considering the first position where they differ.

Sierpiński's argument, also surprisingly simple, provides a deeper insight into the nature of things. The following construction is known as *Sierpiński's partition.*

Proof of Theorem 1.3.13. Let $<$ and $\prec$ be two total orders on X. Denote their characteristic functions by $\chi_<$ and $\chi_\prec$, respectively. The function

$$\chi_< + \chi_\prec \mod 2$$

on X^2 is invariant with regard to the flip $(x,y) \mapsto (y,x)$, and so gives rise to a function $c\colon [X]^2 \to \{0,1\}$; this is a colouring.

Let $A \subseteq X$ be a homogeneous subset. Then the restrictions $< |_A$ and $\prec |_A$ are either isomorphic or anti-isomorphic between themselves. It is, therefore, enough to find two orders on X whose restrictions to every uncountable subset are neither isomorphic nor anti-isomorphic, or, as they say, *incompatible orders*.

In our case, the obvious choice is the usual order on the real line $X = \mathbb{R}$ and a minimal well-ordering. □

The entire series of comments, beginning with Remark 1.3.11, had as its aim namely the presentation of Sierpiński's argument, which we will use in Chapter 5.

1.4 Counter-example: ordered pairs

Here is the third and last major example illustrating the concept of oscillation stability. This time, we will show a G-space that is not oscillation stable.

The space X is $\omega^2 \setminus \Delta_\omega$, that is, the set of all ordered pairs of distinct elements of ω:

$$X = \{(x,y)\colon x, y \in \omega,\ x \neq y\}.$$

We equip X with the discrete uniformity. The group $G = S_\infty$ is acting upon X by double permutations:

$$\tau(x,y) = (\tau x, \tau y).$$

Let $<$ denote the usual linear order on ω identified with the set of natural numbers, and let $\chi_<$ be the characteristic function of the corresponding order relation:

$$\chi_<(x,y) = \begin{cases} 1, & \text{if } x < y, \\ 0, & \text{otherwise.} \end{cases}$$

The function $\chi_<$ is bounded and uniformly continuous as a matter of course. At the same time, $\chi_<$ is not finitely oscillation stable. To see this, define a finite subset of $X = \omega^2 \setminus \Delta_\omega$ by

$$F = \{(0,1), (1,0)\}.$$

For every $\tau \in S_\infty$, the image $\tau(F)$ is of the form

$$\tau(F) = \{(a,b), (b,a)\}, \quad a \neq b,$$

and therefore

$$\operatorname{osc}(f|\tau(F)) = 1.$$

Chapter 2

The fixed point on compacta property

2.1 Extremely amenable groups

Definition 2.1.1. One says that a topological group G is *extremely amenable,* or has the *fixed point on compacta property,* if every continuous action of G on a compact space X admits a fixed point: for some $\xi \in X$ and all $g \in G$, one has $g\xi = \xi$.

Remark 2.1.2. The definition of an extremely amenable group is obtained from the classical definition of an amenable group by removing the underlined parts of the definition: A topological group G is *amenable* if every continuous action of G by affine transformations on a compact convex set X in a locally convex space admits a fixed point.

Thus, extreme amenability is a considerably stronger property. We will say more on the relations between the two notions at the end of this Chapter (section 2.7).

Before giving examples, we will link the above property with finite oscillation stability.

First of all, a convenient property of actions on compact spaces. Recall that the topology of *uniform convergence,* or the *compact-open* *topology*, on the group of homeomorphisms $\mathrm{Homeo}\,(X)$ of a compact space X is determined by the neighbourhoods of the identity of the form

$$\tilde{V} = \{g \in G \colon \forall x \in X, \quad (x, gx) \in V\},$$

where $V \in \mathcal{U}_X$ is an element of the unique compatible uniformity on X.

Exercise 2.1.3. Verify that the above topology on the group $\mathrm{Homeo}\,(X)$ is a group topology, and the standard action of $\mathrm{Homeo}\,(X)$ on X is continuous as a map $\mathrm{Homeo}\,(X) \times X \to X$.

Proposition 2.1.4. *An action of a group G on a compact space X is continuous as a map $G \times X \to X$ if and only if the associated homomorphism from G to the homeomorphism group* $\mathrm{Homeo}\,(X)$*, equipped with the compact-open topology, is continuous.*

Proof. Denote the action in question by γ. We will consider it both as a function of two arguments,

$$\gamma \colon G \times X \ni (g, x) \mapsto \gamma_g(x) \in X, \tag{2.1}$$

and a group homomorphism,

$$G \ni g \mapsto \gamma_g \in \mathrm{Homeo}\,(X). \tag{2.2}$$

Let the mapping in Eq. (2.1) be continuous, and let $V \in \mathcal{U}$. For every $x \in X$ there is a neighbourhood U_x of identity in G and a neighbourhood W_x of x in X such that $U_x \cdot W_x \subseteq V[x]$. Choose a finite subcover W_{x_i}, $i = 1, 2, \ldots, n$ of X by open sets, and let $U = \cap_{i=1}^n U_{x_i}$. Now every $x \in X$ is contained in some $U_{x_i} \subseteq V[x_i]$, and if $g \in U$, then $gx \in \underbrace{V[x_i]}$. Consequently, $(x, gx) \in V \circ V^{-1}$ and so U is contained in $V \circ V^{-1}$. This shows the continuity of the homomorphism in Eq. (2.2). The opposite implication follows from Exercise 2.1.3. □

Now recall that the *left uniform structure* on a topological group, $\mathcal{U}_L(G)$, has as a basis of entourages of the diagonal the sets

$$V_L = \{(x, y) \in G \times G \colon x^{-1}y \in V\}.$$

Lemma 2.1.5. *Let a topological group G act continuously on a compact space X. Then for every $\xi \in X$ the mapping*

$$G \ni g \mapsto g\xi \in X$$

(Fig. 2.1) is right uniformly continuous, while the mapping

$$G \ni g \mapsto g^{-1}\xi \in X$$

is left uniformly continuous.

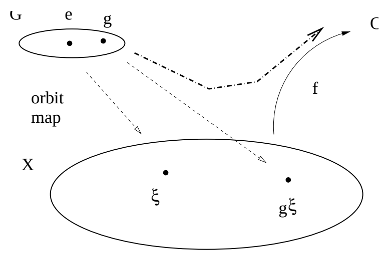

Figure 2.1: To Lemma 2.1.5.

Proof. Let V be an element of the unique compatible uniformity on X. By Proposition 2.1.4, there is a neighbourhood of identity U in G such that for all $x \in X$ and $g \in U$, one has $(x, gx) \in V$. If now $g, h \in G$ and $gh^{-1} \in U$, then $(gx, hx) = (gh^{-1}(hx), hx) \in V$.

The second statement now follows from the fact that the inversion map $i\colon G \to G$, given by $i(g) = g^{-1}$, is an isomorphism between the left and the right uniformities on G. $\square$

Next we are going to show that every bounded right (or left) uniformly continuous function on G arises in this way.

Let E be a normed space. To every isometry u of E one can associate the dual isometry, u^*, of the dual Banach space E^*:

$$E^* \ni f \mapsto u^*(f)(x) = f \circ u \in E^*.$$

However, the correspondence $u \mapsto u^*$ is only an *anti-homomorphism* from the group $\mathrm{Iso}\,(E)$ of all isometries of E into the group $\mathrm{Iso}\,(E^*)$.

To obtain a homomorphism, we associate to every $u \in \mathrm{Iso}\,(E)$ the isometry

$$(u^{-1})^*\colon f \mapsto f \circ u^{-1}.$$

The closed unit ball B_{E^*} of E^* is an invariant subset under isometries, and the correspondence

$$u \mapsto [f \mapsto f \circ u^{-1}|B_{E^*}] \tag{2.3}$$

is easily seen to be a group monomorphism from $\mathrm{Iso}\,(E)$ to the group $\mathrm{Homeo}\,(B_{E^*})$ of self-homeomorphisms of the dual closed unit ball equipped with the weak* topology. It is moreover easy to see that Eq. (2.3) defines a monomorphism of groups. Since the ball is compact, we equip the group $\mathrm{Homeo}\,(B_{E^*})$ with the compact-open topology.

Lemma 2.1.6. *The monomorphism (2.3) is an embedding of topological groups.*

Proof. Let us compare standard basic neighbourhoods of identity in the topological group $\mathrm{Iso}\,(E)$,

$$\{u \in \mathrm{Iso}\,(E)\colon \forall i = 1, 2, \dots, n,\ \ \|u(x_i) - x_i\| < \varepsilon\},\ \ x_i \in E,\ \varepsilon > 0,$$

and in $\mathrm{Homeo}\,(B_{E^*})$,

$$\{u \in \mathrm{Homeo}\,(B_{E^*})\colon \forall f \in B_{E^*},\ \ (u(f), f) \in V\},\ \ V \in \mathcal{U}_X,$$

where $\mathcal{U}_X$ is the (unique) compatible uniformity on B_{E^*}. Applying the Hahn–Banach theorem in the first case and noticing that $\mathcal{U}_X$ is induced by the additive uniformity on E^* equipped with the weak* topology in the second case, we reduce both neighbourhoods to the same form:

$$\{u\colon \forall i = 1, 2, \dots, n,\ \forall f \in B_{E^*},\ \ |f(u(x_i)) - f(x_i)| < \varepsilon\},\ x_i \in E,\ \varepsilon > 0$$

$\square$

Corollary 2.1.7. *Let π be a representation of a group G in a normed space E by isometries. Then π is strongly continuous if and only if the dual action of G on the closed unit ball B_{E^*} equipped with the weak$*$ topology is continuous.* $\square$

The following statement is easy to verify. Recall that for a function f on a G-set X and an element $g \in G$, one denotes by ${}^g f$ the function given by

$$^g f(x) = f(g^{-1}x).$$

Lemma 2.1.8. *The left regular representation λ_G of a topological group G in the space* RUCB(G) *of all right uniformly continuous bounded functions on G,*

$$(g, f) \mapsto {}^g f,$$

is a strongly continuous representation by isometries. □

Denote by $\mathcal{S}(G)$ the Samuel compactification of the right uniform space $(G, \mathcal{U}_R)$, that is, the space of maximal ideals of the commutative C^*-algebra RUCB(G). Every left translation of the space $(G, \mathcal{U}_R)$ by an element $g \in G$, being a uniform isomorphism, extends to a self-homeomorphism of $\mathcal{S}(G)$, and thus G acts on $\mathcal{S}(G)$ in a natural way. Notice that $\mathcal{S}(G)$ is an invariant subset of the dual ball of RUCB(G).

Lemma 2.1.9. *The action of G on $\mathcal{S}(G)$ is the restriction of the action dual to the left regular representation. Consequently, it is continuous, and every right uniformly continuous bounded function on G extends to a continuous function on the compact G-space $\mathcal{S}(G)$.* □

This $\mathcal{S}(G)$ is called the *greatest ambit* of G, and is the maximal compact G-space possessing a dense orbit. We will discuss its properties later, but now we are ready to reformulate the property of extreme amenability in terms of finite oscillation stability.

For every left-invariant bounded pseudometric d on G, denote $H_d = \{x \in G\colon d(x, e) = 0\}$, and let $\hat{d}$ be the metric on the left coset space G/H_d given by $\hat{d}(xH_d, yH_d) = d(x, y)$. The metric $\hat{d}$ is invariant under left translations by elements of G. We will denote the metric space $(G/H_d, \hat{d})$, equipped with the left action of G by isometries, simply by G/d.

Theorem 2.1.10. *For a topological group G, the following are equivalent.*

1. *G is extremely amenable.*

2. *G, equipped with the left uniform structure and the action of G on itself by left translations, is finitely oscillation stable.*

3. *Whenever G acts transitively and continuously by isometries on a metric space X, the G-space X is finitely oscillation stable.*

4. *There is a directed collection of bounded left-invariant continuous pseudometrics d, determining the topology of G and such that every metric G-space G/d is finitely oscillation stable.*

Proof. (2) $\iff$ (4): Denote by $\pi\colon G \to G/H_d$ the quotient map, then a uniformly continuous function f on G/d is finitely oscillation stable if and only the function $f \circ \pi$, uniformly continuous on (G, d), is finitely oscillation stable. Now the equivalence follows from the equivalence of items (1) and (2) in Theorem 1.1.17.

(3) $\implies$ (4): trivial.

(2) $\implies$ (3): if d_X is a metric on X and $\xi \in X$, the pseudometric on G given by $d(x, y) = d_X(x\xi, y\xi)$ is left-invariant and continuous. If now f is a bounded uniformly continuous function on X, the function $G \ni g \mapsto f(g\xi)$ is bounded and uniformly continuous, and therefore finitely oscillation stable, which implies the same conclusion for f.

(2) $\implies$ (1): Let G act continuously on a compact space X. Choose a point $\xi \in X$. The orbit mapping

$$G \ni g \mapsto g^{-1}\xi \in X$$

is left uniformly continuous by Lemma 2.1.5, hence finitely oscillation stable by assumption coupled with condition (4) of Theorem 1.1.17. Therefore, for every entourage $V \in \mathcal{U}_X$ and every finite $F \subseteq G$ there is a $g = g_{V,F} \in G$ such that the set $F^{-1}g^{-1}\xi$ is V-small. The net $(g_{V,F}^{-1}\xi)$ in the compact space X, indexed by all $V \in \mathcal{U}_X$ and all finite $F \subseteq G$, has a cluster point, which is fixed under the action of G.

(Equivalently, for the same V and F, the set $\Phi(V, F)$ of all $\zeta \in X$ such that $F\zeta$ is V-small is non-empty. If V is closed, then so is $\Phi(V, F)$. The system of all sets $\Phi(V, F) \subseteq X$, where V is a closed element of $\mathcal{U}_X$ and $F \subseteq G$ is finite, is centered, and thus contains a common element, which is a G-fixed point.)

(1) $\implies$ (2): assuming G is extremely amenable, there is a fixed point ξ in the greatest ambit $\mathcal{S}(G)$. Let f be a bounded real-valued

left uniformly continuous function on G, let $F \subseteq G$ be finite and symmetric, and let $\varepsilon > 0$. The function $\check{f}\colon x \mapsto f(x^{-1})$ is bounded right uniformly continuous and so extends over the greatest ambit $\mathcal{S}(G)$ by continuity. If $g_\alpha \to \xi$, where $g_\alpha \in G$, then for each $x \in F$, clearly $xg_\alpha \to x\xi = \xi$ as well, therefore for some α and all $x \in F$, $|\check{f}(\xi) - \check{f}(xg_\alpha)| < \varepsilon/2$. This implies $\operatorname{osc}(\check{f}|Fg_\alpha) < \varepsilon$, or equivalently $\operatorname{osc}(f|g_\alpha^{-1}F) < \varepsilon$, and we are done. □

Remarks 2.1.11. Extremely amenable discrete semigroups were studied by Mitchell [124] (who has favoured the terminology "common fixed point on compacta property," also used by Furstenberg) and Granirer [69, 70] (who preferred "extreme amenability"). It seems that extremely amenable topological groups were first discussed in print in Granirer's paper [69], II (as hypothetical objects), and then in Mitchell's paper [125], where the question of their existence was raised.

The compact-open topology on groups of homeomorphisms was studied in [4]. Lemma 2.1.5 is a standard result in abstract topological dynamics [5, 64, 186]. Lemmas 2.1.6 and 2.1.8 are hard to trace back to their inventors, they really belong to the folklore; for instance, they appear in [163], but were most certainly known before Teleman's time. The same can be said of the concept of the greatest ambit $\mathcal{S}(G)$. A standard reference is Brooks' paper [24], see also [43, 5, 186], but the essentials of the construction can be already found in [163]. Theorem 2.1.10 in its present precise form appears in author's paper [142].

2.2 Three main examples

2.2.1 Example: the unitary group

Here we will show that the unitary group $U(\ell^2)$, equipped with the strong operator topology, is extremely amenable. The strong topology is that of simple convergence on ℓ^2 (or on the unit ball), and it is a Polish topology. A standard neighbourhood, $V[\xi_1, \xi_2, \ldots, \xi_n; \varepsilon]$, consists of all $u \in U(\ell^2)$ with the property

$$\|\xi_i - u(\xi_i)\| < \varepsilon \text{ for } i = 1, 2, \ldots, n.$$

It is enough to take ξ_i from a countable subset whose linear span is dense in ℓ^2, for instance, sometimes it is convenient to consider only the standard basic vectors e_i, $i = 1, 2, \ldots$.

Here the fixed point on compacta property is being deduced using the concentration of measure. Let $\mathcal{H}$ be a Hilbert space. For every $m \in \mathbb{N}$, denote by $\mathrm{St}_m(\mathcal{H})$ the *Stiefel manifold* of all orthonormal m-frames in $\mathcal{H}$, that is, the ordered m-tuples of orthonormal vectors. This space is equipped with the uniform metric:

$$d((x,y) = \max_{i=1}^{n} \|x_i - y_i\|.$$

One can easily see that $\mathrm{St}_m(\mathcal{H})$ is a complete metric space and a smooth manifold modelled on the Banach space $\mathcal{H}^m$. We will denote $\mathrm{St}_m(\ell^2)$ by $\mathrm{St}_m(\infty)$, and $\mathrm{St}_m(\ell^2(n))$ by $\mathrm{St}_m(n)$. In particular, $\mathrm{St}_1(\infty) = \mathbb{S}^\infty$, while $\mathrm{St}_1(n) = \mathbb{S}^{n-1}$.

The group $U(\ell^2)$ (respectively, $U(n)$) acts transitivitely and continuously on every manifold $\mathrm{St}_m(\infty)$ (resp., $\mathrm{St}_m(n)$) by isometries. In fact, every $\mathrm{St}_m(\infty)$ can be identified with a homogeneous factor-space of $U(\ell^2)$ by choosing a frame $X \in \mathrm{St}_m(\infty)$, through the factor-map

$$U(\ell^2) \ni u \mapsto uX \in \mathrm{St}_m(\infty).$$

The definition of the strong topology easily implies that the group $U(\ell^2)$ is the inverse limit of the homogeneous metric spaces $\mathrm{St}_m(\infty)$, $m \in \mathbb{N}$. Therefore, according to Theorem 2.1.10, it is enough to prove the following result.

Theorem 2.2.1. *The action of $U(\ell^2)$ on $\mathrm{St}_m(\infty)$, $m \in \mathbb{N}$, is finitely oscillation stable.*

This is a consequence of Theorem 1.2.9, or else Theorem 1.2.12, with finite-dimensional Stiefel manifolds $\mathrm{St}_m(N)$ playing the role of compact subspaces K_n, while $G_N = U(N)$ are still the unitary groups of sufficiently high finite rank N. All one needs, is the following generalization of Theorem 1.2.27.

Theorem 2.2.2. *For every natural number m, the Stiefel manifolds $\mathrm{St}_m(n)$, $n \in \mathbb{N}$, equipped with the rotation-invariant probability Borel measures and the Euclidean distance, form a normal Lévy family.*

This result can be deduced directly from Theorem 1.2.27 by very simple means, especially since we do not care much about the sharpness of the constants of normal concentration.

Let (X, d, μ) be a space with metric and measure, and let $Y \subseteq X$ be a closed subset. Recall that a Borel measure ν on Y is called *measure induced* from X, or else a *Hausdorff* measure, if for every Borel subset $A \subseteq Y$ one has

$$\nu(A) = \lim_{\varepsilon \downarrow 0} \frac{\mu(A_\varepsilon)}{\mu(Y_\varepsilon)}.$$

Example 2.2.3. For natural numbers m, n, embed the manifold $\mathrm{St}_m(n)$ into the Euclidean sphere of dimension nm in a natural way. Namely, an orthonormal m-frame $(x_1, x_2, \ldots, x_m) \in \mathrm{St}_m(n)$ is mapped into the vector whose first n coordinates are those of x_1, followed by the n coordinates of x_2, etc., after which the resulting vector of norm $\sqrt{2n}$ is normalized to one. Then the probability measure on $\mathrm{St}_m(n)$, induced from $\mathbb{S}^{nm-1}$, coincides with the normalized Haar (rotation-invariant) measure on $\mathrm{St}_m(n)$, in view of the uniqueness of the latter. □

The following two results are obvious.

Lemma 2.2.4. *Let (X_n, d_n, μ_n) be a (normal) Lévy family of mm-spaces, and let for every n $Y_n \subseteq X_n$ be a closed subset such that $\mu_n((Y_n)_\varepsilon) \to 1$ (respectively, $\mu_n((Y_n)_\varepsilon) \geq 1 - C_1 \exp(-C_2\varepsilon^2 n)$) for every $\varepsilon > 0$ as $n \to \infty$. Assume further that Y_n supports a measure ν_n induced from X_n. Then the family of mm-spaces $(Y_n, d_n|_{Y_n}, \nu_n)$ forms a (normal) Lévy family.* □

Lemma 2.2.5. *Let X and Y be metric spaces with measure, and let $f\colon X \to Y$ be a 1-Lipschitz map such that $\mu_Y = f_*\mu_X$. Then*

$$\alpha_Y(\varepsilon) \leq \alpha_X(\varepsilon).$$

□

(Here $f_*\mu_X$ is the *push-forward measure* on Y: $(f_*\mu_X)(B) = \mu_X(f^{-1}(B))$.)

Every real-valued measurable function f on a finite measure space has a *median* (*Lévy mean*), that is, a number M with the property

$$\mu\{x \in X\colon f(x) \geq M\} \geq \frac{1}{2} \text{ and } \mu\{x \in X\colon f(x) \leq M\} \geq \frac{1}{2}.$$

In general, the median value of a function need not be unique, though it is such, for instance, for continuous functions on connected mm-spaces (X, d, μ) (where "connected" should be of course understood in the sense that the support $\operatorname{supp}\mu$ of the measure μ is topologically connected).

For a uniformly continuous function f on a metric space X, denote by $\delta = \delta(\varepsilon)$ the *modulus of uniform continuity* of f, that is, a function from $\mathbb{R}_+$ to itself where $\delta(\varepsilon)$ is the largest value such that

$$\forall x, y \in X, \quad d_X(x,y) < \delta(\varepsilon) \implies d_Y(f(x), f(y)) < \varepsilon.$$

For instance, if f is a Lipschitz function, then $\delta(\varepsilon) \leq L^{-1}\varepsilon$, where L is a Lipschitz constant for f.

Theorem 2.2.6. *Let $X = (X, d, \mu)$ be an mm-space, and let f be a uniformly continuous real-valued function on X with the modulus of uniform continuity δ. Denote by $M = M_f$ a median value for f. Then for every $\varepsilon > 0$*

$$\mu\{|f(x) - M| > \varepsilon\} \leq 2\alpha_X(\delta(\varepsilon)).$$

Proof. If we denote

$$A = \{x \in X \mid f(x) \leq M\}$$

and

$$B = \{x \in X \mid f(x) - M \leq \varepsilon\},$$

then clearly

$$A_{\delta(\varepsilon)} \subseteq B$$

and consequently

$$\mu(B) \geq 1 - \alpha(\delta(\varepsilon)).$$

A similar argument shows that

$$\mu\{x \mid M - f(x) \leq \varepsilon\} \geq 1 - \alpha(\delta(\varepsilon)).$$

The two halves come nicely together to imply that

$$\mu\{x\colon M - \varepsilon \leq f(x) \leq M + \varepsilon\} \geq 1 - 2\alpha(\delta(\varepsilon)),$$

which is the statement of the theorem. □

Proof of Theorem 2.2.2. Following V. Milman [114], §4, we will give the proof for $n = 2$, as the extension to higher rank Stiefel manifolds is obvious. For a natural number n, embed the manifold $\mathrm{St}_2(n)$ into the Euclidean sphere of dimension $2n - 1$ as in Example 2.2.3, via

$$i\colon \mathrm{St}_2(n) \ni (x, y) \mapsto \frac{\sqrt{2n}}{2}(x_1, x_2, \ldots, x_{n+1}, y_1, \ldots, y_{n+1}) \in \mathbb{S}^{2n-1}.$$

Consider the function

$$f\colon \mathbb{S}^{2n-1} \ni (x, y) \mapsto \langle x, y \rangle \in \mathbb{R},$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^{n+1}$, and also the function

$$g\colon \mathbb{S}^{2n-1} \ni (x, y) \mapsto \|x\| - \|y\| \in \mathbb{R}.$$

These f and g are Lipschitz, have zero as their median value, and it is easy to verify that the intersection of the inverse images $f^{-1}(-\varepsilon, \varepsilon) \cap g^{-1}(-\varepsilon, \varepsilon)$ is contained in the ε-neighbourhood of the image $i(\mathrm{St}_2(n))$. By Theorem 2.2.6,

$$\mu(i(\mathrm{St}_2(n))_\varepsilon) \geq 1 - 4\alpha_{\mathbb{S}^{2n-1}}(\varepsilon),$$

and Lemma 2.2.4 implies that the spaces $(\mathrm{St}_2(n))_{n\geq 1}$ with the induced measure (the Haar measure on the Stiefel manifold) and the metric inherited from $\mathbb{S}^{2n-1}$ through the embedding i form a Lévy family. Since i is a bi-Lipschitz embedding, Lemma 2.2.5 finishes the proof. □

Remarks 2.2.7. The fixed point on compacta property of the unitary group $U(\ell^2)$ with the strong operator topology was established by Gromov and Milman [78], who remark that the result answers a question by Furstenberg.

Other results and concepts of the section now belong to the inner core of Asymptotic Geometric Analysis and were introduced by Vitali Milman. See the book [123] and surveys [118, 120, 122].

2.2.2 Example: the group $\mathrm{Aut}\,(\mathbb{Q}, \leq)$

The group $\mathrm{Aut}\,(\mathbb{Q}, \leq)$ of all order-preserving self-bijections of the set $\mathbb{Q}$ of rational numbers (with the usual order) is equipped with the natural Polish topology, that is, the topology of simple convergence on the set $\mathbb{Q}$ viewed as discrete. With this topology, $\mathrm{Aut}\,(\mathbb{Q}, \leq)$ forms a Polish group, homeomorphic to a closed subset of $(\mathbb{Q}_d)^{\mathbb{Q}}$.

Theorem 2.2.8. *The Polish group* $\mathrm{Aut}\,(\mathbb{Q}, \leq)$ *is extremely amenable.*

For every finite subset $F \subset \mathbb{Q}$, define a pseudometric d_F on the group $\mathrm{Aut}\,(\mathbb{Q}, \leq)$ by

$$d_F(\sigma, \tau) = \begin{cases} 0, & \text{if } \sigma(F) = \tau(F), \\ 1 & \text{otherwise.} \end{cases} \tag{2.4}$$

Exercise 2.2.9. Verify that every pseudometric d_F as in Eq. (2.4) is bounded, left-invariant, and continuous, and the family of all pseudometrics d_F, where F runs over all finite subsets of $\mathbb{Q}$, determines the topology of $\mathrm{Aut}\,(\mathbb{Q}, \leq)$.

Exercise 2.2.10. Show that for every finite subset $F \subset \mathbb{Q}$, the left $\mathrm{Aut}\,(\mathbb{Q}, \leq)$-space $\mathrm{Aut}\,(\mathbb{Q}, <)/d_F$ is naturally isomorphic, as a metric G-space, to the space $[\mathbb{Q}]^{|F|}$ of all $|F|$-subsets of $\mathbb{Q}$, equipped with the discrete metric.

Proof of Theorem 2.2.8. According to Exercise 2.2.10 and Theorem 1.3.4, for every finite subset $F \subset \mathbb{Q}$ the left $\mathrm{Aut}\,(\mathbb{Q}, \leq)$-space

$$\mathrm{Aut}\,(\mathbb{Q}, <)/d_F \cong [\mathbb{Q}]^{|F|}$$

is finitely oscillation stable. Since the pseudometrics d_F determine the topology of the group $\mathrm{Aut}\,(\mathbb{Q}, \leq)$ (Exercise 2.2.9), Theorem 2.1.10, condition (4) concludes the argument. $\square$

Remark 2.2.11. Notice that extreme amenability of the topological group $\mathrm{Aut}\,(\mathbb{Q},\leq)$ is equivalent to finite Ramsey's theorem.

Exercise 2.2.12. The above argument does not work for the infinite symmetric group S_∞ in place of $\mathrm{Aut}\,(\mathbb{Q},\leq)$. Why?

Theorem 2.2.8 has some further consequences. First of all, an obvious statement.

Lemma 2.2.13. *If $h\colon G\to H$ is a continuous homomorphism between topological groups, having dense image, and G is extremely amenable, then so is H.* □

Let $\mathrm{Homeo}_+[0,1]$ denote the group of all orientation-preserving self-homeomorphisms of the closed unit interval, equipped with the compact-open topology. Similarly, let $\mathrm{Homeo}_+\mathbb{R}$ be the homeomorphism group of the real line (or the open interval $(0,1)$), again with the compact-open topology. Both are Polish topological groups.

Corollary 2.2.14. *Topological groups* $\mathrm{Homeo}_+[0,1]$ *and* $\mathrm{Homeo}_+\mathbb{R}$ *are extremely amenable.*

Proof. Every order-preserving bijection $f\colon\mathbb{Q}\to\mathbb{Q}$ uniquely extends to an order-preserving bijection, $\bar f$, of the Dedekind completion of $\mathbb{Q}$, that is, $\mathbb{R}$. The resulting correspondence

$$\mathrm{Aut}\,(\mathbb{Q},<)\ni f\mapsto \bar f\in \mathrm{Homeo}_+\mathbb{R}$$

is easily verified to be a continuous monomorphism with a dense image. Similarly, it is easy to construct a continuous monomorphism with a dense image from the latter group to $\mathrm{Homeo}_+[0,1]$. Now one applies Lemma 2.2.13. □

Remark 2.2.15. The present subsection is based on results by the author [138], obtained independently from any previous work on extreme amenability of which he was unaware at the time. The motivation was to answer a question of Ellis on the enveloping semigroup of the universal minimal flow of a topological group [43].

2.2.3 Counter-example: the infinite symmetric group

The infinite symmetric group S_∞ also carries a natural Polish group topology, just like the group $\mathrm{Aut}\,(\mathbb{Q}, \leq)$: it is the topology of simple convergence on the infinite set ω with the discrete topology.

Exercise 2.2.16. Verify that with the above defined topology, S_∞ is a Polish group.

Contrary to $\mathrm{Aut}\,(\mathbb{Q}, \leq)$, the Polish group S_∞ is not extremely amenable. Indeed, by Theorem 1.3.1 it is enough to show an example of a metric space upon which S_∞ acts transitively and continuously by isometries, and which is not finitely oscillation stable.

Such a metric space is, as we have already seen, $X = \omega^2 \setminus \Delta_\omega$ equipped with the discrete metric (Section 1.4).

Remark 2.2.17. The observation that S_∞ is not extremely amenable was first made in the paper [138]. As the relation between finite oscillation stability and extreme amenability was not well understood at the time, the proof was not quite so transparent as above.

Fursternerg's question, mentioned in [78], came in two parts: one, about extreme amenability of the unitary group, was answered by Gromov and Milman in the affirmative, and the other, about extreme amenability of the infinite symmetric group, has been answered by our result from [138] in the negative.

The explicit form of any non-trivial minimal flow for the group S_∞ remained unknown for a while, namely until the work of Glasner and Weiss [60], who have not only given very interesting examples of such flows, but have found a universal minimal flow as well. We will describe this result in detail in Section 4.3. Here is an explicit example of a non-trivial compact minimal S_∞-space.

Example 2.2.18 (Glasner and Weiss [60]). Every linear order $\prec$ on the set ω can be identified with the characteristic function of the corresponding relation $\{(x, y) \in \omega \times \omega \colon x \prec y\}$. In this way, the set LO of all linear orders on ω becomes a zero-dimensional compact subspace of $\{0, 1\}^{\omega \times \omega}$. The group S_∞ acts on LO by double permutations:

$$(x \overset{\sigma}{\prec} y) \Leftrightarrow (\sigma^{-1}x \prec \sigma^{-1}y)$$

for all $\prec \in \mathrm{LO}$, $\sigma \in S_\infty$, and $x, y \in \omega$.

In fact, this is the restriction of the regular representation of the infinite symmetric group S_∞ in the space $\ell^\infty(\omega \times \omega)$ to the invariant subset LO consisting of characteristic functions of all linear orders.

Exercise 2.2.19. Verify that the action of the Polish group S_∞ on the compact space LO is continuous and minimal.

2.3 Equivariant compactification

It is an appropriate place where to establish the universal property of the greatest ambit (Corollary 2.3.10 at the end of this Section). However, it is convenient to do it in a slightly more general setting, even if, in general, we present the results on a strictly need-to-know basis. Therefore, we will consider here a general procedure of universal compactification of uniform spaces, equipped with a continuous action of a topological group by uniform isomorphisms.

If a group G acts on a set X, then G also acts on the Banach space $\ell^\infty(X)$ of all bounded complex-valued functions on X by

$$\ell^\infty(X) \ni f \overset{g}{\mapsto} {}^g f \in \ell^\infty(X), \quad {}^g f(x) = f(g^{-1}x).$$

The above action determines a homomorphism from G into the group of isometries of ℓ^∞, the left regular representation of G.

Let now G be a topological group.

Definition 2.3.1. A function $f \in \ell^\infty(X)$ is called *π-uniform* if the orbit map

$$G \ni g \mapsto {}^g f \in \ell^\infty(X) \tag{2.5}$$

is continuous.

If X is a uniform space, then we also require f to be uniformly continuous. If X is a topological space, then we require f to be continuous (that is, uniformly continuous with regard to the finest compatible uniformity on X).

Remark 2.3.2. Strictly speaking, the concept of a π-uniform function depends on whether or not we consider the space X as a uniform space, a topological space, or just a set without topology. However, in every particular situation this will be clear from the context and thus no ambiguity will arise.

Remark 2.3.3. It is useful to keep in mind that the orbit map (2.5) is continuous if and only if it is continuous at the identity, e_G. Indeed, for every $g, h \in G$

$$\begin{aligned} \left\| {}^g f - {}^h f \right\| &= \sup\{|f(g^{-1}x) - f(h^{-1}x)| \colon x \in G\} \\ &= \sup\{|f(y) - f(h^{-1}gy)| \colon y \in G\} \\ &= \left\| {}^e f - {}^{g^{-1}h} f \right\|, \end{aligned}$$

and the latter can be made arbitrarily small by choosing $g^{-1}h$ from a suitably small neighbourhood of identity, V, or, equivalently, if $h \in gV$.

The above argument also shows that if an orbit map (2.5) is continuous, then it is *left* uniformly continuous on G.

Example 2.3.4. Let $X = G$, considered as a topological space (or even just a set) and equipped with the action of G on itself by left translations. In this case, π-uniform functions are exactly *right* uniformly continuous functions.

Indeed, if V is a neighbourhood of identity in G and $v \in V$, then

$$\begin{aligned} \|{}^e f - {}^v f\| &= \sup\{|f(x) - f(vx)| \colon x \in G\} \\ &\leq \sup\{|f(x) - f(y)| \colon yx^{-1} \in V\}, \end{aligned}$$

and consequently

$$\forall \varepsilon > 0, \ \ \exists V \ni e_G, \ \ \forall v, \ \ (v \in V) \implies \|{}^e f - {}^v f\| < \varepsilon$$

is equivalent to the right uniform continuity of f.

Example 2.3.5. If G is a discrete group acting by uniform isomorphisms on a uniform space X, then π-uniform functions on X are just all uniformly continuous functions.

The collection $\mathrm{UNIFB}\,(X)$ of all bounded π-uniform functions forms a closed $*$-subalgebra of $\ell^\infty(X)$, and thus a C^*-algebra. Denote by $\alpha_G X$, or simply αX, the maximal ideal space of $\mathrm{UNIFB}\,(X)$, equipped with the natural continuous action of G (cf. Lemma 2.1.6). There is a natural map $X \to \alpha X$, which is (uniformly) continuous if X is a uniform/topological space, and the image of X in αX is everywhere dense. Moreover, the map i is G-equivariant. Thus, αX is a compactification of the G-space X, which is known as the *maximal equivariant compactification* of X.

Remark 2.3.6. We do not require of a compactification to contain the original space as a homeomorphic subspace, as it the custom in set-theoretic topology. In our approach, a compactification of a G-space X is a pair (K, i) consisting of a compact G-space K and an equivariant continuous map $i\colon X \to K$ whose image is dense in K. This is motivated by the fact that, generally speaking, an equivariant compactification of a G-space X containing X as a topological subspace need not exist. There are moreover examples due to Megrelishvili [109] of non-trivial Polish G-spaces X whose maximal equivariant compactification is a singleton.

Example 2.3.7. The maximal equivariant compactification of the topological group G equipped with the action on itself by left translations is the greatest ambit, $\mathcal{S}(G)$.

Example 2.3.8. If G is a discrete group acting on a uniform space X by uniform isomorphisms, then $\alpha_G X$ is the Samuel compactification σX of X.

The terminology is justified by the following result.

Theorem 2.3.9. *Let G be a topological group, acting continuously by uniform isomorphisms on a uniform space X, and let K be an equivariant compactification of X, that is, a pair consisting of a compact G-space K and a continuous map $\phi\colon X \to K$, commuting with the action of G. Then there is a unique morphism of G-spaces $\bar{\phi}\colon \alpha X \to K$ such that $\bar{\phi}|_X = \phi$.*

Proof. The action of G on $C(K)$ is continuous, and so every function $f \in C(K)$ is π-uniform. Consequently, there is a well-defined morphism of C^*-algebras, commuting with the action of G,

$$i^*\colon C(K) \ni f \mapsto f \circ i \in \mathrm{UNIFB}\,(X).$$

This morphism defines the required map from αX, the maximal ideal space of $\mathrm{UNIFB}\,(X)$, to K, the maximal ideal space of $C(K)$, and it is easy to see that it has all the required properties. $\square$

Corollary 2.3.10. *The greatest ambit $\mathcal{S}(G)$ of a topological group G has the following universal property. If (X, x_0) is a G-ambit, that is, a compact G-space with a distinguished point $x_0 \in X$, there is a unique morphism of G-spaces from $\mathcal{S}(G)$ to X, taking e to x_0.* $\square$

Remark 2.3.11. π-Uniform functions are a convenient and standard tool for study of equivariant compactifications, largely developed through the efforts of de Vries [185]. See also Megrelishvili [109] for some recent developments concerning non-locally compact groups.

2.4 Essential sets and the concentration property

Following Milman and only slightly extending a setting for his definition, let us introduce the following notion. It will be useful later in connection with more general approach to fixed points, but will also provide a conceptual framework for a relatively easy proof of Veech's theorem in the next section.

Definition 2.4.1. Let $X = (X, \mathcal{U}_X)$ be a uniform space, and let F be a family of uniform isomorphisms of X.

A subset $A \subseteq X$ is called *essential* with regard to F, or *F-essential,* if the family of subsets

$$V[fA],\ V \in \mathcal{U}_X,\ f \in F \cup \{\mathrm{Id}\}$$

is centred.

Remark 2.4.2. This means that for every entourage of the diagonal $V \in \mathcal{U}_X$ and every finite collection of transformations $f_1, f_2, \dots, f_n \in F$, $n \in \mathbb{N}$, one has

$$V[A] \cap \bigcap_{i=1}^{n} V[f_i A] \neq \emptyset, \tag{2.6}$$

Since each f_i is a uniform isomorphism, one can instead use the following condition:

$$V[A] \cap \bigcap_{i=1}^{n} f_i V[A] \neq \emptyset. \tag{2.7}$$

Definition 2.4.3. The pair (X, F) as above has the *concentration property* if every finite cover γ of X contains an essential set $A \in \gamma$. (Cf. Fig. 2.2.)

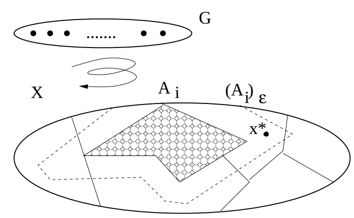

Figure 2.2: To the concept of concentration property.

Here is a very simple example of a G-space without concentration property.

Example 2.4.4. Let $\mathbb{T}$ denote the unit circle, and let $F = SU(1)$ be the family of all rotations of $\mathbb{T}$. The pair $(\mathbb{T}, F)$ does not have the concentration property. Indeed, let R_θ be a rotation through an angle $\theta \neq 0$. Then any finite cover of $\mathbb{T}$ by intervals of geodesic length $\ell < |\theta|$ does not contain an essential set; here the entourage of the diagonal is given by the value $\varepsilon = (|\theta| - \ell)/2$.

Here is a more interesting example, which belongs to Imre Leader (1988, unpublished).

Example 2.4.5. Let $\mathbb{S}^\infty$ denote, as usual, the unit sphere in the space $\ell^2 = \ell^2(\mathbb{N})$, upon which the unitary group $U(\ell^2)$ acts by rotations. Then the pair $(\mathbb{S}^\infty, U(\ell^2))$ does not have the concentration property.

For a $\Gamma \subset \mathbb{N}$, let p_Γ stand for the orthogonal projection of $\ell^2(\mathbb{N})$ onto its subspace $\ell^2(\Gamma)$. Denote by E (respectively, F) the set of all even (resp., odd) natural numbers, and let

$$A = \left\{x \in \mathbb{S}^\infty \colon \|p_E x\| \geq \sqrt{2}/2\right\},$$

$$B = \left\{x \in \mathbb{S}^\infty \colon \|p_E x\| \leq \sqrt{2}/2\right\}.$$

Clearly, $A \cup B = \mathbb{S}^\infty$. At the same time, both A and B are inessential. To see this, let E_1, E_2, E_3 be three arbitrary disjoint infinite subsets of $\mathbb{N}$, and let $\phi_i \colon \mathbb{N} \to \mathbb{N}$ be bijections with $\phi_i(E) = E_i$, $i = 1, 2, 3$. Let g_i denote the unitary operator on $\ell^2(\mathbb{N})$ induced by ϕ_i. Now

$$g_i(A) = \left\{ x \in \mathbb{S}^\infty \colon \; \|p_{E_i} x\| \geq \sqrt{2}/2 \right\},$$

and consequently

$$(g_i(A))_\varepsilon \subseteq \left\{ x \in \mathbb{S}^\infty \colon \; \|p_{E_i} x\| \geq (\sqrt{2}/2) - \varepsilon \right\}.$$

Thus, as long as $\varepsilon < \sqrt{2}/2 - \sqrt{3}/3$, we have

$$\cap_{i=1}^3 (g_i(A))_\varepsilon = \emptyset.$$

The set B is treated in exactly the same fashion.

Every uniformly continuous mapping $f \colon X \to X$ determines a unique continuous mapping $\bar{f} \colon \sigma X \to \sigma X$. If f is a uniform automorphism of X, then $\bar{f}$ is a self-homeomorphism of σX (with the inverse $\bar{f^{-1}}$). In the following result, we will identify f and its extension over σX.

Proposition 2.4.6. *For a family F of automorphisms of a uniform space $X = (X, \mathcal{U}_X)$ the following are equivalent.*

1. *The pair (X, F) has the property of concentration.*
2. *The pair $(\sigma X, F)$ has the property of concentration.*
3. *The family F has a common fixed point in the Samuel compactification of X.*
4. *There exists an F-invariant multiplicative mean on the space* $\mathrm{UCB}\,(X)$.

Proof. (1) $\implies$ (2): obvious from the definition.

(2) $\implies$ (3): emulates a proof of Proposition 4.1 and Theorem 4.2 in [119].

There exists a point $x^* \in \sigma X$ whose every neighbourhood is essential: assuming the contrary, one can cover the compact space σX

with open F-inessential sets and select a finite subcover containing no F-essential sets, a contradiction.

We claim that x^* is a common fixed point for F. Assume it is not so. Then for some $f \in F$ one has $fx^* \neq x^*$. Choose an entourage, W, of the unique compatible uniform structure on σX with the property $W^2[x^*] \cap W^2[fx^*] = \emptyset$. Since f is uniformly continuous, there is a $W_1 \subseteq W$ with $(x,y) \in W_1 \Rightarrow (fx, fy) \in W$ for all x, y. Since $f(W_1[x^*]) \subseteq W[f(x^*)]$, we conclude that $W_1[x^*]$ is F-inessential (with $V = W$), a contradiction.

(3) $\Leftrightarrow$ (4): fixed points in the Gelfand space σX of the commutative C^*-algebra $\mathrm{UCB}\,(X)$ correspond to F-invariant multiplicative means (states) on $\mathrm{UCB}\,(X)$.

(3) $\Rightarrow$ (1): If γ is a finite cover of X, then the closures of all $A \in \gamma$ taken in σX cover the latter space, and so there is an $A \in \gamma$ with $\mathrm{cl}_{\,\sigma X}(A)$ containing an F-fixed point $x^* \in \sigma X$. We claim that A is F-essential in X.

To prove this, we need a simple fact of general topology. Let $B_1, \ldots, B_n$ be subsets of a uniform space X satisfying the condition $V[B_1] \cap \cdots \cap V[B_n] = \emptyset$ for some entourage $V \in \mathcal{U}_X$. Then the closures of B_i, $i = 1, 2, \ldots, n$ in the Samuel compactification σX have no point in common: $\mathrm{cl}_{\sigma X}(B_1) \cap \cdots \cap \mathrm{cl}_{\sigma X}(B_n) = \emptyset$.

[Let ρ be a uniformly continuous bounded pseudometric on X subordinated to the entourage V in the sense that $(x,y) \in V$ whenever $x, y \in X$ and $\rho(x,y) < 1$. For each $i = 1, \ldots, n$ and $x \in X$ set $d_i(x) = \inf\{\rho(b,x) \colon b \in B_i\}$. The real-valued functions d_i are uniformly continuous (indeed ρ-Lipschitz-1) and bounded on X, and therefore extend to (unique) continuous functions $\tilde{d}_i$ on σX. If there existed a common point, x^*, for the closures of all B_i in σX, then all $\tilde{d}_i$ would vanish at x^* and consequently for any given $\varepsilon > 0$ there would exist an $x \in X$ with $d_i(x) < \varepsilon$, and in particular $x \in V[B_i]$, for all i. However, for every $x \in X$, there is an i with $x \notin V[B_i]$.]

Now assume that $\cap_{f \in F_1} V[fA] = \emptyset$ for some $V \in \mathcal{U}_X$ and a finite subfamily F_1 of F.

The above observation from uniform topology implies that

$$\cap_{f \in F_1} \mathrm{cl}_{\sigma X}(f(A)) = \emptyset,$$

and, since extensions of f to σX are homeomorphisms,

$$\cap_{f \in F_1} f(\mathrm{cl}_{\sigma X}(A)) = \emptyset,$$

a contradiction. $\square$

Remark 2.4.7. The reader may be wondering at this point what is the relationship between the concentration property (Definition 2.4.3) and finite oscillation stability (Definition 1.1.10) of a uniform G-space X.

A loose answer is that oscillation stability is the concept suited for the pairs of the form (*left* uniformity, *left* action), while the concentration property works for the pairs (*right* uniformity, *left* action).

For example, if $X = G$ is the topological group itself, then $(G, \mathcal{U}_L)$ is oscillation stable iff $(G, \mathcal{U}_R)$ has the concentration property iff G is extremely amenable.

However, in general there is no link between the two properties of the same uniform G-space.

Example 2.4.8. The unit sphere $\mathbb{S}^\infty$ equipped with the action of the unitary group $U(\ell^2)$ is finitely oscillation stable (Milman's theorem 1.2.5), but does not have the concentration property (I. Leader's example 2.4.5).

On the other hand, every non-trivial uniform space equipped with a trivial action of a group has the concentration property but is not oscillation stable, for "trivial" reasons.

Here is a more interesting example, where the action of the group G is even transitive.

Example 2.4.9. Consider the left action of the group $U(\ell^2)$ upon itself, equipped with the right uniform structure (with regard to the strong topology). The pair $(U(\ell^2), (U(\ell^2), \mathcal{U}_R))$ has the concentration property, since the group is extremely amenable. Finite oscillation stability of the above pair would lead to the concentration property of the pair $(U(\ell^2), (U(\ell^2), \mathcal{U}_L))$, and consequently of the pair $(U(\ell^2), \mathbb{S}^\infty)$, contrary to Example 2.4.5. (We leave the details of this argument as an exercise.)

Remark 2.4.10. The concepts of an essential set, as well as the concentration property were introduced by Vitali Milman [119, 120] as a follow-up to his results with Gromov about extreme amenability of the unitary group, as well as the concentration property of the unit sphere $\mathbb{S}^\infty$ equipped with an action of an abelian or a compact group by isometries [78]. Milman stresses in [119] that this is "just a different interpretation of the same approach as in [78]." See [122] for an interesting explanation of the reasons behind this approach ("concentration without a measure").

A question asked in [120] – does the pair $(\mathbb{S}^\infty, U(\ell^2))$ have the concentration property? — was proposed by Bollobas to his then Ph.D. student Imre Leader, who had almost immedaitely answered it in the negative. Leader has sent his example 2.4.5 to Milman, but has chosen to never publish it.

A deeper reading of the same question is of course "for which groups of unitary transformations G does the pair $(\mathbb{S}^\infty, G)$ have the concentration property?," and it was completely answered in [140, 142]. This is yet another development that is missing from these notes and which should be included in any future edition.

2.5 Veech theorem

One of the reasons why extremely amenable groups have only been discovered relatively recently, is that extreme amenability is an essentially "infinite-dimensional" phenomenon: no locally compact group is extremely amenable.

First of all, it was Ellis [42] who had proved that every discrete group G acts freely on a compact space. His argument is sufficiently simple, and as warm-up, we propose to the reader to reconstruct it.

Recall that an action of a group G on a set X is *free* (or *strongly effective*) if no element $g \neq e$ has fixed points: for each $x \in X$, one has $gx \neq x$.

Exercise 2.5.1. Let G be a discrete group and let $g \in G$, $g \neq e$. Construct a bounded function $f \colon G \to \mathbb{Z}$ with the property that for every $x \in G$, one has

$$\operatorname{osc}(f|_{\{x,gx\}}) \geq 1.$$

By extending the function f by continuity over the Stone–Čech compactification βG, conclude that the action of G on βG is free.

Then Granirer and Lau [71] showed that every locally compact group admits an effective minimal action on a compact space. Finally, Veech [177] has proved an even stronger result.

Theorem 2.5.2 (Veech). *Every locally compact group admits a free action on a compact space.*

Remark 2.5.3. Of course, as an immediate corollary, every locally compact group admits a free *minimal* action on a compact space.

Remark 2.5.4. The proof of Veech theorem can be modelled after the proof of Ellis's result for discrete groups as in Exercise 2.5.1 above, and indeed this was Veech's original argument. (Cf. Fig. 2.3.)

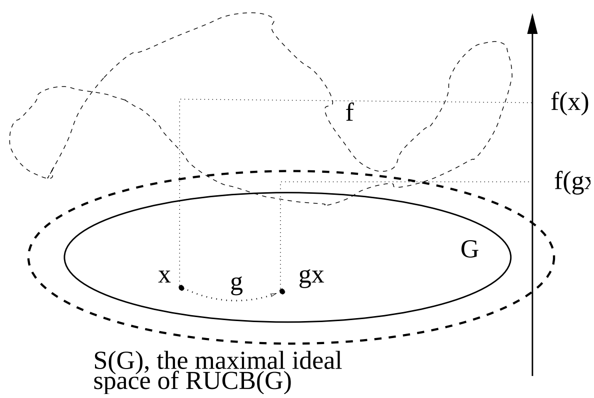

Figure 2.3: To the original proof of Veech theorem.

The difficulty here is that one cannot use βG as a compactification of G, but rather the greatest ambit $\mathcal{S}(G)$, that is, the greatest compactification on which the action of G remains continuous. In order to be extendable over $\mathcal{S}(G)$, a function $f\colon G \to \mathbb{R}^N$ must be right uniformly continuous, and while constructing such a function on a single orbit of $\langle g \rangle$ is easy, coalescing those functions together so as to get a uniformly continuous one is very technically involved. An easier proof was later given by Pym [148], and independently a yet different proof was proposed (for second-countable groups) by Adams and Stuck [1]. A further simplification was achieved in [99]. Here we are proposing to simplify the proof even further, and the concept of an essential set comes very handy for this purpose.

Let us say that a set is *g-inessential* if it is F-inessential in the sense of Definition 2.4.1 for $F = \{g\}$

Lemma 2.5.5. *Let g be a uniform isomorphism of a uniform space $(X, \mathcal{U}_X)$. The following are equivalent.*

1. *The extension of g over the Samuel compactification σX is fixed point-free.*
2. *X can be covered with finitely many g-inessential sets.*
3. *There is a uniformly continuous bounded function f from X to the space $\ell^\infty(n)$ of a suitable finite dimension n such that for every $x \in X$, $\operatorname{osc}(f|\{x, gx\}) \geq 1$.*

Proof. Denote by $\bar{g}$ the extension of g over σX.

(1) $\Longrightarrow$ (2): the set $G = \{(x, \bar{g}x) \colon x \in \sigma X\}$ is closed in $\sigma X \times \sigma X$, being the image of the compact σX under the diagonal product $\mathrm{I}d \times \bar{g}$ of two continuous mappings. Since G is disjoint from the diagonal, its complement in $\sigma X \times \sigma X$ is an element of the unique compatible uniformity $\mathcal{U}_{\sigma X}$. By Lemma 1.1.14, there are a finite cover γ of X and an element $V \in \mathcal{U}_X$ such that the set $\widetilde{\gamma_V}$ is contained in $X \times X \setminus G$, that is, if $A \in \gamma$ and $(x, y) \in V[A]$, then $(x, y) \in X \times X \setminus G$. It follows that no element of γ is essential for g.

(2) $\Longrightarrow$ (3): let γ be a finite cover of X with inessential sets with regard to $\{g\}$, and let $V \in \mathcal{U}_X$ be such that for every $A \in \gamma$, $V[A] \cap V[gA] = \emptyset$. Choose a uniformly continuous bounded pseudometric ρ on X such that $(x, y) \in V$ whenever $\rho(x, y) < 1$. For each $A \in \gamma$, the function $f_A(x) = \rho(A, x)$ is in $\mathrm{UCB}(X)$ and so extends to a continuous function $\bar{f}_A \in C(\sigma X)$. Define a bounded uniformly continuous function $f \colon \sigma X \to \ell^\infty(|\gamma|)$ whose components are f_A, $A \in \gamma$. Clearly, for every $x \in X$ one has $\|f(x) - f(gx)\|_\infty \geq 1$.

(3) $\Longrightarrow$ (1): a bounded uniformly continuous function $f \colon X \to \ell^\infty(n)$ with the property $\operatorname{osc}(f|\{x, gx\}) \geq 1$ for every $x \in X$ extends

by continuity to a continuous function $\bar{f}\colon \sigma X \to \ell^\infty(n)$ which, again by continuity, has the same property,

$$\|f(x) - f(\bar{g}x)\|_\infty \geq 1,$$

this time for every $x \in \sigma X$. We conclude: no point of σX is fixed under $\bar{g}$. $\square$

Corollary 2.5.6. *For a topological group G the following are equivalent.*

1. *G admits a free action on a compact space.*
2. *G acts freely on the greatest ambit $\mathcal{S}(G)$.*
3. *For every $g \in G$, $g \neq e_G$ the group G, equipped with the right uniformity and the action on itself by left translations, admits a finite cover by g-inessential sets.*
4. *For every two-element subset A of G there are $n \in \mathbb{N}_+$ and a function $f \in \mathrm{LUCB}\,(G; \ell^\infty(n))$ having large oscillations on all left translates of A.*

Proof. (1) $\Longrightarrow$ (2): if G acts freely on a compact space X, then, according to Corollary 2.3.10, there is a morphism of G-spaces $\mathcal{S}(G) \to X$, and clearly the action of G on $\mathcal{S}(G)$ is free as well.

(2) $\Longrightarrow$ (1) in a trivial way.

The equivalence (2) $\iff$ (3) follows from Lemma 2.5.5 (equivalence of (1) and (2)) once we recall that the greatest ambit $\mathcal{S}(G)$ is the Samuel compactification of the right uniform space of G.

Similarly, the equivalence (2) $\iff$ (4) follows from Lemma 2.5.5, equivalence of (1) and (3), if we replace a right uniformly continuous bounded function f with the left uniformly continuous bounded function $\check{f}$, $\check{f}(x) = f(x^{-1})$. $\square$

We also need the following often-used result on graph colouring.

Lemma 2.5.7. *Let $\Gamma = (V, E)$ be a simple graph (no loops) whose degree at every vertex is bounded by $d \in \mathbb{N}$. Then there is a colouring of the set V of vertices with at most $d+1$ colours in such a way that no two adjacent vertices are of the same colour.*

Proof. By Zorn's lemma, there exist a partial colouring c of V with $d+1$ colours whose graph is maximal with the property that no two adjacent vertices are of the same colour. Assuing that the domain of c is not all of V, it is easy to see that the colouring can be extended further, contradicting the maximality. $\square$

After all the preparation, the proof of the Veech theorem 2.5.2 becomes simple.

Proof of the Veech theorem. Let $g \in G$, $g \neq e$ be arbitrary. Choose a symmetric compact neighbourhood of the identity, K, with the property $g \notin K^6$. Let Γ be a maximal subset of G with the property that for every $a, b \in \Gamma$, $a \neq b$, one has $Ka \cap Kb = \emptyset$. As a consequence, $K^2\Gamma = G$; indeed, if some $g \notin K^2\Gamma$, then $Kg \cap K\Gamma = \emptyset$, contrary to the maximality of Γ. Make Γ into a graph as follows: $a, b \in \Gamma$ are adjacent if and only if either $gK^3a \cap K^3b \neq \emptyset$, or vice versa, that is, $gK^3b \cap K^3a \neq \emptyset$. The condition $g \notin K^6$ assures that Γ has no loops. Equivalently, a, b are adjacent if and only if $ba^{-1} \in K^3 \cdot \{g, g^{-1}\} \cdot K^3$. Denote $C = K^3 \cdot \{g, g^{-1}\} \cdot K^3$.

Denote by N a natural number with the property that the compact set C can be covered with N right translates of K:

$$C \subseteq \cup_{j=1}^{N} Kg_j.$$

We claim that the graph Γ has degree bounded by N. Indeed, let a vertex $a \in \Gamma$ be adjacent to $b_1, b_2, \ldots, b_n$. Then $b_1a^{-1}, b_2a^{-1}, \ldots, b_na^{-1} \in C$, and for each i there is a $j = 1, 2, \ldots, N$ with $b_ia^{-1} \in Kg_j$. At the same time, for each j there is at most one i with this property, because the assumptions $b_ia^{-1} = \kappa g_j$ and $b_{i'}a^{-1} = \kappa' g_j$, $\kappa, \kappa' \in K$, imply that Kb_i and $Kb_{i'}$ have the point g_ja in common, and therefore $i = i'$ by the choice of Γ. We conclude: $n \leq N$.

Apply Lemma 2.5.7 to partition Γ into subsets Γ_i, $1 \leq i \leq k \leq N+1$, so that no two elements from the same set are adjacent to each other. The sets $K^2\Gamma_i$, $i = 1, \ldots, k$ form a finite cover of G. Each of them is g-inessential with regard to the right uniformity on G and the action by left multiplication. We verify this using the following element of the right uniformity

$$K_R = \{(x, y) \in G \times G \colon xy^{-1} \in K\}.$$

Indeed, for every i

$$K_R[K^2\Gamma_i] = K^3\Gamma_i, \quad gK_R[K^2\Gamma_i] = gK^3\Gamma_i,$$

and the two above sets are disjoint, for otherwise two different elements $a, b \in \Gamma_i$ would be adjacent to each other. Now Corollary 2.5.6(3) finishes the proof. $\square$

2.6 Big sets

Definition 2.6.1. A subset A of a group G is called *left syndetic,* if there is a compact set $K \subseteq G$ with

$$G = KA.$$

If A is left syndetic with regard to the discrete topology on G, then A is called *discretely left syndetic,* or else *big* (*big on the left*). This is simply the case where finitely many left translates of A cover G.

The right versions of the above are defined in an obvious way.

For instance, a subset A of the group of integers $\mathbb{Z}$ with the discrete topology is big if and only if A is relatively dense, that is, the size of "gaps" between elements of A is bounded above: for some $N \in \mathbb{N}$, every interval $[n, n+N]$ meets A.

Exercise 2.6.2. Let S be a left syndetic subset of a topological group. Show that for every neighbourhood V of the identity the set VS is big on the left.

In topological dynamics, big sets are closely linked to minimal flows.

Definition 2.6.3. Let G be a topological group, continuously acting on a compact space X. Such an X is called a *minimal G-space,* or a *minimal flow,* if it contains no proper compact G-subspace. Equivalently: if the orbit of every point $x \in X$ is dense in X.

As a simple convequence of Zorn's Lemma, every compact G-space contains a minimal G-subspace. The following is immediate.

Proposition 2.6.4. *A topological group G is extremely amenable if and only if every minimal G-flow is a singleton.* $\square$

Here is a standard observation in abstract topological dynamics.

Proposition 2.6.5. *Let G be a topological group, acting continuously and minimally on a compact space X. Let $x \in X$ and let $V \subseteq X$ be open. Then the open set*

$$\widetilde{V} = \{g \in G \colon gx \in V\}$$

is discretely left syndetic in G.

Proof. The translates hV, $h \in G$ form an open cover of X: assuming a $y \in X$ is not in their union, the orbit of y would miss V, contrary to everywhere density in X. Choose a finite subcover $\{fV \colon f \in F\}$, where $F \subseteq G$, $|F| < \infty$. It remains to notice that $\widetilde{fV} = f\widetilde{V}$ for each $f \in G$ and therefore $F\widetilde{V} = G$. $\square$

Remark 2.6.6. To see that the set $\widetilde{V}$ in the above Proposition is not necessarily right syndetic, just consider the action of the homeomorphism group Homeo $(\mathbb{S}^1)$ on the circle $\mathbb{S}^1$.

Let us obtain an intrinsic characterization of extremely amenable groups.

Exercise 2.6.7. Show that a subset X of a topological group G is dense in G if and only if $VX = G$ for every neighbourhood V of the identity.

Theorem 2.6.8. *For a topological group G the following are equivalent.*

1. *G is extremely amenable.*
2. *For every set $S \subseteq G$, big on the left, the set SS^{-1} is everywhere dense in G.*
3. *For every left syndetic set $S \subseteq G$, the set SS^{-1} is everywhere dense in G.*

Proof. (1) $\Longrightarrow$ (2): assume that G is extremely amenable, and let S be a big subset of G such that $\mathrm{cl}\, SS^{-1} \neq G$. By replacing S with S^{-1} we can assume that S is big on the right, that is, $SF = G$ for some finite $F \subseteq G$, and that $\mathrm{cl}\, S^{-1}S \neq G$. Choose an element $g \in G$ and a bounded continuous left-invariant pseudometric d on G such that $d(g, S^{-1}S) = 1$. For every $a \in F$, define a function

$$f_a(x) = d(xa^{-1}, S).$$

This f_a is bounded left uniformly continuous, as well as the following map:

$$G \ni x \mapsto f(x) = (f_a(x))_{a\in F} \in \ell^\infty(|F|).$$

Now define a finite set

$$\Phi = \{a^{-1}ga \colon a \in F\} \cup \{e\}.$$

We claim that f has oscillation 1 on every left translate, $x\Phi$, of Φ. Indeed, if $a \in F$ is such that $x = sa \in Sa$, then $f_a(x) = d(s, S) = 0$, yet $a^{-1}ga \in \Phi$ and

$$\begin{aligned} f_a(xa^{-1}ga) &= d(sg, S) \\ &= d(g, s^{-1}S) \\ &\geq d(g, S^{-1}S) \\ &= 1. \end{aligned}$$

Thus, the bounded left uniformly continuous function f taking values in a finite dimensional normed space is not oscillation stable. According to Theorem 1.1.17 (equivalence of (1) and (4)), the group G, equipped with its left uniformity and acting on itself by left translations, is not finitely oscillation stable. This contradicts Theorem 2.1.10.

(2) $\Longrightarrow$ (3): let S be a left syndetic set, that is, $KS = G$ for a compact K. Let V be an arbitrary symmetric neighbourhood of identity in G, and let $g \in G$ be any. Choose a finite $F \subseteq G$ with $FV \supseteq K$. It follows that the set VS is discretely left syndetic, and therefore by assumption (2) and Exercise 2.6.7 one has $VSS^{-1} \ni g$, or $Vg \cap SS^{-1} \neq \emptyset$. Since the sets Vg form a neighbourhood basis of g, we conclude that $g \in \mathrm{cl}\,(SS^{-1})$.

(3) $\implies$ (1): Assuming G is not extremely amenable, there is a minimal action of G on a compact space $X \neq \{*\}$. Let $g \in G$ and $x \in X$ be such that $gx \neq x$. Let $\mathcal{U}_X$ denote the unique compatible uniformity on X. For every $E \in \mathcal{U}_X$, let

$$\tilde{E} = \{(g,h) \in G \times G \colon (gx, hx) \in E\}.$$

The entourages $\tilde{E}$, $E \in \mathcal{U}_X$ form a basis of a uniform structure, $\mathcal{V}$, on G. As a consequence of Lemma 2.1.5, $\mathcal{V}$ is coarser than the right uniformity $\mathcal{U}_R(G)$.

For some $E \in \mathcal{U}_X$ one has $E(x) \cap E(gx) = \emptyset$. Find a symmetric neighbourhood of identity, $\mathcal{O}$, and an $E_1 \in \mathcal{V}$, such that $\mathcal{O}_R \circ E_1 \subseteq E$. (Here

$$\mathcal{O}_R = \{(x,y) \in G \times G \colon xy^{-1} \in \mathcal{O}\}$$

is the standard basic element of the right uniformity on G.) As a consequence,

$$\mathcal{O}E_1(x) \cap \mathcal{O}E_1(gx) = \emptyset.$$

Set $V = E_1(x) \cap g^{-1}E_1(gx)$. This is a neighbourhood of x and $\mathcal{O}V \cap gV = \emptyset$. The (open) set $S = \widetilde{V} \subseteq G$ is big on the left (Prop. 2.6.5), in particular left syndetic, and $\mathcal{O}\widetilde{V} = \widetilde{\mathcal{O}V}$, which implies that $\mathcal{O}S \cap gS = \emptyset$, that is, $g \notin \overline{SS^{-1}}$. $\square$

Recall that an action of a group G on a set X is *effective* if for every $g \in G$, $g \neq e$ the g-motion $x \mapsto gx$ is nontrivial, that is, $gx \neq x$ for some $x \in X$. Equivalently, the associated homomorphism $G \to S_X$ is a monomorphism.

More or less the same arguments establish the following "dual" version of Theorem 2.6.8.

Theorem 2.6.9. *For a topological group G the following are equivalent.*

1. *G acts effectively on a minimal compact space.*
2. *For every $g \in G$, $g \neq e$, there is a set $S \subseteq G$, big on the left, and such that $g \notin \mathrm{cl}\, SS^{-1}$.*
3. *For every $g \in G$, $g \neq e$, there is a left syndetic set $S \subseteq G$ such that $g \notin \mathrm{cl}\, SS^{-1}$.*

□

Recall that a topological group G is *minimally almost periodic* if every continuous finite-dimensional unitary representation of G (a continuous homomorphism into a unitary group $U(n)$ of finite rank $n \in \mathbb{N}_+$) is trivial. An equivalent requirement is that every continuous homomorphism to a compact group is trivial. (Indeed, every compact group admits a separating family of finite-dimensional unitary representations.) It is easy to see that every extremely amenable group is minimally almost periodic. In fact, the following is a corollary of the Veech theorem (2.5.2).

Corollary 2.6.10. *Every continuous homomorphism from an extremely amenable group to a locally compact group has a trivial image,* $\{e\}$. □

Since there exist minimally almost periodic locally compact – and even discrete — groups, such as for example $SL_n(\mathbb{C})$, not every minimally almost periodic group is extremely amenable. However, for abelian topological groups the question remains open. A particularly interesting case is that of *monothetic* topological groups, that is, groups containing an everywhere dense cyclic subgroup.

The following question has been asked by Eli Glasner [59]: *Does there exist a monothetic topological group that is minimally almost periodic but not extremely amenable?*

The significance of this question lies in its link with an open problem in combinatorial number theory. The *Bohr topology* on a topological group G is the finest totally bounded topology coarser than the topology of G. The Bohr topology on a discrete group is simply the finest totally bounded topology. Successive efforts of Bogoliuboff, Følner [48], Cotlar and Ricabarra [34], Ellis and Keynes [44] have shown that if S is a big subset of $\mathbb{Z}$, then the set $S - S + S$ is a Bohr neighbourhood of zero. It is also known that in such a case $S - S$ differs from a Bohr neighbourhood of zero by a set of upper Banach density zero. However, it remains unknown [177] if $S - S$ itself is a Bohr neighbourhood of zero. Cf. also [187].

Suppose the answer to the above Glasner's question is, as many expect it to be, yes, so there is a group topology τ making the group of integers into a minimally almost periodic but non-extremely amenable group. By Theorem 2.6.8, there is a big subset $S \subseteq \mathbb{Z}$ such that the τ-closure of $S - S$ is not all of $\mathbb{Z}$. Then $S - S$ is not a Bohr neighbourhood of zero, in view of the following.

Lemma 2.6.11. *A topological group* (G, τ) *is minimally almost periodic if and only if every Bohr neighbourhood of identity in* $(G, \tau_{discrete})$ *is everywhere dense in* G*.*

Proof. $\Rightarrow$: The collection of all sets of the form $V + U$, where $V \in \tau$ and U is open in the Bohr topology on the discrete group G, forms a basis for a group topology β on G. If there is a U with $\mathrm{cl}_{\tau} U \neq G$, the topology β is not indiscrete. Since β is clearly precompact and coarser than τ, we conclude that (G, τ) is not minimally almost periodic.

$\Leftarrow$: obvious. $\square$

We will see in the next Chapters examples of monothetic extremely amenable groups. There are also very numerous examples of monothetic minimally almost periodic groups known, for many of which we don't know if they are extremely amenable or not, see, for example, [2, 37, 147, 189]. Also, maybe the contruction of Banaszczyk [7, 8] who can obtain extremely amenable groups as factor-groups of locally convex spaces by discrete subgroups can be modified so as to produce a minimally almost periodic group that is not extremely amenable?

Remark 2.6.12. Syndetic sets are very commonly used in dynamics, and Proposition 2.6.5 can be found in every textbook on the subject. Theorem 2.6.8 was proved by the present author in [139], in response to a problem of finding an intrinsic description of extremely amenable groups asked by Granirer in 1967 [69], II. Theorem 2.6.9 and its proof can also be found in [139].

2.7 Invariant means

Definition 2.7.1. Let a group G act on a set X, and let $\mathscr{F}$ be a linear subspace of ℓ^{∞}, containing the function 1. A *mean* on the space $\mathscr{F}$ is a linear functional $\phi\colon \mathscr{F} \to \mathbb{C}$ that is bounded of norm 1, positive, and sending the function 1 to the element 1. A mean ϕ is called *invariant* if it is invariant under the action of G:

$$^g\phi = \phi \text{ for all } g \in G.$$

Here $^g\phi(f) = \phi(^{g^{-1}}f)$.

Every probability measure μ on a compact space X can be identified, by the F. Riesz representation theorem, with a mean on the space $C(X)$. In this context, a Dirac measure (point mass) on X is simply the evaluation functional $\delta_x\colon f \mapsto f(x)$. The space of all probability measures on X, denoted $P(X)$, is thus the closed convex hull of X inside of the dual space $C(X)^*$ equipped with the weak* topology.

The following is thus a direct consequence of Lemma 2.1.6.

Proposition 2.7.2. *If a topological group G acts continuously on a compact space X, this action extends to a continuous action of G on the space $P(X)$ of probability measures on X.* □

Invariant measures correspond to invariant means, while Dirac measures of fixed points in X correspond to invariant *multiplicative* means, that is, those means ϕ with the property

$$\phi(fg) = \phi(f)\phi(g) \text{ for all } f, g \in C(X).$$

The fixed point on compacta property of a topological group G can be now restated as follows.

Theorem 2.7.3. *For a topological group G, the following are equivalent.*

1. *G is extremely amenable.*
2. *There exists a left-invariant multiplicative mean on the space* $\mathrm{RUCB}\,(G)$.
3. *Every compact G-space supports an invariant Dirac measure.*

□

Each of the equivalent conditions above can be naturally relaxed in the following way.

Definition 2.7.4. A topological group G is called *amenable* if every compact G-space admits an invariant probability measure.

The following equivalences are rather straightforward and can be obtained by standard arguments involving such familiar objects as the greatest ambit $\mathcal{S}(G)$, the C^*-algebra $\mathrm{RUCB}\,(G)$, and the space $P(\mathcal{S}(G))$ of all probability measures on the greatest ambit. We leave the proof of the following result as an exercise, given that all the necessary tools are aleady at reader's disposal.

Theorem 2.7.5. *For a topological group G, the following are equivalent.*

1. *G is amenable.*
2. *There exists a left-invariant mean on the space* $\mathrm{RUCB}\,(G)$.
3. *Every continuous action by affine maps on a compact convex subspace of a locally convex space admits a fixed point.*

□

Clearly, extreme amenability implies amenability, but the converse is not true. For example, every abelian topological group is amenable.

For locally compact groups, the list of equivalences is considerably longer. Most of them either make no sense for large groups (such as, for instance, the formulations in terms of the spaces $L^p(G)$), or else are false (for instance, the existence of an invariant mean on the space $\mathrm{CB}\,(G)$ of all continuous bounded functions on G).

For a standard references on amenability of topological groups, see [72], [137].

In this section we will show that some of the well-known infinite-dimensional topological groups are not amenable, thus *ipso facto* not extremely amenable. The following concept provides (among other things) a convenient framework for disproving amenability of large topological groups.

Definition 2.7.6. Let a group G act by uniform isomorphisms on a uniform space X. Say that the action of G is *Eymard–Greenleaf* *amenable,* or that X is an *Eymard–Greenleaf amenable uniform G-space*, if there exists a G-invariant mean on the space $\mathrm{UCB}\,(X)$. Equivalently (by the Riesz representation theorem), there exists an invariant probability measure on the Samuel compactification σX.

The particular case where the acting group G is locally compact and $X = G/H$ is the factor-space equipped with the right uniform structure was thoroughly studied by Eymard [46] and Greenleaf [73].

Remark 2.7.7. In particular, notice that if a uniform G-space X has the concentration property, then it is Eymard–Greenleaf amenable. Interestingly, in some important situations where the phase space X is infinite-dimensional (even if the acting group G is not), the converse is also true. This development also goes back to Milman and Gromov [78], cf. also e.g. the work [141] by the author. We have to leave this topic out, but it will feature prominently in any future re-edition of the present notes, should it ever come to fruition.

The most common way to deduce non-amenability of a locally compact group G is to show that G contains, as a closed subgroup, a copy of the free group F_2 on two generators. Indeed, F_2 is non-amenable (which fact lies at the root of the Banach–Tarski paradox), and amenability is inherited by closed subgroups of locally compact groups. (Though notice that not every non-amenable locally compact, in fact discrete, group contains a copy of F_2.)

However, amenability is not inherited by closed subgroups of infinite-dimensional groups: for instance, the (extremely) amenable group $U(\ell^2)$ with the strong topology contains a closed copy of F_2 (acting by permutations on the coordinate vectors). As a result, showing that a given infinite-dimensional group is non-amenable is slightly more complicated.

A general strategy consists in finding a well-understood locally compact subgroup H of G and a uniform G-space X with the following two properties: (i) amenability of G would imply amenability of X, and (ii) considered as an H-space, X non-amenable.

The forthcoming result (Lemma 2.7.10) provides the main technical tool needed. We need the following notion from theory of topological transformation groups.

Definition 2.7.8. Let a group G act by uniform isomorphisms on a uniform space $X = (X, \mathcal{U}_X)$. The action is called *bounded* [186] (or else *motion equicontinuous* [64]) if for every $U \in \mathcal{U}_X$ there is a neighbourhood of the identity, $V \ni e_G$, such that $(x, g\cdot x) \in U$ for all $g \in V$ and $x \in X$.

Notice that every bounded action is continuous.

Example 2.7.9. Let E be a Banach space. The action of the general linear group $\mathrm{GL}(E)$, equipped with the uniform operator topology (given by the operator norm on $\mathrm{GL}(E) \subset L(E,E)$) on the unit sphere $\mathbb{S}_E$ of a Banach space E is bounded, in fact by the very definition of the uniform operator topology.

For the notion of a uniformly equicontinuous action, see Exercise 1.1.8.

Lemma 2.7.10. *Let a group G act by uniform isomorphisms on a uniform space X. Then there is an equivariant positive linear operator of norm one, $\psi\colon \mathrm{UCB}\,(X) \to \ell^\infty(G)$, sending 1 to 1.*

If G is a topological group and the action on X is continuous, then the image of ψ is contained in $\mathrm{CB}\,(G)$.

If in addition the action of G is uniformly equicontinuous (for instance, an action by isometries on a metric space), then the image of ψ is contained in $\mathrm{LUCB}\,(G)$.

If the action of G on X is bounded, then the image of ψ is contained in $\mathrm{RUCB}\,(G)$.

Proof. Choose a point $x_0 \in X$ and set, for each $f \in \ell^\infty(X)$ and every $g \in X$,
$$\psi(f)(g) := f(gx_0).$$
The function $\psi(f)\colon G \to \mathbb{C}$ so defined is bounded, and clearly the operator $\psi\colon \ell^\infty(X) \to \ell^\infty(G)$ is linear of norm one, positive, and sends the function 1 to 1. For each $h \in G$,
$$\begin{aligned}\psi({}^h f)(g) &= {}^h f(gx_0)\\ &= f(h^{-1}gx_0)\\ &= \psi(f)(h^{-1}g)\\ &= {}^h(\psi(f))(g),\end{aligned}$$
that is, $\psi({}^h f) = {}^h(\psi(f))$ and the operator
$$\psi\colon \mathrm{UCB}\,(X) \ni f \mapsto \tilde{f} \in \mathrm{CB}\,(G)$$

is G-equivariant.

Suppose now G is a topological group continuously acting on X. Every function $\psi(f)$ is continuous, as the composition of two continuous maps: the orbit map $g \mapsto gx_0$ and the function $f: X \to \mathbb{C}$. Thus, $\psi(f) \in \mathrm{CB}\,(G)$.

Let the action of G on X be in addition uniformly equicontinuous. Let $\varepsilon > 0$. Find a $W \in \mathcal{U}_X$ such that $|f(x) - f(y)| < \varepsilon$ whenever $(x,y) \in W$, and further a $U \in \mathcal{U}_X$ with the property that $(x,y) \in U \implies (gx, gy) \in W$ for all $g \in G$. Due to the continuity of the action of G on X, there is a neighbourhood of the identity, V, in G such that, whenever $g \in V$, one has $(x_0, gx_0) \in U$. If now $g,h \in G$ are such that $g^{-1}h \in V$, then $(g^{-1}hx_0, x_0) \in U$ and consequently $(hx_0, gx_0) \in W$, and

$$\begin{aligned} |\psi(f)(g) - \psi(f)(h)| &= |f(gx_0) - f(hx_0)| \\ &< \varepsilon, \end{aligned}$$

that is, the function $\psi(f)$ is left uniformly continuous.

Let now the action of a topological group G on X be bounded. Let $f \in \mathrm{UCB}\,(X)$. By a given $\varepsilon > 0$, choose a $U \in \mathcal{U}_X$ using the uniform continuity of f, and a symmetric neighbourhood $V \ni e_G$ so that (i) $|f(x) - f(y)| < \varepsilon$ whenever $(x,y) \in U$, and (ii) $(x, g\cdot x) \in U$ once $g \in V$ and $x \in X$. If now $g,h \in G$ are such that $gh^{-1} \in V$, one has

$$\begin{aligned} |\psi(f)(g) - \psi(f)(h)| &= |f(gh^{-1}(hx)) - f(hx)| \\ &< \varepsilon. \end{aligned}$$

□

Corollary 2.7.11. *Every continuous action of an amenable locally compact group G on a uniform space X by uniform isomorphisms is amenable.*

Proof. Let $\psi: \mathrm{UCB}\,(X) \to \mathrm{CB}\,(G)$ be an equivariant positive linear operator of norm one, sending 1 to 1, as in Lemma 2.7.10. Since G is amenable and locally compact, there exists a left-invariant mean ϕ on the space $\mathrm{CB}\,(G)$, and the composition $\phi \circ \psi$ is a G-invariant mean on $\mathrm{UCB}\,(X)$. □

This result is no longer true for more general topological groups, cf. Remark 2.7.18 below.

In a similar way, one establishes the following result.

Corollary 2.7.12. *Let a topological group G act in a bounded way on a uniform space X. If G is amenable, then X is an Eymard–Greenleaf amenable uniform G-space.* $\square$

This can be used to deduce non-amenability of the general linear groups of a number of Banach spaces, equipped with the uniform operator topology. Here the uniform space X upon which the groups act, will be the projective space.

Let E be a (complex or real) Banach space. Denote by $\mathbb{P}\mathrm{r}_E$ the projective space of E. If we think of $\mathbb{P}\mathrm{r}_E$ as a factor-space of the unit sphere $\mathbb{S}_E$ of E, then $\mathbb{P}\mathrm{r}_E$ becomes a metric space via the rule

$$d(x,y) = \inf\{\|\xi - \zeta\| : \xi, \zeta \in \mathbb{S}_{\mathcal{H}}, p(\xi) = x, p(\zeta) = y\},$$

where $p\colon \mathbb{S}_E \to \mathbb{P}\mathrm{r}_E$ is the canonical factor-map. Notice that the infimum in the formula above is in fact minimum. The proof of the triangle inequality is based on the invariance of the norm distance on the sphere under multiplication by scalars. The above metric on the projective space is complete.

Let $T \in \mathrm{GL}(E)$ be a bounded linear invertible operator on a Banach space E. Define a mapping $\tilde{T}$ from the projective space $\mathbb{P}\mathrm{r}_E$ to itself as follows: for every $\xi \in \mathbb{S}_E$ set

$$\tilde{T}(p(\xi)) = p\left(\frac{T(\xi)}{\|T(\xi)\|}\right).$$

The above definition is clearly independent on the choice of a representative, ξ, of an element of the projective space $x \in \mathbb{P}\mathrm{r}_E$.

Lemma 2.7.13. *The mapping $\tilde{T}$ is a uniform isomorphism (and even a bi-Lipschitz isomorphism) of the projective space $\mathbb{P}\mathrm{r}_E$.*

Proof. It is enough to show that $\tilde{T}$ is uniformly continuous, because $\widetilde{TS} = \tilde{T}\tilde{S}$ and so $\widetilde{T^{-1}} = \tilde{T}^{-1}$. Let $x, y \in \mathbb{P}\mathrm{r}_E$, and let $\xi, \zeta \in \mathbb{S}_E$ be such that $p(\xi) = x$, $p(\zeta) = y$, and $\|\xi - \zeta\| = d(x, y)$. Both $\|T(\xi)\|$ and $\|T(\zeta)\|$ are bounded below by $\left\|T^{-1}\right\|^{-1}$, and therefore

$$\begin{aligned} d(\tilde{T}(x), \tilde{T}(y)) &\leq \frac{\pi}{2}\left\|T^{-1}\right\| \cdot \|T(\xi) - T(\zeta)\| \\ &\leq \frac{\pi}{2}\left\|T^{-1}\right\| \cdot \|T\| \cdot \|\xi - \zeta\| \\ &= \frac{\pi}{2}\left\|T^{-1}\right\| \cdot \|T\| \, d(x, y). \end{aligned}$$

□

Lemma 2.7.14. *The correspondence*

$$\mathrm{GL}(E) \ni T \mapsto [\tilde{T} \colon \mathbb{P}\mathrm{r}_E \to \mathbb{P}\mathrm{r}_E]$$

determines an action of the general linear group $\mathrm{GL}(E)$ *on the projective space* $\mathbb{P}\mathrm{r}_E$ *by uniform isomorphisms. With respect to the uniform operator topology on* $\mathrm{GL}(E)$*, the action is bounded.*

Proof. The first part of the statement is easy to check using Lemma 2.7.13. As to the second, if $\|T - \mathbb{I}\| < \varepsilon$, then for every $\xi \in \mathbb{S}_E$

$$\begin{aligned} \left\|\tilde{T}(x) - x\right\| &\leq \frac{\pi}{2}\left\|T^{-1}\right\| \cdot \|T(x) - x\| \\ &< \frac{\pi\varepsilon}{2(1-\varepsilon)}. \end{aligned}$$

□

Definition 2.7.15. Say that a representation π of a group G by bounded linear operators in a normed space E is *amenable in the sense of Bekka* if the action of G by isometries on the projective space $\mathbb{P}\mathrm{r}_E$, associated to π as in Lemma 2.7.14, is Eymard–Greenleaf amenable in the sense of Definition 2.7.6. In other words, there exists an invariant mean on the space $\mathrm{UCB}(\mathbb{P}\mathrm{r}_E)$, or, equivalently, an invariant probability measure on the Samuel compactification of the projective space of E.

The following Fig. 2.4 illustrates the concept of an amenable representation π in the case where π is unitary and so one can replace the hard-to-depict projective space with a sphere $\mathbb{S}^\infty$, which is invariant under the action.

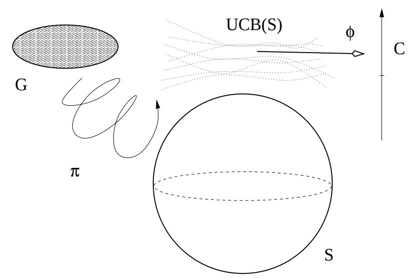

Figure 2.4: To the concept of an amenable unitary representation.

Remark 2.7.16. The original definition by Bekka [12] was also given for the case where π is a unitary representation of a group G in a Hilbert space, but in different terms. Namelly, Bekka calls such a π amenable if there exists a state, ϕ, on the von Neumann algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators on the space $\mathcal{H}$ of representation, which is invariant under the action of G by inner automorphisms: $\phi(\pi(g)T\pi(g)^{-1}) = \phi(T)$ for every $T \in B(\mathcal{H})$ and every $g \in G$.

In the case where π is a unitary representation, Definition 2.7.15 is equivalent to Bekka's concept. This was established by the present author in [141], cf. also [144]. The proof of the fact that Bekka-amenable representation satisfies Def. 2.7.15 relies on some results by Bekka deduced using subtle techniques of Connes, and we will not reproduce it here.

By contrast, if a unitary representation π is amenable in the sense of Def. 2.7.15, it is amenable in the sense of Bekka for the following simple reason. Let ϕ be a G-invariant mean on $\mathrm{UCB}\,(\mathbb{S}_{\mathcal{H}})$. Every bounded linear operator T on $\mathcal{H}$ defines a bounded uniformly continuous (in fact, even Lipschitz) function $f_T\colon \mathbb{S}_{\mathcal{H}} \to \mathbb{C}$ by the rule

$$\mathbb{S}_{\mathcal{H}} \ni \xi \mapsto f_T(\xi) := \langle T\xi, \xi\rangle \in \mathbb{C}.$$

This function is symmetric, and therefore factors through the projective space $\mathbb{P}\mathrm{r}_{\mathcal{H}}$; denote the corresponding function on the projective space by $\tilde{f}_T$. Now set $\phi'(T) := \phi(\tilde{f}_T)$. This ϕ' is a G-invariant state on $\mathcal{B}(\mathcal{H})$.

Example 2.7.17. Let G be a locally compact group, and let μ be a quasi-invariant measure on G. Let $1 \leq p < \infty$. The left quasi-regular representation of G in $L^p(\mu)$ is amenable if and only if G is amenable.

Recall that left quasi-regular representation, γ, of G in $L^p(\mu)$ is given by the formula

$$^g f(x) = \left(\frac{d(\mu \circ g^{-1})}{d\mu} \right)^{\frac{1}{p}} f(g^{-1}x),$$

where $d/d\mu$ is the Radon-Nykodim derivative. It is a strongly continuous representation by isometries. Sufficiency ($\Leftarrow$) thus follows at once from Corollary 2.7.11.

To prove necessity ($\Rightarrow$), assume γ is amenable. Then there exists an invariant mean, ϕ, on $\mathrm{UCB}\,(\mathbb{S}_p)$, where $\mathbb{S}_p$ stands for the unit sphere in $L^p(\mu)$. For every Borel subset $A \subseteq G$, define a function $f_A \colon \mathbb{S}_p \to \mathbb{C}$ by letting for each $\xi \in \mathbb{S}_p$

$$f_A(\xi) = \|\xi \cdot \chi_A\|^p \,,$$

where χ_A is the characteristic function of A. The function f_A is bounded and uniformly continuous on $\mathbb{S}_p$. For every $g \in G$,

$$\begin{aligned} {}^g f_A(\xi) &= f_A\left({}^{g^{-1}}\xi \right) \\ &= \int_A \left| {}^{g^{-1}}\xi(x) \right|^p d\mu(x) \\ &= \int_A \frac{d\mu \circ g}{d\mu} \left|\xi(gx)\right|^p d\mu(x) \\ &= \int_{gA} |\xi(y)|^p \, d\mu(y) \\ &= f_{gA}(\xi), \end{aligned}$$

that is, ${}^g f_A = f_{gA}$. It is now easily seen that $m(A) := \phi(f_A)$ is a finitely additive left-invariant measure on G, vanishing on locally null sets, and so G is amenable.

Remark 2.7.18. In contrast with Proposition 2.7.11, even a strongly continuous unitary representation of an amenable non-locally compact topological group need not be amenable. The simplest example is the standard representation of the full unitary group $U(\mathcal{H})_s$ of an infinite-dimensional Hilbert space, equipped with the strong topology. It is not amenable because it contains, as a subrepresentation, the left regular representation of a free nonabelian group, which is not amenable.

A part of the explanation is that when a topological group G acts continuously by uniform isomorphisms on a uniform space X, the resulting representation of G by isometries in $\mathrm{UCB}\,(X)$ need not be continuous. (This is the case, for instance, in the same example $G = U(\mathcal{H})_s$, $X = \mathbb{S}_{\mathcal{H}}$.) Equivalently, the extension of the action of G to the Samuel (uniform) compactification σX is discontinuous, and therefore one cannot deduce the existence of an invariant measure on σX from the assumed amenability of G.

The following are immediate consequences of Corollary 2.7.12 and the notions introduced above.

Corollary 2.7.19. *Let π be a uniformly continuous representation of a topological group in a Banach space E. If G is amenable, then π is an amenable representation.* □

Corollary 2.7.20. *Let E be a Banach space, and let G be a topological subgroup of $\mathrm{GL}(E)$ (equipped with the uniform operator topology). If H is a subgroup of G and the restriction of the standard representation of $\mathrm{GL}(E)$ in E to H is non-amenable, then G is a non-amenable topological group.* □

Example 2.7.21. The general linear groups $\mathrm{GL}(L^p)$ and $\mathrm{GL}(\ell^p)$, $1 \leq p < \infty$, with the uniform operator topology are non-amenable.

Both spaces L^p and ℓ^p can be realized in the form of $L^p(H,\mu)$, where μ is a quasi-invariant measure on a non-amenable locally compact group H. (For instance, $H = \mathrm{SL}(2,\mathbb{C})$ for the continuous case and $H = \mathrm{SL}(2,\mathbb{Z})$ for the purely atomic one.) Identify H with an (abstract, non-topological) subgroup of $\mathrm{GL}(L^p(\mu))$ via the left quasi-regular representation, γ. The restriction of the standard representation of $\mathrm{GL}(L^p(\mu))$ to H is γ, which is a non-amenable representation (Theorem 2.7.17), and Corollary 2.7.20 applies.

Example 2.7.22. The same result establishes non-amenability of the unitary group $U(\ell^2)$ equipped with the uniform operator topology. This was first proved by Pierre de la Harpe [86].

Compare the above result with the following.

Example 2.7.23. The isometry group $\mathrm{Iso}(\ell^p)$, $1 \leq p < \infty$, $p \neq 2$, equipped with the strong operator topology, is amenable, but not extremely amenable.

The isometry groups in question, as abstract groups, have been described by Banach in his classical 1932 treatise [6] (Chap. XI, §5, pp. 178–179). For $p > 1$, $p \neq 2$ the group $\mathrm{Iso}(\ell^p)$ is isomorphic to the semidirect product of the group of permutations S_∞ and the countable power $U(1)^{\mathbb{N}}$ (in the complex case) or $\{1,-1\}^{\mathbb{Z}}$ (in the real case). Here the group of permutations acts on ℓ^p by permuting coordinates, while the group of sequences of scalars of absolute value one acts by coordinate-wise multiplication. The semidirect product is formed with regard to an obvious action of S_∞ on $U(1)^{\mathbb{N}}$ (in the real case, $\{1,-1\}^{\mathbb{N}}$).

The strong operator topology restricted to the group S_∞ is the standard Polish topology, and restricted to the product group, it is the standard product topology. Thus, $\mathrm{Iso}(\ell^p) \cong S_\infty \ltimes U(1)^{\mathbb{N}}$ (correspondingly, $S_\infty \ltimes \{1,-1\}^{\mathbb{N}}$) is the semidirect product of a Polish group with a compact metric group. Since S_∞ is an amenable topological group, so is $\mathrm{Iso}(\ell^p)$. Since the non-extremely amenable group S_∞ is a topological factor-group of $\mathrm{Iso}(\ell^p)$, the latter group is not extremely amenable either.

Here are the last remaining two corollaries of Lemma 2.7.10.

Corollary 2.7.24. *If a topological group G admits a left-invariant mean on the space* $\mathrm{LUCB}\,(G)$ *of all left uniformly continuous bounded functions on G, then every strongly continuous representation of G by isometries in a Banach space is amenable.* □

If G is a SIN group, that is, the left and the right uniformities on G coincide, then $\mathrm{LUCB}\,(G) = \mathrm{RUCB}\,(G)$ and we obtain the following.

Corollary 2.7.25. *Every strongly continuous representation of an amenable SIN group by isometries in a Banach space is amenable.* □

Remark 2.7.26. In general, the condition that G is SIN cannot be dropped here, unless we presume that G is locally compact. For instance, the infinite unitary group $U(\ell_2)$ with the strong operator topology is (extremely) amenable, while the standard unitary representation of $U(\ell_2)$ is non-amenable.

We will use Corollary 2.7.25 later, in Subsection 3.3.2, in order to show that the group $\mathrm{Aut}\,(X, \mu)$ of measure-preserving transformations of the standard Lebesgue measure space, equipped with the uniform topology, is non-amenable.

Remark 2.7.27. The material of this section is mostly based on papers [141] (Corollaries 2.7.11, 2.7.12, 2.7.19, 2.7.20, Definition 2.7.15, Examples 2.7.17, 2.7.21, 2.7.23) and [57, 58] (Corollaries 2.7.24, 2.7.25).

Chapter 3

Lévy groups

3.1 Unitary group with the strong topology

Analysis of the proof of extreme amenability of the unitary group $U(\ell^2)$ with the strong topology (subsection 2.2.1), based on Theorem 1.2.9, suggests the following notion.

Definition 3.1.1. A topological group G is a *Lévy group* if there is a family of compact subgroups (G_α) of G, directed by inclusion, having an everywhere dense union and such that the normalized Haar measures on G_α concentrate with regard to the right (or left) uniform structure on G. △

Remark 3.1.2. The original definition by Gromov and Milman [78] only applied to metrizable topological groups G, in which case it is equivalent to the existence of an increasing chain of compact subgroups

$$G_1 < G_2 < \ldots < G_n < \ldots,$$

having everywhere dense union in G and such that for some right-invariant compatible metric d on G the groups G_n, equipped with the normalized Haar measures and the restrictions of the metric d, form a Lévy family. In these notes, we will most often deal with Polish Lévy groups.

Theorem 3.1.3. *Every Lévy group is extremely amenable.*

Proof. According to Theorem 2.1.10, it is enough to verify that the group G, equipped with the left uniform structure and the action of G on itself by left translations, is finitely oscillation stable. But this follows from Theorem 1.2.9. $\square$

One can easily deduce that the group $U(\ell^2)$ with the strong operator topology is Lévy, using the following approach.

Let G be a group acting by isometries on a metric space X, and suppose G is equipped with the topology of simple convergence on X. Equivalently, G is a topological subgroup of $\mathrm{Iso}\,(X)$.

For a metric subspace K of X, denote $X^{\hookleftarrow K}$ the set of all isometric embeddings of K into X, equipped with the uniform metric

$$d(i,j) = \sup\{d_X(i(k), j(k))\colon k \in K\}.$$

For example, if $X = \mathbb{S}^\infty$ (with the geodesic distance) and K is a two-element metric space of diameter $\pi/2$, then $X^{\hookleftarrow K} \cong \mathrm{St}_2(\ell^2)$ is the 2-Stiefel manifold.

The group G naturally acts on $X^{\hookleftarrow K}$ by $(g \cdot i)(x) = g \cdot i(x)$, and this action is isometric and continuous.

Now assume in addition that G admits an increasing sequence of compact subgroups G_n with an everywhere dense union.

Lemma 3.1.4. *Under the above assumptions, G is a Lévy group provided, given a neighbourhood of the identity V in G, there is a finite subset $K \subseteq X$ with the properties*

1. *there is an $\varepsilon > 0$ such that every $g \in G$ moving K in $X^{\hookleftarrow K}$ by less than ε is contained in V, and*
2. *the family of orbits $G_n \cdot K$ in $X^{\hookleftarrow K}$, equipped with the push-forward normalised Haar measure and the induced metric, forms a Lévy family.*

Proof. Let $A_n \subseteq G_n$ have the property $\liminf \mu_n(A_n) > 0$, where μ_n stand for the normalised Haar measures on G_n. Let V be a neighbourhood of identity in G. Find an $\varepsilon > 0$ and a K as in the assumptions of the Lemma. Then for every n $VA_n \cap G_n$ contains the inverse image of the ε-neighbourhood of $A_n \cdot K$ under the orbit map $G \ni g \mapsto gK \in X^{\hookleftarrow K}$. Since the values of push-forward measures of the ε-neighbourhoods of $A_n \cdot K$ go to 1, the same is true of the values of $\mu_n(VA_n \cap G_n)$. □

Recall that the strong topology on the unitary group $U(\ell^2)$ is that of pointwise convergence with regard to the action of this group on $\mathbb{S}^\infty$, and moreover that of pointwise convergence on the standard basic vectors alone. As a consequence, it is enough to consider in the above Lemma 3.1.4 the actions of $U(\ell^2)$ on the Stiefel manifolds

$$\mathrm{St}_n(\ell^2) \cong (\mathbb{S}^\infty)^{\hookleftarrow\{e_1,e_2,\dots,e_n\}}.$$

Corollary 3.1.5. *The unitary group $U(\ell^2)$ with the strong operator topology is a Lévy group.* □

Remark 3.1.6. A similar argument establishes in fact that the unitary group $U(\mathcal{H})$ of every infinite-dimensional Hilbert space $\mathcal{H}$, equipped with the strong operator topology, is a Lévy group in the sense of more general Definition 3.1.1.

Remark 3.1.7. It follows from the proof of Lemma 3.1.4, applied to the group $U(\ell^2)$, that any approximating sequence of unitary subgroups $U(n)$ of finite rank, embedded into $U(\ell^2)$ and equipped with normalized Haar measures forms a (normal) Lévy family with regard to the right uniform structure generated by the strong operator topology on $U(\ell^2)$.

A considerably finer result is however true.

Theorem 3.1.8 (Gromov and Milman [78]). *The family of the special unitary groups $SU(n)$ of finite rank, equipped with normalized Haar measure and the Hilbert–Schmidt metric*

$$d_{HS}(u,v) = \left(\sum_{i,j=1}^{n} |u_{ij} - v_{ij}|\right)^{1/2},$$

forms a normal Lévy family. □

The proof is based on theory of isoperimetric inequalities on Riemannian manifolds, and we refer the reader to [123, 76, 102].

Corollary 3.1.9 (Gromov and Milman [78]). *The group $U(\infty)_2$ of all operators of the form $\mathbb{I}+K$, where K is of Schatten class 2, equipped with the Hilbert-Schmidt operator metric, is a Lévy group.* □

3.2 Group of measurable maps

3.2.1 Construction and example

For a topological group G, consider the group $L^0(X,\mu;G)$ of all μ-equivalence classes of measurable maps from the standard separable Polish space X, equipped with a non-atomic probability measure μ, to G.

Remark 3.2.1. To simplify the task, assume G is separable; for more general groups, one needs to tread measurability very carefully indeed. A mapping f from a locally compact space X equipped with a positive measure μ to a topological space Y is called *strongly measurable,* or *measurable in the sense of Bourbaki,* if for every compact subset $K \subseteq X$ and every $\varepsilon > 0$ there is a compact $C \subseteq X$ such that $\mu(K\Delta C) < \varepsilon$ and $f|_C$ is continuous. For instance, if Y is a Banach space, then f is measurable in the sense of Bourbaki if and only if for every continuous linear functional $\phi \in E^*$ the function $\phi \circ f$ is measurable, and for every compact $K \subseteq X$ there is a separable subspace $Z \subseteq Y$ containing $f(x)$ for almost all $x \in K$. (Cf. [52], p. 357, or [22].) In some situations, such generality pays off. However, in our notes we will stay within the second-countable case, and this comment can be safely forgotten. △

Equip the group $L^0(X,\mu;G)$ with *topology of convergence in measure,* where for every neighbourhood of identity V in G and every $\varepsilon > 0$, a standard basic neighbourhood of identity in $L^0(X,\mu;G)$ is of the form

$$[V,\varepsilon] = \{f \in L^0(X,\mu;G)\colon \mu\{x \in X\colon f(x) \notin V\} < \varepsilon\}.$$

Clearly, G is contained within $L^0(X,\mu;G)$ as a closed topological subgroup consisting of all constant maps. This construction was apparently first considered by Hartman and Mysielski [89], whose purpose was to demontrate that an arbitrary topological group G embeds as a topological subgroup into a path-connected (moreover, contractible) topological group.

The following is, historically, the second ever known example of a Lévy group.

Theorem 3.2.2 (Glasner; Furstenberg and Weiss). *Let G be a compact metric group. The group $L^0(X,\mu;G)$ is a Lévy group.*

Remark 3.2.3. The above result was proved many years ago, soon after Gromov and Milman have established the Lévy property of the unitary group $U(\infty)_2$. Glasner has only published it much later [59], while Furstenberg and Weiss have left it unpublished. One can prove, by using the concept of a *generalized Lévy group,* that the group $L^0(X,\mu;G)$ is extremely amenable if G is an amenable locally compact group [142].

We leave the following as an exercise.

Lemma 3.2.4. *Let K be a compact group, equipped with Haar measure μ. For every $1 \leq p < \infty$ the family of L_p-pseudometrics on $L^0(X,\mu;K)$, given by*

$$d_p(f,g) = \left(\int_X d(f(x),g(x))^p \, d\mu(x)\right)^{1/p},$$

induces the topology of convergence in measure as d runs over a family of continuous invariant pseudometrics inducing the topology of K. □

We will be using the L_1-distance d_1.

Consider a refining sequence of partitions of the standard separable non-atomic measure space (X,μ) which is generating the sigma-algebra and such that the n-th partition $\mathcal{P}_n$ consists of sets of measure 2^{-n} each. Denote by K_n the subgroup of $L^0(X,\mu;K)$ consisting of all functions constant on elements of the partition $\mathcal{P}_n$. As an abstract group, K_n is isomorphic to the 2^n-th power K^{2^n}. The restriction of every pseudometric d_1 to K_n is the *Hamming distance* normalized to one: for $x,y \in K_n$, one has

$$d_1(x,y) = \frac{1}{2^n}\sum_{i=1}^{2^n} d(x_i,y_i).$$

By the basic measure theory, the union of the increasing chain of subgroups K_n is everywhere dense in $L^0(X, \mu; K)$.

Before proceeding to the proof of Theorem 3.2.2, let us first derive the following.

Corollary 3.2.5 (Glasner [59]). *The group $L^0(X, \mu; \mathbb{T})$ is a monothetic Lévy group.*

Let us introduce the following *ad hoc* notion.

Definition 3.2.6. We call an abelian topological group G an *ω-torus* if it is topologically generated by the union of countably infinitely many subgroups topologically isomorphic to the circle group $\mathbb{T}$.

For example, the group $L^0(X, \mu; \mathbb{T})$ is an ω-torus.

The following was essentially proved by Rolewicz [153]. Even though he did not state the result in its full generality, the proof is his. The present rendering of the result is taken from the paper by Morris and the author [127].

Theorem 3.2.7. *Every Polish ω-torus is monothetic.*

The proof is based on the following classical result.

Theorem 3.2.8 (Kronecker's Lemma). *If $x_1, \ldots, x_n$ are rationally independent real numbers, then the n-tuple $(x_1', \ldots, x_n')$ made up of their images under the quotient homomorphism $\mathbb{R} \to \mathbb{R}/\mathbb{Z} \cong \mathbb{T}$ to the circle group generates an everywhere dense subgroup in the n-torus $\mathbb{T}^n$.* □

Exercise 3.2.9. Let G be a Polish ω-torus, generated by the subgroups $\mathbb{T}_i \cong \mathbb{T}$, $i \in \mathbb{N}$. Denote by d a translation invariant complete metric, generating the topology on G.

(1) Show that for each $i = 1, 2, \ldots$ one can choose recursively a number $n_i \in \mathbb{N}$ and an element $x_i \in \mathbb{T}_i$ so that:

1. $d(x_i, 0) < 2^{-i}$.
2. The first n_i powers of the product $x_1 \cdot x_2 \cdot \cdots \cdot x_i$ form a 2^{-i}-net in the compact subgroup $\mathbb{T}_1 \cdot \mathbb{T}_2 \cdot \cdots \cdot \mathbb{T}_i$ of G.
3. Whenever $j > i$, the first n_i powers of the element x_j are contained in the d-ball around zero of radius 2^{-j}.

(2) Now show that the element $x = \prod_{l=1}^{\omega} x_l$ is well-defined and forms a topological generator for the group G, thus proving Corollary 3.2.5.

(If necessary, details of the proof can be found in [127].)

To accomplish the proof of Theorem 3.2.2, it remains to establish the following result on concentration of measure in products.

Theorem 3.2.10. *Let (X_i, d_i, μ_i), $i = 1, 2, \dots, n$ be metric spaces with measure, each having finite diameter a_i. Equip the product $X = \prod_{i=1}^{n} X_i$ with the product measure $\otimes_{i=1}^{n}\mu_i$ and the ℓ_1-type metric*

$$d(x, y) = \sum_{i=1}^{n} d_i(x_i, y_i).$$

Then the concentration function of X satisfies

$$\alpha_X(\varepsilon) \leq 2e^{-\varepsilon^2/16 \sum_{i=1}^{n} a_i^2}.$$

This is being done using the martingale technique, which also leads to a number of other interesting examples of Lévy groups. Our presentation in the following subsection 3.2.2 essentially follows [123], though in slightly greater generality.

3.2.2 Martingales

Definition 3.2.11. Let (X, Σ, μ) be a measure space, and let Σ_1 be a sub-sigma-algebra of Σ. For an integrable function f on X, denote by $E(f \mid \Sigma_1)$ the *conditional expectation* of f with regard to Σ_1.

Recall the definition of the conditional expectation. The measure $f \cdot \mu$, restricted to Σ_1, is obviously absolutely continuous with regard to the restriction of μ to Σ_1, that is, every $\mu|_{\Sigma_1}$-null set is a $f \cdot \mu|_{\Sigma_1}$-null set. By the Radon–Nikodym theorem, there is a Σ_1-measurable function g such that $f \cdot \mu|_{\Sigma_1} = g \cdot (\mu|_{\Sigma_1})$. This g is exactly the conditional expectation $E(f \mid \Sigma_1)$.

Example 3.2.12. An important case is where Σ_1 is generated by the elements of a finite measurable partition Ω of X, that is, Σ_1 is the smallest sub-sigma-algebra of Σ containing Ω as a subset. Here, on each element $A \in \Omega$ the function $E(f \mid \Omega)$ takes a constant value

$$\int_A f(x)\ d\mu(x).$$

For instance, the conditional expectation of f with respect to the trivial partition of X is a constant function assuming at all points the value $E(f)$. This is why we will identify:

$$E(f \mid \{X\}) = E(f).$$

If X is finite and Ω_f is the finest partition of X (the one with singletons), then

$$E(f \mid \Omega_f(X)) = f.$$

The following properties of conditional expectation are easy to verify.

Proposition 3.2.13. *Let f and g be real-valued functions on a measure space (X, Σ, μ), and let Σ_1 be a sub-sigma-algebra of Σ.*

1. *If $f \leq g$, then $E(f \mid \Sigma_1) \leq E(g \mid \Sigma_1)$.*
2. *If $g = E(g \mid \Sigma_1)$ (that is, g is Σ_1-measurable), then $E(gf \mid \Sigma_1) = gE(f \mid \Sigma_1)$.*
3. *In particular, $E(\lambda f \mid \Sigma_1) = \lambda E(f \mid \Sigma_1)$.*
4. *$E(f + g \mid \Sigma_1) = E(f \mid \Sigma_1) + E(g \mid \Sigma_1)$.*

□

Definition 3.2.14. Let

$$\{\emptyset, X\} = \Sigma_0 \subseteq \Sigma_1 \subseteq \ldots \subseteq \Sigma_n = \Sigma \tag{3.1}$$

be an increasing sequence (*filtration*) of sub-sigma-algebras of Σ. A *martingale* (formed with respect to the above filtration) is a collection of measurable functions $(f_0, f_1, \ldots, f_n)$ on X such that for every $i = 1, 2, \ldots, n$

$$E(f_i \mid \Sigma_{i-1}) = f_{i-1}.$$

We have, by obvious finite induction:

Lemma 3.2.15. *Under the assumptions of Def. 3.2.14, whenever $i \leq j$,*

$$E(f_j \mid \Sigma_i) = f_i.$$

This immediately leads to the following result.

Proposition 3.2.16. *If f is a function on X, then*

$$f_i = E(f \mid \Sigma_i), \quad i = 0, 1, \ldots, n$$

defines a martingale on X. Moreover, every martingale is obtained in this form from $f = f_n$. □

Example 3.2.17. Denote by $\Sigma^n = \{0,1\}^n$ the *Hamming cube,* equipped with the normalized counting measure and the normalized Hamming distance

$$d(x,y) = \frac{1}{n}|\{i \colon x_i \neq y_i\}|$$

The *k-th standard partition* Ω_k of the Hamming cube Σ^n consists of all sets

$$A_\tau = \{\sigma \in \Sigma^n \mid \pi^n_k(\sigma) = \tau\},$$

where $\tau \in \Sigma^k$. In other words, elements of the partition Ω_k are sets of all strings whose first k bits are the same. Easy calculations show that for every $k = 0, 1, \ldots, n$

$$E(\bar{w} \mid \Omega_k)(\sigma) = \bar{w}_k(\pi^n_k(\sigma)) + \frac{n-k}{2}.$$

Here $\bar{w}_k$ is the *normalized Hamming weight* on the cube Σ^k:

$$\bar{w}_k(x) = \frac{x_1 + x_2 + \ldots + x_k}{k}.$$

Definition 3.2.18. Let $(f_0, f_1, \ldots, f_n)$ be a martingale. The *martingale differences* are the functions

$$d_i = f_i - f_{i-1}.$$

Remark 3.2.19. Let f be a function on a measure space X equipped with a filtration of sigma-algebras. Form the martingale as in Prop. 3.2.16. Then

$$f = E(f) + d_1 + d_2 + \ldots + d_{n-1} + d_n.$$

Proposition 3.2.20. *The martingale differences satisfy the property*

$$E(d_i \mid \Sigma_{i-1}) = 0.$$

Proof. $E(d_i \mid \Sigma_{i-1}) = E(f_i \mid \Sigma_{i-1}) - E(f_{i-1} \mid \Sigma_{i-1}) = f_{i-1} - f_{i-1}$. □

The following inequality is at the heart of martingale techniques.

Proposition 3.2.21. *For every $x \in \mathbb{R}$,*

$$e^x \leq x + e^{x^2}. \tag{3.2}$$

Proof. (Communicated to me by Mike Doherty.) The function

$$f(x) = e^{x^2} + x - e^x$$

is infinitely smooth, and by the Taylor formula, at any point $x \in \mathbb{R}$, for some θ between 0 and x,

$$\begin{aligned} f(x) &= f(0) + f'(0)x + \frac{1}{2} f''(\theta)x^2 \\ &= \frac{1}{2} f''(\theta)x^2, \end{aligned} \tag{3.3}$$

as $f(0) = f'(0) = 0$. The second derivative,

$$f''(x) = 2(1 + 2x^2)e^{x^2} - e^x,$$

is always positive. Indeed,

$$\begin{aligned} f''(x) > 0 \quad &\Leftrightarrow \quad 2(1 + 2x^2)e^{x^2 - x} > 1 \\ &\Leftrightarrow \quad 2(1 + 2x^2)e^{(x - \frac{1}{2})^2 - \frac{1}{4}} > 1, \end{aligned}$$

and

$$\begin{aligned} 2(1+2x^2)e^{(x-\frac{1}{2})^2-\frac{1}{4}} &\geq 2e^{-\frac{1}{4}} \\ &\geq 2\left(1-\frac{1}{4}\right) \\ &> 1, \end{aligned}$$

as $e^{-\frac{1}{4}} > 1-\frac{1}{4}$ by an application of Taylor's theorem.

It follows from (3.3) that $f(x) \geq 0$, as required. □

We will also need the following fact, easy but useful.

Lemma 3.2.22 (Chebyshev's inequality). *Let f be a measurable real-valued function on a measure space X taking non-negative values only. Then*

$$\mu\{x \in X \mid f(x) \geq 1\} \leq E(f).$$

□

Here comes the main technical result of the present Section, evaluating the probability that the value of the function f on a measure space at a random point $x \in X$ deviates from the expected value of f by more than a given constant $c > 0$.

Lemma 3.2.23 (Azema's inequality). *Let $f\colon X \to \mathbb{R}$ be an intergrable function on a measure space X, equipped with a filtration of sub-sigma-algebras as in Eq. (3.1), and let $(f_0, f_1, \ldots, f_n)$ be the corresponding martingale obtained from f as in 3.2.16, with martingale differences $(d_1, d_2, \ldots, d_n)$. Then for every $c > 0$*

$$\mu(\{x \in X \mid |f(x) - E(f)| \geq c\} \leq 2\exp\left(-\frac{c^2}{4\sum_{i=1}^n \|d_i\|_\infty^2}\right).$$

Proof. Let $\lambda > 0$ be arbitrary. According to Lemma 3.2 (applied with $x = \lambda d_i$), one has for every $i = 1, 2, \ldots, n$ μ-a.e.

$$e^{\lambda d_i} \leq \lambda d_i + e^{\lambda^2 d_i^2}, \tag{3.4}$$

and hence (Proposition 3.2.13.(1))

$$\begin{aligned}
E(e^{\lambda d_i} \mid \Sigma_{i-1}) &\leq E(\lambda d_i \mid \Sigma_{i-1}) + E(e^{\lambda^2 d_i^2} \mid \Sigma_{i-1}) \\
\text{(by Prop. 3.2.20)} \quad &= E(e^{\lambda^2 d_i^2} \mid \Sigma_{i-1}) \\
&\leq \left\| e^{\lambda^2 d_i^2} \right\|_\infty \\
&\leq e^{\lambda^2 \|d_i\|_\infty^2}. \qquad (3.5)
\end{aligned}$$

For each $i = 1, 2, \ldots, n$ one gets, using Lemma 3.2.15,

$$\begin{aligned}
E\left(e^{\lambda \sum_{j=1}^{i} d_j}\right) &= E\left(E\left(e^{\lambda \sum_{j=1}^{i} d_j} \mid \Sigma_{i-1}\right)\right) \\
\text{(by Prop. 3.2.13.(2))} \quad &= E\left(\left(e^{\lambda \sum_{j=1}^{i-1} d_j}\right) E\left(e^{\lambda d_i} \mid \Sigma_{i-1}\right)\right) \\
\text{(by inequality (3.5))} \quad &\leq E\left(e^{\lambda \sum_{j=1}^{i-1} d_j}\right) e^{\lambda^2 \|d_i\|_\infty^2}. \qquad (3.6)
\end{aligned}$$

Taking into account Remark 3.2.19, observe that

$$\begin{aligned}
\mu\{\sigma \mid f(\sigma) - E(f) \geq c\} &= \mu\left\{\sigma \mid \sum_{i=1}^{n} d_i(\sigma) \geq c\right\} \\
&= \mu\left\{\sigma \mid \lambda \sum_{i=1}^{n} d_i(\sigma) - \lambda c \geq 0\right\} \\
&= \mu\left\{\sigma \mid e^{\lambda \sum_{i=1}^{n} d_i(\sigma) - \lambda c} \geq 1\right\} \\
\text{(by Lemma 3.2.22)} \quad &\leq E\left(e^{\lambda \sum_{i=1}^{n} d_i - \lambda c}\right) \\
&= E\left(e^{\lambda d_1} e^{\lambda d_2} \ldots e^{\lambda d_{n-1}} e^{\lambda d_n}\right) e^{-\lambda c} \\
\text{(by inequality (3.6))} \quad &\leq E\left(e^{\lambda d_1} e^{\lambda d_2} \ldots e^{\lambda d_{n-1}}\right) e^{\lambda^2 \|d_n\|_\infty^2} e^{-\lambda c} \\
\text{(applying (3.6)) } n \text{ times)} \quad &\leq \ldots \\
&\leq e^{\lambda^2 \|d_1\|_\infty^2} e^{\lambda^2 \|d_2\|_\infty^2} \cdots e^{\lambda^2 \|d_n\|_\infty^2} e^{-\lambda c} \\
&= e^{\lambda^2 \sum_{j=1}^{n} \|d_j\|_\infty^2 - \lambda c}. \qquad (3.7)
\end{aligned}$$

We substitute into (3.7) the value

$$\lambda = \frac{c}{2 \sum_{j=1}^{n} \|d_j\|_\infty^2}$$

to conclude that

$$\mu\{\sigma\colon f(\sigma) - E(f) \geq c\} \leq e^{-\frac{c^2}{4\sum_{j=1}^n \|d_j\|_\infty^2}}. \tag{3.8}$$

Repeating everything all over again with only very minor adjustments, one obtains the twin inequality

$$\mu\{\sigma\colon E(f) - f(\sigma) \geq c\} \leq e^{-\frac{c^2}{4\sum_{j=1}^n \|d_j\|_\infty^2}}. \tag{3.9}$$

Combined together, the equations (3.8) and (3.9) obviously imply

$$\mu\{\sigma\colon |E(f) - f(\sigma)| \geq c\} \leq 2e^{-\frac{c^2}{4\sum_{j=1}^n \|d_j\|_\infty^2}}, \tag{3.10}$$

as desired. □

Before giving our next concept, that of the length of a metric space with measure, let us remind a classical result [152].

Theorem 3.2.24 (Rokhlin). *Let Ω be a partition of a probability measure space (X, Σ, μ) by measurable sets, such that the quotient space X/Ω is separated (up to a set of measure zero) by a countable number of measurable sets. Then for almost all $x \in X$ the equivalence class $\Omega(x)$ containing x supports a probability measure μ_x (called* conditional measure*) with the following property. Denote by $\mathscr{B}_\Omega$ the sub-sigma-algebra of Σ whose elements are Ω-saturated. Then for every measurable set $A \subseteq X$ the function*

$$X \ni x \mapsto \mu_x(A \cap \Omega(x)) \in \mathbb{R}$$

is $\mathscr{B}_\Omega$-measurable, and

$$\mu(A) = \int \mu_x(A \cap \Omega(x))\ d\mu(x).$$

□

Definition 3.2.25. Under an *isomorphism of mm-spaces* we mean a measure-preserving isometry between subsets of full measure.

Definition 3.2.26. Let (X, μ, d) be a metric space with measure admitting a refining sequence of separated measurable partitions

$$\Omega_0 = \{X\} \prec \Omega_1 \prec \ldots \prec \Omega_n = \{\{x\} \colon x \in X\}.$$

Suppose for every $i = 1, 2, \ldots, n$ there is a number $a_i \geq 0$ such that for μ-a.e. $x \in X$ and μ_x-a.e. $y, z \in \Omega_{i-1}(x)$ there is an isomorphism ϕ of mm-spaces between $\Omega_i(y)$ and $\Omega_i(z)$ such that for all $a \in \Omega_i(y)$,

$$d(a, \phi(a)) \leq a_i.$$

Then we say that the *length* of X is $\leq \left(\sum_{i=1}^n a_i^2\right)^{1/2}$. Accordingly, we define the length of X, denoted $\ell(X)$, as the infimum of all the expressions as above.

Define the *essential oscillation* of a measurable function f on a measure space (X, μ) as

$$\text{ess osc}(f|X) = \text{ess sup}_X(f) - \text{ess inf}_X(f).$$

Lemma 3.2.27. *Let $(X, d.\mu)$ be an mm-space of length $\leq \ell$, and let $f \colon X \to \mathbb{R}$ be a 1-Lipschitz function. Then for every $\varepsilon > 0$*

$$\mu\{x \colon |f(x) - E(f)| \geq \varepsilon\} \leq 2e^{-\frac{\varepsilon^2}{4\ell^2}}.$$

Proof. Choose a sequence of partitions Ω_i as in Definition 3.2.26, and let $\mathscr{B}_i$ denote the sigma-algebra formed by all Borel sets which are unions of elements of Ω_i. Those sigma-algebras form a filtration of X. Let $(f_i)_{i=0}^n$ be the corresponding martingale determined by the function f (Prop. 3.2.16). In order to use the Azema inequality (Lemma 3.2.23), one needs to prove that $\|d_i\|_\infty \leq a_i$. It is enough to verify that for μ-a.e. $x \in X$, the essential oscilation of f_i on the equivalence class $\Omega_{i-1}(x)$ does not exceed a_i. Let $x, y \in \Omega_{i-1}(x)$. Denote $A = \Omega_i(y)$, $B = \Omega_i(z)$. There is, μ_x-almost surely, an isomorphism of mm-spaces $\phi \colon A \to B$ with the property $d(x, \phi(x)) \leq a_i$ for all $x \in A$. Denote by μ_A, μ_B the conditional measures on A, B

respectively. One has

$$\begin{aligned} f_i(x) - f_i(y) &= \int_A f(x)\ d\mu_A(x) - \int_B f(x)\ d\mu_B(x) \\ &= \int_A (f(x) - f(\phi(x)))\ d\mu_A(x) \\ &\leq a_i. \end{aligned}$$

□

Theorem 3.2.28. *If a metric space with measure (X, d, μ) has length $\leq \ell$, then the concentration function of X satisfies*

$$\alpha_X(\varepsilon) \leq 2e^{-\varepsilon^2/16\ell^2}. \tag{3.11}$$

Proof. Let $A \subseteq X$ be such that $\mu(A) \geq 1/2$. The distance function, d_A, from A,

$$X \ni x \mapsto d_A(x) \equiv d(x, A) \in \mathbb{R},$$

is 1-Lipschitz. By Lemma 3.2.27, for every $\varepsilon > 0$ one has

$$\mu\{x\colon |d_A(x) - E(d_A)| \geq \varepsilon\} \leq 2e^{-\varepsilon^2/4\ell^2}. \tag{3.12}$$

Case 1. $E(d_A) \leq \varepsilon$.

If now $|d_A(x) - E(d_A)| < \varepsilon$, then $d_A(x) < 2\varepsilon$, so that

$$\{x \in X\colon d_A(x) > 2\varepsilon\} \subseteq \{x \in X\colon |d_A(x) - E(d_A)| \geq \varepsilon\},$$

and

$$\begin{aligned} \mu\{x \in X\colon d_A(x) > 2\varepsilon\} &\leq \mu\{x \in X\colon |d_A(x) - E(d_A)| \geq \varepsilon\} \\ &\leq 2e^{-\varepsilon^2/4\ell^2}. \end{aligned} \tag{3.13}$$

Now the desired inequality (3.11) follows by substituting into Eq. (3.13) $\varepsilon/2$ in place of ε.

Case 2. $E(d_A) \geq \varepsilon$.

In this case, for each $x \in A$ one has $|d_A(x) - E(d_A)| = E(d_A) \geq \varepsilon$, and in view of Eq. (3.12),

$$\frac{1}{2} \leq \mu(A) \leq 2e^{-\varepsilon^2/4\ell^2}$$

and therefore

$$1 - \mu(A_\varepsilon) \le \frac{1}{2} \le 2e^{-\varepsilon^2/4\ell^2},$$

and the formula (3.11) also follows. $\square$

Proof of Theorem 3.2.10. For each $i = 1, 2, \dots, n$ let $\pi_i \colon X \to \prod_{j=i}^n X_j$ be the canonical projection, and denote by Ω_i the partition of X with the sets $\pi_i^{-1}(x)$, $x \in \prod_{j=i}^n X_j$. Every such set, endowed with the conditional measure and induced metric, is isomorphic to the ℓ_1-type product $\prod_{j=1}^{i-1} X_j$, and the length of the product is estimated easily: it satisfies

$$\ell(X) \le \left(\sum_{i=1}^n a_i^2 \right)^{1/2}.$$

Theorem 3.2.28 finishes the proof. $\square$

However, the range of applications of Theorem 3.2.28 is wider than just the result on concentration of measure in products (Theorem 3.2.10), and we will point out at some applications in the following Sections.

3.2.3 Unitary groups of operator algebras

The two examples of Lévy groups treated in Sections 3.1 and 3.2, that is, the unitary group $U(\ell^2)$ with the strong topology and the group of measurable maps $L^0(X,\mu;\mathbb{T})$ with the topology of convergence in measure, can be viewed as two extreme cases in a family of Lévy groups coming from theory of operator algebras: they are both unitary groups of approximately finite dimensional von Neumann algebras equipped with the ultraweak topology. For the unitary group $U(\ell^2)$, the corresponding von Neumann algebra is $\mathcal{L}(\ell^2)$, the algebra of all bounded linear operators on ℓ^2. For the group of maps $L^0(X,\mu;\mathbb{T})$, the corresponding von Neumann algebra is $L^\infty(X,\mu)$, the algebra of all essentially bounded measurable complex-valued functions on the standard Lebesgue space.

There is a class of von Neumann algebras, the so-called approximately finite-dimensional (AFD) ones, including both of the above examples and for which one can prove that the unitary group is the product of an extremely amenable group and a compact group. In particular, if an AFD algebra has no finite-dimensional factors, its unitary group is extremely amenable.

We will just outline the concepts, proofs, and results in this Section.

For a crash course in operator algebras, we refer the reader, for example, to Chapter V in Connes' *Noncommutative Geometry* [32]. For a more detailed treatment, we recommend for instance the books by Sakai [155] and Takesaki [161].

Recall that a von Neumann algebra M is a unital C^*-algebra which, regarded as a Banach space, is a dual space. In this case, the predual M_* of M turns out to be unique, and the $\sigma(M, M_*)$-topology on M is called the *ultraweak topology.* The *unitary group* of M is given by the condition

$$U(M) = \{u \in M \colon uu^* = u^*u = 1\}.$$

Equipped with the ultraweak topology, the group $U(M)$ is a complete (in the two-sided uniformity) topological group, which is Polish if and only if M has separable predual.

For example, the predual of $\mathcal{L}(\mathcal{H})$ is the space of all trace class operators:

$$\mathcal{L}(\mathcal{H})_* = \{T \in \mathcal{L}(\mathcal{H}) \colon \mathrm{tr}\,|T| < \infty\},$$

where

$$|T| = (T^*T)^{1/2}$$

and

$$\mathrm{tr}\,(T) = \sum_{\xi} \langle T\xi, \xi\rangle,$$

where the summation is taken over all elements of an orthonormal basis in $\mathcal{H}$. (The choise of basis is irrelevant.) The Banach space duality between $\mathcal{L}(\mathcal{H})$ and its predual is established by the pairing

$$\langle T, S\rangle = \mathrm{tr}\,(TS).$$

It can be shown that the restriction of the ultraweak topology to the unitary group $U(\ell^2) = U(\mathcal{L}(\mathcal{H}))$ coincides with the strong operator topology.

The predual of $L^\infty(X,\mu)$ is the Banach space $L^1(X,\mu)$. The unitary group of the algebra $L^\infty(X,\mu)$ is the familiar group of all (equivalence classes of) measurable maps, $L^0(X,\mu;\mathbb{T})$.

Exercise 3.2.29. Prove that the restriction of the ultraweak topology to the unitary group $L^0(X,\mu;\mathbb{T})$ coincides with the topology of convergence in measure.

(*Hint:* either use the uniform convexity of the unit circle and establish the result directly, or else employ Banach's Open Mapping Theorem: a continuous homomorphism onto between two Polish topological groups is always open, cf. e.g. [95], Cor. 3, p. 98 and Th. 3, p. 90.)

Alternatively, von Neumann algebras can be characterized as those C^*-algebras isomorphic to the ultraweakly closed sub-C^*-algebras of $\mathcal{L}(\mathcal{H})$, or else — which statement forms the subject of the von Neumann's bicommutant theorem — to those sub-C^*-algebras A of $\mathcal{L}(\mathcal{H})$ coinciding with its own double commutant:

$$A'' = A,$$

where for a subset $X \in \mathcal{L}(\mathcal{H})$

$$X' = \{T \in \mathcal{L}(\mathcal{H})\colon \forall S \in X, \quad ST = TS\}.$$

A von Neumann algebra M acting on a Hilbert space $\mathcal{H}$ is called *injective* if there is a projection of norm 1 from the Banach space $\mathcal{L}(\mathcal{H})$ onto M regarded as a Banach subspace. This property does not depend on the space $\mathcal{H}$ on which M acts (that is, on a realization of M as a von Neumann subalgebra of $\mathcal{L}(\mathcal{H})$), and is in fact indeed equivalent to the injectivity of M in the usual category-theoretic sense in a suitable category, namely that of von Neumann algebras and the so-called completely positive mappings. Two examples of injective von Neumann algebras are $\mathcal{L}(\mathcal{H})$ and $L^\infty(X,\mu)$. Every finite-dimensional matrix algebra is injective as well.

A von Neumann algebra with a separable predual is called *hyperfinite* if it is generated, as a von Neumann algebra, by an increasing sequence of finite-dimensional subalgebras. By a celebrated result of Connes [31], injectivity and hyperfiniteness (as well as a host of other properties) are all equivalent between themselves for von Neumann algebras with separable predual.

Here is the main result whose proof we want to briefly outline in this Subsection.

Theorem 3.2.30 (Giordano and Pestov [58]). *Let M be a von Neumann algebra with separable predual. The following conditions are equivalent.*

1. *M is injective.*
2. *The unitary group $U(M)$ with the ultraweak topology is the product of a compact group and a Lévy group.*

The implication (2) $\implies$ (1) is known since the work of de la Harpe [87], who has shown that the amenability of the unitary group $U(M)$ with the ultraweak topology implies the so-called Schwartz property (P), which in turn is equivalent to the injectivity of M by Connes' result.

The proof of the implication (1) $\implies$ (2) uses methods already developed in the present notes, and is based on the reduction theory for von Neumann algebras.

A von Neumann algebra M is called a *factor* if the centre of M is trivial, that is, consists of scalar multiples of 1. We recall the following result of von Neumann. Let $\mathfrak{F}$ be the set of all factors contained in $\mathcal{L}(\mathcal{H})$. This set can be given a natural Polish topology: for instance, such is the Vietoris topology on the set of all intersections of factors with the unit ball of $\mathcal{L}(\mathcal{H})$ equipped with the (compact metrizable) ultraweak topology. In this way, $\mathfrak{F}$ supports the structure of a standard Borel space. Let now

$$X \ni t \mapsto M(t) \in \mathfrak{F}$$

be a Borel mapping from a standard Borel space X equipped with a probability measure μ to the Borel space $\mathfrak{F}$ of factors. The *direct integral* of the family of factors $M(t)$, $t \in X$ is the set of equivalence classes of all essentially bounded Borel sections

$$X \ni t \mapsto x(t) \in M(t),$$

where two sections are identified if they coincide μ-a.e., and are equipped with the pointwise operations and the norm

$$\|x\| = \operatorname{ess\,sup}_{t\in X} \|x(t)\|_{M_t}.$$

The direct integral is denoted

$$M = \int_X M(t)\ d\mu(t).$$

Theorem 3.2.31 (von Neumann). *Every von Neumann algebra M acting on a separable Hilbert space is isomorphic to a direct integral of a family of factors $M(t)$, $t \in X$.*

Moreover, it is known that M is injective if and only if the factor $M(t)$ is injective for μ-a.e. t.

The unitary group of the direct integral can be characterized as the direct integral of the unitary groups of factors $M(t)$, equipped with the suitably defined topology of convergence in measure.

If the continuous part X_c of (X, μ) is non-empty, then the direct sub-integral taken over X_c results in a Lévy group no matter what the AFD factors $M(t)$ are. The central observation here is that, using an approximating sequence, (A_n), of finite-dimensional subalgebras of M, one can choose approximating sequences of finite-dimensional subalgebras in the factors $M(t)$ for μ-a.e. t in a measurable fashion, and then use simple functions with values in finite-dimensional unitary groups in a way modelling Theorem 3.2.2 in order to show that the unitary group of $\int_{X_c} M(t)\ d\mu(t)$ is Lévy.

Every atom $\{t\}$ in the index space (X, μ), to which a finite-dimensional factor is associated, results in a compact group factor of $U(M)$.

Therefore, it only remains to verify the Lévy property of the unitary group $U(M)$ of infinite-dimensional factors corresponding to atoms. In this case, the unitary groups of approximating finite-dimensional subalgebras can be shown to always form a Lévy family.

The proof of Theorem 3.2.30 is concluded by observing that a countable direct product of Lévy groups is again a Lévy group.

We refer to the paper [58] for a detailed (though slightly different) proof, with all the exact references.

3.3 Groups of transformations of measure spaces

3.3.1 Maurey's theorem

As an application of the martingale technique (Theorem 3.2.28), we obtain the following result [108].

Theorem 3.3.1 (Maurey). *Symmetric groups S_n of rank n, equipped with the normalized Hamming distance*

$$d_{ham}(\sigma,\tau) := \frac{1}{n}|\{i\colon \sigma(i)\neq\tau(i)|$$

and the normalized counting measure

$$\mu(A) := \frac{|A|}{n!}$$

form a normal Lévy family, with the concentration functions satisfying the estimate

$$\alpha_{S_n}(\varepsilon)\leq \exp(-\varepsilon^2 n/32). \tag{3.14}$$

To better understand the proof, it makes sense to state the result in a more general way.

Lemma 3.3.2. *Let H be a closed subgroup of a compact group G, equipped with a bi-invariant metric d. The formula*

$$\bar{d}(xH,yH) = \inf_{h_1,h_2\in H} d(xh_1,yh_2)$$

defines a left-invariant metric on the factor-space G/H.

Proof. Since a bi-invariant metric satisfies $d(a,b)=d(a^{-1},b^{-1})$, one has

$$\begin{aligned}\bar{d}(xH,yH) &= \inf_{h_1,h_2\in H} d(h_1x,h_2y)\\ &= \inf_{h\in H} d(hx,y),\end{aligned}$$

and the triangle inequality follows from the fact that, for all $h' \in H$,

$$\begin{aligned}
\bar{d}(xH, yH) &= \inf_{h\in H} d(hx, y) \\
&\leq \inf_{h\in H} [d(hx, h'z) + d(h'z, y)] \\
&= \inf_{h\in H} d(hx, h'z) + d(h'z, y) \\
&= \inf_{h\in H} d(h'^{-1}hx, z) + d(h'z, y) \\
&= \bar{d}(xH, zH) + d(h'z, y),
\end{aligned}$$

where the infimum of the r.h.s. taken over all $h' \in H$ equals $\bar{d}(xH, zH)+\bar{d}(zH, yH)$. The fact that $\bar{d}$ is a metric, follows from compactness, as every two cosets are at a positive distance from each other. Left-invariance of $\bar{d}$ (with regard to the standard action of G on G/H) is obvious. □

Theorem 3.3.3. *Let G be a compact group, metrized by a bi-invariant metric d, and let*

$$\{e\} = G_0 < G_1 < G_2 < \ldots < G_n = G$$

be a chain of subgroups. Denote by a_i the diameter of G_i/G_{i-1}, $i = 1, 2, \ldots, n$, with regard to the factor-metric. Then the concentration function of the mm-space (G, d, μ), where μ is the normalized Haar measure, satisfies

$$\alpha_G(\varepsilon) \leq 2\exp\left(-\frac{\varepsilon^2}{16\sum_{i=0}^{n-1} {a_i}^2}\right).$$

Proof. Let Ω_i be the measurable partition of G into the left G_i-cosets. If two cosets xG_i, yG_i are contained in the coset zG_{i-1}, the mapping

$$\iota\colon xG_i \ni xh \mapsto yh \in yG_i$$

is an isomorphism between mm-spaces (as the conditional measure on each coset coincides with the translate of the normalized Haar measure), with the property $d(xh, yh) = d(x, y) \leq a_i$. The proof is finished by applying, as before, the concept of the length of an mm-space and Theorem 3.2.28. □

Proof of Maurey's theorem 3.3.1. The chain of subgroups

$$S_0 < S_1 < S_2 < \ldots < S_n,$$

canonically embedded each one into the next via

$$S_i \ni \sigma \mapsto \begin{pmatrix} 1 & 2 & 3 & \ldots & i & i+1 \\ \sigma_1 & \sigma_2 & \sigma_3 & \ldots & \sigma_i & i+1 \end{pmatrix} \in S_{i+1},$$

satisfies the assumptions of Theorem 3.3.3, where the value of a_i is easily estimated as

$$\operatorname{diam} S_i/S_{i-1} = \frac{2}{n}.$$

□

3.3.2 Group of measure-preserving transformations

As a consequence of Maurey's theorem, every topological group approximated from within by an increasing sequence of the symmetric groups S_n of finite rank, is a Lévy group, as long as there is a left-invariant metric on G whose restrictions to S_n are *uniformly* equivalent to the Hamming distance (that is, the Lipschitz constants establishing the equivalence are the same for all n).

The most obvious candidate is the infinite symmetric group S_∞ with its standard Polish topology. However, we know (Subsection 2.2.3) that S_∞ is not extremely amenable, and therefore it cannot be a Lévy group.

The reason is that there is no compatible left-invariant metric on S_∞ whose restriction on S_n would coincide with the normalized Hamming distance.

Exercise 3.3.4. Show directly that the groups S_n under the standard embedding $S_n < S_\infty$ do not concentrate with regard to the normalized Haar measures and the left uniformity on S_∞. In fact, for every neighbourhood of identity V in S_∞ one can produce explicitely a sequence of subsets $A_n \subset S_n$ each containing at least half of the points, such that the normalized counting measures of $VA_n \cap S_n$ in S_n remain uniformly bounded away from 1 as $n \to \infty$.

In spite of this fact, Maurey's theorem is not wasted on infinite-dimensional groups, and a Lévy group which is approximated by finite symmetric groups "in the right way" does exist. This is the group $\mathrm{Aut}\,(X,\mu)$ of all measure-preserving transformations of the standard non-atomic probability space, equipped with the weak topology.

It will be convenient to describe at the same time a somewhat larger group, which will appear in the next Section. This is the group $\mathrm{Aut}^*(X,\mu)$, consising of all non-singular (measure class preserving) transformations of (X,μ). In other words, it consists of all transformations τ taking null-sets to null-sets, and whose inverses also have this property.

Exercise 3.3.5. Show that if the measure space (X,μ) is complete (that is, the sigma-algebra includes all Lebesgue measurable sets rather than just Borel sets), then every bi-measurable transformation τ of (X,μ) is measure class preserving.

The group $\mathrm{Aut}\,(X,\mu)$, formed by all measure preserving transformations, is a proper subgroup of $\mathrm{Aut}^*(X,\mu)$.

Recall that for every $p\in[1,\infty)$ the *quasi-regular representation* $\gamma=\gamma_p$ of the group $\mathrm{Aut}^*(X,\mu)$ in the Banach space $L^p(X)$ is defined for every $\tau\in\mathrm{Aut}^*(X,\mu)$ and $f\in L^p(X)$ by

$$\gamma_\tau(f)(x)=f(\tau^{-1}x)\sqrt[p]{\frac{d\mu\circ\tau^{-1}}{d\mu}(x)},$$

where $\mu\circ\tau^{-1}=\tau_*(\mu)$ is the pushforward measure, and $d/d\mu$ denotes the Radon–Nikodym derivative.

The strong operator topology on the isometry group of $L^p(X)$ induces a Polish group topology on $\mathrm{Aut}^*(X,\mu)$.

Exercise 3.3.6. Verify that for different values $1\le p<\infty$ these topologies on $\mathrm{Aut}^*(X,\mu)$ coincide with each other and make $\mathrm{Aut}^*(X,\mu)$ into a Polish (separable completely metrizable) group.

This is the *weak topology* on $\mathrm{Aut}^*(X,\mu)$ (also called the *coarse topology*).

Exercise 3.3.7. Show that the restriction of the weak topology to $\mathrm{Aut}\,(X,\mu)$ is the weakest topology making continuous every function

$$\mathrm{Aut}^*(X,\mu)\ni\tau\mapsto\mu(A\cap\tau(A))\in\mathbb{R},$$

where $A \subseteq X$ is a measurable subset. (It is of course enough to consider elements of a generating family of measurable sets.)

For the group $\mathrm{Aut}^*(X,\mu)$, an analogous assertion is no longer true.

Exercise 3.3.8. Verify that for different values $1 \le p < \infty$ these topologies coincide with each other and with the weak topology on $\mathrm{Aut}^*(X,\mu)$.

Define a metric $d = d_{unif}$ on $\mathrm{Aut}^*(X,\mu)$ as follows:

$$d_{unif}(\tau,\sigma) = \mu\{x \in X \colon \tau(x) \neq \sigma(x)\}.$$

This metric is left-invariant:

$$\begin{aligned} d(\eta\sigma,\eta\tau) &= \mu\{x \in X \mid \eta\sigma(x) \neq \eta\tau(x)\} \\ &= \mu\{x \in X \mid \sigma(x) \neq \tau(x)\} \\ &= d(\sigma,\tau). \end{aligned}$$

The topology induced on $\mathrm{Aut}^*(X,\mu)$ by the metric d is a group topology. Indeed, since d is left-invariant, it remains to verify two conditions:
(1) every ε-ball $\mathcal{O}_\varepsilon(e)$ around the identity in $\mathrm{Aut}^*(X,\mu)$, formed with regard to the metric, is symmetric, which is obvious, as one has for a $\tau \in \mathrm{Aut}^*(X,\mu)$:

$$\begin{aligned} \tau \in \mathcal{O}_\varepsilon(e) &\Leftrightarrow \mu\{x \in X \colon \tau(x) \neq x\} < \varepsilon \\ &\Leftrightarrow \mu\{x \in X \colon \tau(x) = x\} > 1-\varepsilon \\ &\Leftrightarrow \mu\{x \in X \colon x = \tau^{-1}(x)\} > 1-\varepsilon \\ &\Leftrightarrow \mu\{x \in X \colon x \neq \tau^{-1}(x)\} < \varepsilon \\ &\Leftrightarrow \tau^{-1} \in \mathcal{O}_\varepsilon(e). \end{aligned}$$

(2) For every $\varepsilon > 0$ and every $\tau \in \mathrm{Aut}^*(X,\mu)$ there is a $\delta > 0$ such that $\tau^{-1}\mathcal{O}_\delta(e)\tau \subseteq \mathcal{O}_\varepsilon(e)$. This clearly follows from the fact that both τ and τ^{-1} are non-singular transformations.

This group topology is known as the *uniform topology.*

Exercise 3.3.9. Verify that the group $\mathrm{Aut}\,(X,\mu)$, equipped with the uniform topology, contains an uncountable uniformly discrete subset, that is, a subset X such that, for some positive ε and all $x, y \in X$, one has $d(x,y) \geq \varepsilon$. Conclude that the uniform topology is strictly finer than the weak topology.

Exercise 3.3.10. Show that the uniform topology makes $\mathrm{Aut}^*(X,\mu)$ into a complete (in two-sided uniformity) and path-connected group.

Exercise 3.3.11. Verify that both the weak topology and the uniform topology only depend on the equivalence class of the measure μ, rather than on μ itself. (The reference here is [30], Rem. 3, p. 373, but the proof is rather straightforward in any case.)

Sometimes we will indicate the uniform topology by the subscript u, as in $\mathrm{Aut}\,(X,\mu)_u$. Similarly, for the weak topology the subscript w will be used.

Remark 3.3.12. One can also define a right-invariant metric $d^\wedge$:

$$d^\wedge(\tau,\sigma) = \mu\{x \in X : \tau^{-1}(x) \neq \sigma^{-1}(x)\}.$$

It determines the same uniform topology.

Remark 3.3.13. Note that the restriction of d to $\mathrm{Aut}\,(X,\mu)$ is bi-invariant, and so the topological group $\mathrm{Aut}\,(X,\mu)_u$ is SIN (= has small invariant neighbourhoods), that is, the left and the right uniform structures coincide.

Exercise 3.3.14. Show that the uniform operator topology induced on $\mathrm{Aut}\,(X,\mu)$ through the quasi-regular representation in $L^2(X,\mu)$ is in fact discrete, and consequently the uniform topology on $\mathrm{Aut}\,(X,\mu)$ (and $\mathrm{Aut}^*(X,\mu)$ as well) is not induced by the uniform operator topology on the unitary group $U(L^2(X))$.

Now we turn our attention to the group $\mathrm{Aut}\,(X,\mu)$. Notice that the finite symmetric groups embed into $\mathrm{Aut}\,(X,\mu)$ in a natural way. Namely, identify (X,μ) with the unit interval $\mathbb{I}$ equipped with the Lebesgue measure λ, and subdivide it for every $n \in \mathbb{N}$ into *dyadic intervals of rank* n: $[i2^{-n},(i+1)2^{-n}]$, $i = 0,1,\ldots,2^n-1$. Now one can think of the symmetric group S_{2^n} of rank 2^n as consisting of all transformations of $\mathbb{I}$ mapping each dyadic interval of rank n onto a dyadic interval of rank n via a translation (the *interval exchange transformations*).

The following result is a well-known consequence of the Rokhlin–Kakutani Lemma (cf. e.g. [83], pp. 65–68).

Theorem 3.3.15 (Weak Approximation Theorem). *The union of the subgroups S_{2^n}, $n \in \mathbb{N}$ is everywhere dense in* $\mathrm{Aut}\,(\mathbb{I})$ *with respect to the weak topology.* □

Together with Maurey theorem 3.3.1, the above result leads to the following [57].

Theorem 3.3.16 (Giordano and Pestov). *The group* $\mathrm{Aut}\,(X,\mu)$ *with the weak topology is a Lévy group.*

In order to accomplish the argument, we still need to show that the approximating chain of subgroups $(S_{2^n}) < \mathrm{Aut}\,(X,\mu)$ forms a Lévy family. To see this fact clearly, notice the following.

Exercise 3.3.17. Show that the restriction of the uniform metric d_{unif} to each S_{2^n} embedded into $\mathrm{Aut}\,(X,\mu)$ as above coincides with the normalized Hamming distance.

Now by force of Maurey theorem 3.3.1 the chain (S_{2^n}) forms a Lévy family with regard to the uniform topology (and the associated one-sided uniform structure) on $\mathrm{Aut}\,(X,\mu)$. The identity map from $\mathrm{Aut}\,(X,\mu)_u$ onto $\mathrm{Aut}\,(X,\mu)_w$ being uniformly continuous, it follows that (S_{2^n}) is a Lévy family with respect to the weak topology on $\mathrm{Aut}\,(X,\mu)$ as well. Theorem 3.3.16 is now proved. □

In conclusion, we are going to show that with regard to the uniform topology, the group $\mathrm{Aut}\,(X,\mu)$ is non-amenable, using the machinery developed in Section 2.7. First, let us remind some classical concepts from representation theory.

Definition 3.3.18. A unitary representation π of a topological group G *almost has invariant vectors* if for every compact $F \subseteq G$ and every $\varepsilon > 0$ there is a ξ in the space of representation $\mathcal{H}_\xi$ with $\|\xi\| = 1$ and $\|\pi_g\xi - \xi\| < \varepsilon$ for every $g \in F$.

(As can be easily seen at the level of definitions, if a representation π has almost invariant vectors, then the system $(\mathbb{S}_{\mathcal{H}}, G, \pi)$ has the concentration property. The converse is not true.)

Definition 3.3.19. One says that a topological group G has *Kazhdan's property (T)* if, whenever a unitary representation of G almost has invariant vectors, it has an invariant vector of norm one.

For an excellent account of the theory, we refer the reader to the book [88] and especially its considerably extended and updated English version, currently in preparation and made available on-line [14]. See [13] for the beginnings of the theory for infinite-dimensional groups.

We only need two facts here. First, the groups $\mathrm{SL}(\mathbb{R}, n)$ have Kazhdan's property for $n \geq 3$. Second, according to a criterion obtained in [12, 15], a locally compact second-countable group has property (T) if and only if every amenable strongly continuous unitary representation of G contains a finite-dimensional subrepresentation.

Example 3.3.20 (Giordano and Pestov [57, 58]). Let (X, μ) be the standard Lebesgue measure space. The group $\mathrm{Aut}\,(X, \mu)$ equipped with the uniform topology is non-amenable.

Let $V = L^2_0(X)$ be the closed unitary G-submodule of $L^2(X)$ consisting of all functions $f \in L^2(X)$ with

$$\int_X f(x)\, d\mu(x) = 0.$$

Recall that with regard to the uniform topology the group $\mathrm{Aut}\,(X, \mu)$ is a SIN group. According to Proposition 2.7.24, if $\mathrm{Aut}\,(X, \mu)_u$ were amenable, then so would be the regular representation γ of $\mathrm{Aut}\,(X, \mu)_u$ in $L^2_0(X)$. In particular, the restriction of γ to any subgroup G of $\mathrm{Aut}\,(X, \mu)_u$ would be amenable as well. It is, therefore, enough to discover a group G acting on a standard non-atomic Lebesgue space X by measure-preserving transformations in such a manner that the associated regular representation of G in $L^2_0(X)$ is not amenable in the sense of Bekka.

Let G be a minimally almost periodic semisimple Lie group with property (T), such as $SL_3(\mathbb{R})$. Let Γ denote a lattice in G, that is, a discrete subgroup such that the Haar measure induces an invariant finite measure on G/Γ. Set $X = G/\Gamma$.

Assume that the standard representation π of G in $L^2_0(X) = L^2_0(G/\Gamma)$ is amenable. Since G is a Kazhdan group, π must have a finite-dimensional subrepresentation. Minimal almost periodicity of G means that any such subrepresentation is trivial (one-dimensional). But $L^2_0(X)$ has no nontrivial G-invariant vectors: since the action of G on X is transitive, such vectors must be constant functions.

3.3.3 Full groups

Definition 3.3.21. An equivalence relation $\mathscr{R}$ on a standard Borel space X is called *Borel*, or *measurable,* if $\mathscr{R}$ is a Borel subset of $X \times X$.

Example 3.3.22. Let $X = \{0,1\}^{\mathbb{N}}$, equipped with the product topology. The *tail equivalence relation* on X is given by the following condition:

$$x \sim y \iff \exists n \in \mathbb{N}, \quad \forall i \geq n, \quad x_i = y_i,$$

that is, x and y are equivalent if and only if their coordinates eventually coincide. This equivalence relation is denoted by E_0.

Definition 3.3.23. An *automorphism* of a Borel equivalence relation $\mathscr{R}$ on a standard Borel space X, or an *$\mathscr{R}$-automorphism,* is a Borel map $\sigma\colon X \to X$ with the property that $(x, \sigma(x)) \in \mathscr{R}$ for all $x \in X$. A probability measure μ on X is *$\mathscr{R}$-invariant*, respectively *$\mathscr{R}$-quasi-invariant*, if μ is invariant (resp., quasi-invariant) under every $\mathscr{R}$-automorphism of X. In this case, one says that the Borel equivalence relation $\mathscr{R}$ on (X,μ) is *measure preserving* (resp. *measure class preserving*).

Definition 3.3.24. Let $\mathscr{R}$ be a Borel equivalence relation on a standard Borel space X, equipped with a quasi-invariant measure μ. The *full group* of $\mathscr{R}$ in the sense of Dye [40], denoted by $[\mathscr{R}]$, is a subgroup of $\mathrm{Aut}^*(X,\mu)$ consisting of all transformations σ preserving each equivalence class of $\mathscr{R}$, that is, such σ that $(x,\sigma(x)) \in \mathscr{R}$ for μ-a.e. $x \in X$.

Exercise 3.3.25. Show that the full group $[\mathscr{R}]$ is a closed subgroup of the group $\mathrm{Aut}^*(X,\mu)$ with regard to the uniform topology.

(The reader may also consult [111], Lemma 1.2.)

The theory has most substance in the case where the relation $\mathscr{R}$ is *countable* in the sense that for μ-a.e. x the class $\mathscr{R}[x]$ is countable.

A major source of such equivalence relations is given by actions of countable groups.

Let a group G act on (X, μ) by measure class preserving transformations, defined μ-a.e. Thus, a group action of G can be interpreted as a group homomorphism $G \to \mathrm{Aut}^*(X, \mu)$. This action is *free* if it is free μ-a.e., that is, fixed points of every motion $X \ni x \mapsto gx \in X$ form a null set.

Definition 3.3.26. The *orbit equivalence relation* $\mathscr{R}_G$ is defined as follows:

$$(x, y) \in \mathscr{R}_G \iff \exists g \in G, \quad gx = y,$$

that is, the equivalence classes of $\mathscr{R}_G$ are the orbits of the action of G. The full group $[\mathscr{R}_G]$ of the orbit equivalence relation $\mathscr{R}_G$ is also denoted $[G]$.

In the case where the acting group is countable, the orbit equivalence relation preserves a remarkable amount of information about the group action.

Exercise 3.3.27. Assume the acting group G is countable and the action of G on X is free μ-a.e. Prove that a non-singular transformation $\tau \in \mathrm{Aut}^*(X, \mu)$ is contained in the full group $[G]$ if and only if there is a measurable partition $\{X_g : g \in G\}$ such that $\{gX_g : g \in G\}$ also forms a partition of X and

$$\forall x \in X_g, \quad \tau(g) = gx.$$

Note that for the actions of uncountable groups the above is no longer true (sufficiency cannot be guaranteed).

Exercise 3.3.28. Deduce from Exercise 3.3.27 that if G acts on (X, μ) freely by measure-preserving transformations, then the full group $[G]$ consists of measure-preserving transformations as well, that is, the orbit equivalence relation $\mathscr{R}_G$ is measure preserving.

Example 3.3.29. Let $\mathbb{Z}_2^{\oplus \mathbb{N}}$ denote the restricted direct power of the group $\mathbb{Z}_2$ (that is, the free group in the variety of all groups of order 2). Since $\mathbb{Z}_2^{\oplus \mathbb{N}}$ is a subgroup of the compact group $X = \mathbb{Z}_2^{\mathbb{N}}$, it acts on the latter by multiplication. The tail equivalence relation E_0 from Example 3.3.22 is the orbit equivalence relation determined by the above action. With regard to the normalized Haar measure on the group $X = \mathbb{Z}_2^{\mathbb{N}}$, the relation E_0 is measure-preserving.

Example 3.3.30. The tail equivalence relation is also determined by a suitable action of the group $\mathbb{Z}$ on the same Cantor set $\mathbb{Z}_2^{\mathbb{N}}$, the so-called *odometer action*. In this case, one thinks of $\mathbb{Z}_2^{\mathbb{N}}$ as the set of all infinite binary expansions of the 2-adic integers,

$$\mathbb{Z}_2^{\mathbb{N}} \ni x \mapsto \sum_{i=0}^{\infty} 2^i x_i \in \mathbb{A}_2,$$

and the action of the generator of the group $\mathbb{Z}$ consists in adding 1 with carry. Notice that the corresponding orbit equivalence relation here coincides with the tail equivalence relation μ-a.e., because the orbits of the elements x whose coordinates are eventually 1 are not the same, but of course the measure of the set of such x is zero.

Remark 3.3.31. As one reference on orbit equivalence, we recommend the recent book [98].

Remark 3.3.32. It follows from Exercise 3.3.25 that if the acting group G is countable, the full group $[G]$ with the uniform topology is Polish.

Not so for the full group with regard to the weak topology.

Exercise 3.3.33. Use the Weak Approximation Theorem to construct an example of an action of a countable group on the interval with the Lebesgue measure whose full group is everywhere dense in $\mathrm{Aut}\,(\mathbb{I}, \lambda)$ with the weak topology.

Exercise 3.3.34. Let the symmetric group S_{2^n} act on the space $X = \mathbb{Z}_2^{\mathbb{N}}$ by permutations of all "n-heads" of infinite sequences. In other words, if $x \in X$, then write $x = (x', x'')$, where $x' \in \mathbb{Z}_2^{\{1,2,\dots,n\}}$, $x'' \in \mathbb{Z}_2^{\mathbb{N}\setminus\{1,2,\dots,n\}}$, and set for a $\sigma \in S_{2^n}$

$$\sigma \cdot x = (\sigma(x'), x'') \in X.$$

1. Verify that in this way, the group S_{2^n} embeds into the full group $[E_0]$ of the tail equivalence relation from Example 3.3.29.
2. Show that the union of the increasing chain of subgroups S_{2^n}, $n \in \mathbb{N}$ is uniformly dense in the full group $[E_0]$, using Exercise 3.3.28 and Exercise 3.3.27.

3. Now deduce that the full group $[E_0]$ is a Lévy group.
   (*Hint:* Have another look at the proof of Theorem 3.3.16.)

A refinement of the same argument can be used to obtain a class of full groups of measure class preserving Borel equivalence relations that are Lévy groups.

Definition 3.3.35. A Borel equivalence relation $\mathscr{R}$ is called

- *aperiodic*, if the equivalence class $\mathscr{R}[x]$ is infinite for μ-a.e. x,
- *finite*, if the equivalence class $\mathscr{R}[x]$ is finite for μ-a.e. x,
- *hyperfinite,* if $\mathscr{R}$ is the union of an increasing chain of finite equivalence relations, and
- *ergodic* if every $\mathscr{R}$-saturated measurable set of strictly positive measure has measure 1.

For example, the equivalence relation E_0 is aperiodic, hyperfinite and ergodic.

Here is the main result of this subsection.

Theorem 3.3.36 (Giordano and Pestov [58]). *An ergodic Borel equivalence relation $\mathscr{R}$ on a standard Borel space X equipped with a quasi-invariant probability measure μ is hyperfinite if and only if the full group $[\mathscr{R}]$ with the uniform measure is a Lévy group.*

The proof of sufficiency ($\Leftarrow$) is using the concept of an amenable equivalence relation, due to Zimmer [190], which is equivalent to the concept of a hyperfinite relation by force of the result of Connes, Feldman and Weiss [33]. Now the amenability of $\mathscr{R}$ is deduced from the assumed Lévy property, in fact just amenability, of the full group $[\mathscr{R}]$. We will not discuss this implication here.

To establish necessity, one again utilises a result established by Connes, Feldman and Weiss [33], according to which one can assume, without loss in generality, that the hyperfinite equivalence relation $\mathscr{R}$ is just the tail equivalence relation E_0 on the compact group $X = \{0,1\}^{\mathbb{N}}$, equipped with a suitable ergodic non-atomic quasi-invariant probability measure μ. Now the proof follows the same steps as the proof in the measure-preserving case, outlined in Exercise 3.3.34, though it requires more technical dexterity.

Step 1 is exactly the same. Step 2 is again based on Exercise 3.3.27, and the reader might wish to try and reconstruct it on their own before resorting to help [58] (Lemma 5.2). The last Step 3, however, requires the following variation of Maurey's theorem 3.3.1.

Let X be a standard Borel space equipped with a (not necessarily non-atomic) probability measure μ. In particular, if X has finitely many points and the measure has full support, then $\mathrm{Aut}^*(X,\mu)$ is, as an abstract group, just the symmetric group of rank $n = |X|$.

Theorem 3.3.37. *Let $X = (X,\mu)$ be a probability space with finitely many points. The concentration function α of the automorphism group $\mathrm{Aut}^*(X,\mu)$, equipped with the uniform metric and the normalized counting measure, satisfies*

$$\alpha(\varepsilon) \leq \exp\left(-\frac{\varepsilon^2}{32\sum_{x\in X}\mu(\{x\})^2}\right). \tag{3.15}$$

Proof. Let $X = \{x_1, x_2, \ldots, x_n\}$, where

$$\mu(\{x_1\}) \geq \mu(\{x_2\}) \geq \ldots \geq \mu(\{x_{n-1}\}) \geq \mu(\{x_n\}).$$

For $k = 0, 1, \ldots, n$, let H_k be the subgroup stabilizing each element $x_1, \ldots, x_k$. Thus,

$$\mathrm{Aut}^*(X,\mu) = H_0 > H_1 > H_2 > \ldots > H_n = \{e\}.$$

Let Ω^k be the partition of $G = \mathrm{Aut}^*(X,\mu)$ into left H_k-cosets σH_k, $\sigma \in G$.

Suppose $A = \sigma H_k$ and $B = \tau H_k$ are contained in the same left H_{k-1}-coset. Then $\sigma(x_i) = \tau(x_i) = a_i$ for $i = 1, 2, \ldots, k-1$, while $a = \sigma(x_k)$ and $b = \tau(x_k)$ need not coincide. Thus, elements π of A are defined by the conditions

$$\pi(x_1) = a_1,\ \pi(x_2) = a_2,\ \ldots,\ \pi(x_{k-1}) = a_{k-1},\ \pi(x_k) = a,$$

while elements $\pi \in B$ are defined by the conditions

$$\pi(x_1) = a_1,\ \pi(x_2) = a_2,\ \ldots,\ \pi(x_{k-1}) = a_{k-1},\ \pi(x_k) = b.$$

Let $t_{a,b}$ denote the transposition of a and b. Consider the map

$$\varphi\colon \sigma H_k \ni j \mapsto t_{a,b} \circ j \in \tau H_k. \tag{3.16}$$

Clearly, φ is a bijection between $A = \sigma H_k$ and $B = \tau H_k$. The values of j and $t_{a,b} \circ j$ differ in at most two inputs, $x_k = j^{-1}(a)$ and $j^{-1}(b)$. Since $a, b \notin \{a_1, a_2, \ldots, a_{k-1}\}$, it follows that $j^{-1}(a), j^{-1}(b) \notin \{x_1, x_2, \ldots, x_{k-1}\}$ and $\mu(\{j^{-1}(b)\}) \leq \mu\{(x_k\})$. We conclude: for every $j \in A = \sigma H_k$

$$d_{unif}(j, \varphi(j)) \leq 2\mu(\{x_k\}).$$

Consequently, the metric space $\mathrm{Aut}^*(X, \mu)$ has length at most $\ell = 2\left(\sum_{i=1}^n \mu(\{x_i\}^2)\right)^{1/2}$, and Theorem 3.2.28 accomplishes the proof. □

Corollary 3.3.38. *Let (X_n, μ_n) be a sequence of probability spaces with finitely many points in each, having the property that the mass of the largest atom in X_n goes to zero as $n \to \infty$. Then the family of automorphism groups $\mathrm{Aut}^*(X_n, \mu_n)$, $n \in \mathbb{N}$, equipped with the uniform metric and the normalized counting measure, is Lévy.*

Proof. Let $v_n = (\mu_n(\{x\}))_{x \in X_n}$. By Hölder's inequality,

$$\begin{aligned} \sum_{x \in X_n} \mu_n(\{x\})^2 &= \langle v_n, v_n \rangle \\ &\leq \|v_n\|_1 \cdot \|v_n\|_\infty \\ &= \max_{x \in X_n} \mu_n(\{x\}) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Theorem 3.3.37 now applies. □

Theorem 3.3.36 has some interesting further consequences.

Corollary 3.3.39 (Giordano and Pestov [58]). *Let (X, μ) be a standard non-atomic Lebesgue space. The group $\mathrm{Aut}^*(X, \mu)$ with the weak topology is a Lévy group.*

To prove the corollary, it is enough to find a hyperfinite non-singular equivalence relation $\mathscr{R}$ on the standard Lebesgue measure space (X,μ) such that the full group $[\mathscr{R}]$ is everywhere dense in $\mathrm{Aut}^*(X,\mu)$ equipped with the weak topology.

This is the tail equivalence relation E_0 on the compact abelian group $X=\mathbb{Z}_2^{\mathbb{N}}$, equipped with the product measure given by $\mu = \otimes\mu_n$, with $\mu_{2n}(i)=\frac{\lambda^i}{1+\lambda}$ and $\mu_{2n+1}(i)=\frac{\rho^i}{1+\rho}$, for $i\in\{0,1\}$, where λ and ρ are positive numbers such that $\log\lambda$ and $\log\rho$ are rationally independent. (Cf. [84], Sect. I.3, Example 1.)

Now the proof is based on the following two approximation results. The first is Lemma 32 in [84].

Lemma 3.3.40 (Hamachi and Osikawa). *Let A and B be two measurable subsets of (X,μ) as above of strictly positive measure. Then for every $\varepsilon>0$ there exists an element σ of the full group $[E_0]$ such that*

1. *$\sigma(A)=\sigma(B)$,*
2. *the Radon–Nikodym derivative of σ differs by less than ε from $\mu(B)/\mu(A)$ at μ-a.e. $x\in A$.*

□

The second is a general result true of any standard non-atomic Lebesgue space (X,μ) (Tulcea, [167], Theorem 1; cf. also [111]).

Theorem 3.3.41 (Tulcea). *Given a transformation $\tau\in\mathrm{Aut}^*(X,\mu)$, an $\varepsilon>0$, and a finite measurable partition $\mathcal{Q}$ of X, there exists a finite partition $\mathcal{Q}_1\prec\mathcal{Q}$ and a transformation $\sigma\in\mathrm{Aut}^*(X,\mu)$ with the following properties:*

1. *for every $A\in\mathcal{Q}_1$, the set $B=\sigma(A)$ is in $\mathcal{Q}_1$, and μ-a.e. on A the Radon–Nikodym derivative of σ is $\mu(B)/\mu(A)$.*
2. *if $\sigma(A)=A$, then $\sigma|_A=\mathrm{Id}_A$.*
3. *$d_{unif}(\tau,\sigma)<\varepsilon$.*

□

Exercise 3.3.42. Let V be a weak neighbourhood of the identity of $\mathrm{Aut}^*(X,\mu)$, determined by an $\varepsilon>0$ and a finite measurable partition $\mathcal{Q}$ of X, so that

$$V=\{\beta\in\mathrm{Aut}^*(X,\mu)\colon\ \|\chi_A-\beta\chi_A\|_1<\varepsilon \text{ for each } A\in\mathcal{Q}\}.$$

Let $\tau \in \mathrm{Aut}^*(X,\mu)$. Using the two approximation results above, construct an element σ of the full group $[E_0]$ with $\sigma^{-1}\tau \in V$, thus accomplishing the proof of Corollary 3.3.39.

The second application concerns the isometry groups of spaces $L^p(0,1)$.

Theorem 3.3.43 (Giordano and Pestov [58]). *Let (X,μ) be a standard Borel space with a non-atomic measure. Then for every $1 \leq p < \infty$. the group of isometries $\mathrm{Iso}\,(L^p(X,\mu))$ with the strong operator topology is a Lévy group.*

Of course the case $p = 2$, studied by Gromov and Milman, is the most prominent example and the point of departure of the entire theory. The case $p \neq 2$, however, has to be treated differently. One needs the following description of the isometries of $L^p(X,\mu)$, stated by S. Banach in his classical treatise [6] (théorème 11.5.I, p. 178).

Theorem 3.3.44 (Banach). *Let $1 \leq p < \infty$, $p \neq 2$, let (X,μ) be a finite measure space, and let T be a surjective isometry of $L^p(X,\mu)$. Then there is an invertible measure class preserving transformation σ of X and a measurable function h with $|h| = 1$ such that*

$$Tf = h\ {}^{\sigma}f. \tag{3.17}$$

□

It seems that Banach has never published the proof of this result. The first available proof — of a more general theorem — belongs to Lamperti [101], whose statement of the main result (Theorem 3.1) was however not quite correct, as noticed in [29], cf. also [82].

Let $(X,\mathcal{F},\mu)$ be a finite measure space. A set map $\phi\colon \mathcal{F} \to \mathcal{F}$, defined modulo zero measure sets, is called a *regular set isomorphism* if

1. $\phi(X \setminus A) = \phi(X) \setminus \phi(A)$, $A \in \mathcal{F}$,
2. $\phi(\cup_{n=1}^{\infty} A_n) = \cup_{n=1}^{\infty}\phi(A_n)$ for disjoint $A_n \in \mathcal{F}$,
3. $\mu(\phi(A)) = 0$ if and only if $\mu(A) = 0$, $A \in \mathcal{F}$.

A regular set isomorphism ϕ determines a positive linear operator on $L^p(X,\mu)$, denoted by the same letter ϕ, by the condition $\phi(\chi_A) = \chi_{\phi(A)}$.

Here is a corrected form of the result proved in [101], cf. [29],[82].

Theorem 3.3.45 (Banach–Lamperti). *Let $1 \leq p < \infty$, $p \neq 2$. Then every isometry, T, of $L^p(X,\mathcal{F},\mu)$ (not necessarily onto) is of the form*

$$(Tf)(x) = h(x)(\phi(f))(x), \tag{3.18}$$

where ϕ is a regular set isomorphism and $E\{|h|^p \colon \phi(\mathcal{F})\} = d(\mu \circ \phi^{-1})/d\mu$, that is,

$$\int_{\phi(A)} |h|^p \, d\mu = \int_{\phi(A)} \frac{d(\mu\circ\phi^{-1})}{d\mu} d\mu \quad \text{for all } A \in \mathcal{F}.$$

$\square$

Of course Theorem 3.3.44 now follows.

Recall that $\mathrm{Aut}^*(X,\mu)_w$ can be identified with a (closed) topological subgroup of the group $\mathrm{Iso}\,(L^p(X,\mu))$, equipped with the strong operator topology, through the left quasi-regular representation:

$${}^{\sigma}f(x) = f(\sigma^{-1}x)\left(\frac{d(\mu\circ\sigma^{-1})}{d\mu}(x)\right)^{\frac{1}{p}}, \quad \sigma \in \mathrm{Aut}^*(X,\mu).$$

Invertible (the same as onto) isometries of $L^p(X,\mu)$ correspond to the invertible regular set isomorphisms, that is, invertible measure class preserving transformations of (X,μ), in which case $h\phi(f) = {}^{\phi^{-1}}f$.

Corollary 3.3.46. *Let $1 \leq p < \infty$, $p \neq 2$, let (X,μ) be a finite measure space. The group $\mathrm{Iso}\,(L^p(X,\mu))$, equipped with the strong operator topology, is isomorphic to the semidirect product of the group $\mathrm{Aut}^*(X,\mu)_w$ and the normal group $L(X,\mu;U(1))$ of all measurable maps from X to the circle rotation group equipped with the topology of convergence in measure.*

Proof. According to Banach's theorem 3.3.44, every element $T \in \mathrm{Iso}\,(L^p(X,\mu))$ admits a (clearly unique) decomposition of the form $Tf = h \cdot {}^{\sigma}f$, for $f \in L^p(X,\mu)$, where $h \in L(X,\mu;U(1))$ and $\sigma \in$

$\mathrm{Aut}^*(X,\mu)$. To establish the algebraic isomorphism between the group of isometries and the semidirect product group, it is therefore enough to prove that $L(X,\mu;U(1))$ is normal in $\mathrm{Iso}\,(L^p(X,\mu))$ or, which is the same, invariant under inner automorphisms generated by the elements of $\mathrm{Aut}^*(X,\mu)$. Let $h \in L(X,\mu;U(1))$ and $\sigma \in \mathrm{Aut}^*(X,\mu)$. One has

$$\begin{aligned}
(\sigma^{-1}h\sigma)(f)(x) &\equiv {}^{\sigma^{-1}}(h\cdot {}^{\sigma}f)\,(x) \\
&= (h\cdot {}^{\sigma}f)\,(\sigma x)\left(\frac{d(\mu\circ\sigma)}{d\mu}(x)\right)^{\frac{1}{p}} \\
&= h(\sigma x)f(\sigma^{-1}\sigma x)\left(\frac{d(\mu\circ\sigma^{-1})}{d\mu}(\sigma x)\right)^{\frac{1}{p}}\left(\frac{d(\mu\circ\sigma)}{d\mu}(x)\right)^{\frac{1}{p}} \\
&= \sigma^{-1}(h)(x)f(x),
\end{aligned}$$

and therefore $\sigma^{-1}h\sigma = \sigma^{-1}(h) \in L(X,\mu;U(1))$.

We have already noted that $\mathrm{Aut}^*(X,\mu)_w$ is a closed topological subgroup of $\mathrm{Iso}\,(L^p(X,\mu))$. The same is true of $L^0(X,\mu;U(1))$. Using the fact that every L^p-metric, $1 \le p < \infty$, induces the topology of convergence in measure on $L(X,\mu;U(1))$, it is easy to verify that, first, for every $f \in L^p(X,\mu)$ the orbit map of the multiplication action

$$L^0(X,\mu;U(1)) \ni h \mapsto h\cdot f \in L^p(X,\mu)$$

is continuous, while for $f(x)\equiv 1$ it is a homeomorphic embedding.

As a corollary (which can be also checked directly), the action of $\mathrm{Aut}^*(X,\mu)_w$ on $L^0(X,\mu;U(1))$, defined by $(\sigma,h)\mapsto\sigma^{-1}(h)$, is continuous as a map $\mathrm{Aut}^*(X,\mu)_w\times L^0(X,\mu;U(1)) \to L^0(X,\mu;U(1))$. Therefore, the semi-direct product $\mathrm{Aut}^*(X,\mu)_w \ltimes L^0(X,\mu;U(1))$ is a (Polish) topological group.

The algebraic automorphism given by the multiplication map,

$$\mathrm{Aut}^*(X,\mu)_w \ltimes L^0(X,\mu;U(1)) \to \mathrm{Iso}\,(L^p(X,\mu)),$$

is continuous, and since all groups involved are Polish, it is a topological isomorphism by the standard Open Mapping Theorem for Polish groups, cf. [95], Corollary 3, p. 98 and Theorem 3, p. 90. ☐

Sketch of proof of Theorem 3.3.43. Just like in the proof of Theorem 3.3.36, we will identify the measure space (X,μ) with the compact abelian group $X = \{0,1\}^{\mathbb{N}}$ equipped with the product measure of a special type. Let L_n denote the subgroup of $L(X,\mu;U(1))$ consisting of functions constant on cylinder sets determined by the first n coordinates. It is clear that the group L_n is invariant under all inner automorphisms generated by elements of the permutation group S_{2^n}.

The normalized Haar measures on S_{2^n} concentrate in $\mathrm{Aut}^*(X,\mu)_w$ by Corollary 3.3.38. Similarly, one can prove that the normalized Haar measures on L_n, $n \in \mathbb{N}$, concentrate in $L^0(X,\mu;U(1))$.

The semidirect product groups $S_{2^n} \ltimes L_n$ are compact, and the normalized Haar measures on them are products of Haar measures, respectively, on S_{2^n} and on L_n. It is not hard to verify that the product measures form a Lévy family as well.

Finally, the union of the sequence of subgroups $S_{2^n} \ltimes L_n$ is weakly dense in $\mathrm{Aut}^*(X,\mu)$. □

3.4 Group of isometries of the Urysohn space

3.4.1 Urysohn universal metric space

Definition 3.4.1. The *Urysohn metric space* $\mathbb{U}$ is defined by three conditions:

1. $\mathbb{U}$ is a complete separable metric space;

2. $\mathbb{U}$ is ultrahomogeneous, that is, every isometry between two finite metric subspaces of $\mathbb{U}$ extends to a global isometry of $\mathbb{U}$ onto itself;

3. $\mathbb{U}$ is universal, that is, it contains an isometric copy of every separable metric space.

Theorem 3.4.2. *The universal separable metric space* $\mathbb{U}$ *exists and is unique up to an isometry.*

Both the proof of existence and that of uniqueness are based on the following alternative characterization of the space $\mathbb{U}$.

Definition 3.4.3. Let us say that a metric space X has a *one-point extension property,* or is a *generalized Urysohn metric space*, if

($*$) whenever $A \subseteq X$ is a finite metric subspace of X and $A' = A \cup \{b\}$ is an abstract one-point metric space extension of A, the embedding $A \hookrightarrow X$ extends to an isometric embedding $A' \hookrightarrow X$.

Theorem 3.4.4. *A complete separable metric space X is Urysohn if and only if it has the one-point extension property.*

Lemma 3.4.5. *Let X and Y be two separable complete metric spaces each of which satisfies the extension property ($*$). Then X and Y are isometric. Moreover, if A and B are finite metric subspaces of X and Y respectively, and $i\colon A \to B$ is an isometry, then i extends to an isometry $f\colon X \to Y$.*

Proof. The proof is based on the so-called *back-and-forth* (or *shuttle*) argument. This is a recursive construction which will appear in these lectures more than once.

Choose countable everywhere dense subsets $\tilde{X}$ of X and $\tilde{Y}$ of Y, and enumerate them: $X = \{x_i\}_{i=0}^{\infty}$ and $Y = \{y_i\}_{i=0}^{\infty}$.

Let $f_0 = i$ be the basis of recursion. The outcome of n-th recursive step will be an isometry f_n with finite domain, extending the isometry f_{n-1} and such that $\{x_0, x_1, \ldots, x_n\} \subseteq \operatorname{dom} f_n$ and $\{y_0, y_1, \ldots, y_n\} \subseteq \operatorname{im} f_n$. Once this is achieved, the "limit" mapping f, defined by $f(x_n) = f_n(x_n)$ for all $n \in \mathbb{N}$, is an isometry between $\tilde{X}$ and $\tilde{Y}$ whose restriction to A is i. The unique extension of f by continuity to X establishes an isometry between X and Y with the desired properties.

To perform a step of recursion, suppose an isometry f_n has been already constructed in such a way that the domain $\operatorname{dom} f_n$ is finite and contains the set $\{x_0, x_1, \ldots, x_n\} \cup \operatorname{dom} f_{n-1}$, the restriction of f_n to the domain of f_{n-1} coincides with the latter isometry, and the image $\operatorname{im} f_n$ contains $\{y_0, y_1, \ldots, y_n\}$. We actually split the n-th step in two.

If $x_{n+1} \notin \operatorname{dom} f_n$, then, using the extension property ($\star$), one can select an $y \in Y \setminus \operatorname{im} f_n$ in such a way that the mapping $\tilde{f}_n$, defined by

$$\tilde{f}_n(a) = \begin{cases} f_n(a), & \text{if } a \in \operatorname{dom} f_n, \\ \tilde{f}_n(x_{n+1}), & \text{if } a = x_{n+1} \end{cases}$$

is an isometry. If $x_{n+1} \in \operatorname{dom} f_n$, then nothing happens and we set $\tilde{f}_n = f_n$.

If $y_{n+1} \notin \operatorname{im}(\tilde{f}_n)$ then, again by force of the one-point extension property, one can find an $x \in X \setminus \operatorname{dom} \tilde{f}_n$ so that the bijection

$$f_{n+1}(a) = \begin{cases} \tilde{f}_n(a), & \text{if } a \in \operatorname{dom} f_n, \\ y_{n+1}, & \text{if } a = x \end{cases}$$

is distance preserving. Again, in the case where $y_{n+1} \in \operatorname{im}(\tilde{f}_n)$, we simply put $f_{n+1} = \tilde{f}_n$.

Clearly, $\operatorname{dom} f_n$ is finite, $f_n|\operatorname{dom} f_{n-1} = f_{n-1}$, and $\{x_0, x_1, \ldots, x_{n+1}\} \subseteq \operatorname{dom} f_{n+1}$ and $\{y_0, y_1, \ldots, y_{n+1}\} \subseteq \operatorname{im} f_{n+1}$, which finishes the proof. □

Proof of Theorem 3.4.4. Necessity. Suppose X is an Urysohn space. Let $A \subseteq X$ be a finite metric subspace, and let $A' = A \cup \{a\}$ be a one-point metric space extension of A. Because of universality of X, there is a metric copy A'' of A' in X. The restriction of an isometry $A'' \cong A'$ to A is an isometry between two finite subspaces of X, and therefore it extends to a global isometry j of X. This is the desired metric embedding of A'.

Sufficiency. Assume X has the one-point extension property. According to Lemma 3.4.5, X is ultrahomogeneous. If Y is a countable metric space, a metric embedding $Y \hookrightarrow X$ is constructed by an obvious recursion using the one-point extension property. Finally, such an embedding can be extended from a countable everywhere dense subspace to an arbitrary separable metric space, and so X is universal. □

Now we can also give

Proof of uniqueness part of Theorem 3.4.2. Let X and Y be two Urysohn metric spaces. By Theorem 3.4.4, they possess the one-point extension property and, starting with the trivial isometry between any two singletons in X and Y and using Lemma 3.4.5, one concludes that there is a global isometry between X and Y. □

The existence part depends on the Lemma 3.4.10 below. It is useful to make the following observation first.

Exercise 3.4.6. Given two metric spaces X, Y and an isometry i between subspaces $A \subset X$, $B \subset Y$, there exists a metric space $Z = X \coprod_i Y$, the *coproduct of X and Y amalgamated along i*, such that X and Y are both metric subspaces of Z and for all $a \in A$ the images of a and of $i(a)$ in Z coincide. Moreover, the metric on Z is the maximal among all metrics satisfying the above property.

Hint: form the union $X \cup Y$ and identify all pairs of points $(a, i(a))$, $a \in A$ between themselves to obtain Z, and define the metric on Z as follows: on the image of X and of Y it is d_X and d_Y correspondingly, while in the case where $x \in X \setminus Y$ and $y \in Y \setminus X$, let

$$d_Z(x, y) = \inf\{d_X(x, a) + d_Y(a, y)\}.$$

Definition 3.4.7. Let us say that a metric space X has the *approximate one-point extension property* if for every finite metric subspace $A \subseteq X$, every abstract one-point metric space extension $A' = A \cup \{b\}$, and each $\varepsilon > 0$, there exists $b' \in X$ such that for every $a \in A$ one has

$$|d_X(a, b) - d_{A'}(a, b)| < \varepsilon.$$

Lemma 3.4.8. *The completion of a metric space X with the approximate one-point extension property is again such.*

Proof. Let A be a finite metric subspace of the completion $\hat{X}$ of X, and let $A' = A \cup \{b\}$ be an abstract one-point extension of A. Let $\varepsilon > 0$. Find a bijection $i \colon A \to X$ such that $d(a, i(a)) < \varepsilon/3$ for all $a \in A$. Amalgamate the metric spaces A' and $A \cup i(A)$ along A, and let $B = i(A) \cup \{b\}$ considered as a metric subspace of the amalgam. Since X has the approximate one-point extension property, there is an element $b' \in X$ with $|d(b', i(a)) - d(b, i(a))| < \varepsilon/3$ for all $a \in A$. Consequently, $|d(b', a) - d(b, a)| < \varepsilon$ for all $a \in A$. $\square$

Lemma 3.4.9. *A complete metric space X having the approximate one-point extension property has the one-point extension property.*

Proof. Let $A \subseteq X$ be finite, and let $A' = A \cup \{b\}$ be an abstract one-point extension of A. It is clearly enough to choose a Cauchy sequence (b_i) of elements of X in such a way that for all $a \in A$ the distances $d_X(a, b_i)$ converge to $d_X(a, b)$. This is achieved by recursion. As a basis, choose $b_1 \in X$ so that the one-point extension

$A \cup \{b_1\}$ approximates A' to within $\varepsilon = 1/2$. Now assume that the elements $b_i \in X$, $i \leq n$ have been chosen so that every space $A \cup \{b_i\}$ approximates A' to within $\varepsilon = 2^{-i}$. Form the metric coproduct, Z_n, of $A \cup \{b_1, b_2, \ldots, b_n\}$ with $A' = A \cup \{b\}$ amalgamated along A, and consider it as an abstract one-point metric extension of $A \cup \{b_1, b_2, \ldots, b_n\}$. Using the assumption on X, choose a $b_{i+1} \in X$ so that the distances $d_X(x, b_{i+1})$ and $d_{Z_n}(x, b)$ differ by less than 2^{i+1} for all $x \in A \cup \{b_1, b_2, \ldots, b_n\}$. Clearly, the sequence (b_i) has the desired properties. □

As a consequence of Lemma 3.4.8 and Lemma 3.4.9, we obtain:

Lemma 3.4.10. *The completion of a metric space X with the approximate one-point extension property has the one-point extension property.* □

Proof of existence part of Theorem 3.4.2. The Urysohn space will be constructed by a recursive procedure with a subsequent completion.

As a base for the recursion, start with a singleton metric space $X_1 = \{*\}$. Form a sequence of one-point extensions of X_1 with metrics taking *rational* distances in such a way that every such possible one-point extension appears on the list *infinitely many times*. Denote this list by $\mathcal{L}_1$. Thus, $\mathcal{L}_1 = (\mathcal{L}_1(k))_{k \geq 1}$, where each $\mathcal{L}_1(k)$ is a metric space of the form $X_1 \cup \{b_n\}$.

Suppose a metric space X_n with n elements and a sequence $\mathcal{L}_n = (\mathcal{L}_n(k))_{k \geq n}$ of abstract one-point extensions of X_n with metrics taking rational distances has been formed, so that every such extension appears infinitely often. Set $X_{n+1} = \mathcal{L}_n(n)$. Now form a sequence of all abstract one-point extensions of X_{n+1}, denoted $\mathcal{L}_{n+1} = (X_{n+1} \cup \{b_k\})_{k \geq n+1}$, in such a way that

- for every $k \geq n+1$, the metric subspace $X_n \cup \{b_k\}$ of $X_{n+1} \cup \{b_k\}$ is isometric to the space $\mathcal{L}_n(k)$;
- every abstract one-point extension of X_{n+1} appears on the list infinitely many times.

Clearly, both can be achieved using Exercise 3.4.6 and the recursion hypothesis.

At the end of the recursion, form the union $\cup_{n\geq 1}X_n$, which we will denote by $\mathbb{U}_{\mathbb{Q}}$.

The latter metric space has the one-point extension property provided all distances take rational values. Indeed, let A be a finite metric subspace of $\cup_{n\geq 1}X_n$, and let $A' = A\cup\{b\}$ be an abstract one-point extension of A. Let n be such that $A\subseteq X_{n+1} = \mathcal{L}_n(n)$. For some $k\geq n+1$, the one-point extension $X_{n+1}\cup\{b_k\} = \mathcal{L}_{n+1}(k)$ is isometric to the amalgam of X_{n+1} and A' along A, under the correspondence sending b to b_k. The metric space $\mathcal{L}_{n+1}(k)$ is contained, as a metric subspace, in $\mathcal{L}_k(k) = X_{k+1}$ and consequently

$$A\cup\{b_k\}\subset X_{k+1}\subset\mathbb{U}_{\mathbb{Q}},$$

and $A\cup\{b_k\}$ is isometric to A' under the correspondence $b\mapsto b_k$, as required.

As a consequence, $\mathbb{U}_{\mathbb{Q}}$ has an approximate one-point extension property, and by Lemma 3.4.10 the metric completion of $\mathbb{U}_{\mathbb{Q}}$ has the exact one-point extension property, and since it is separable, it is the Urysohn metric space by Theorem 3.4.4. □

Remark 3.4.11. The countable metric space $\mathbb{U}_{\mathbb{Q}}$ constructed in the above proof is quite remarkable on its own right. This is the so-called *rational Urysohn space*, and it is ultrahomogeneous and universal for all countable metric spaces with rational distances. The Urysohn metric space $\mathbb{U}$ is the metric completion of $\mathbb{U}_{\mathbb{Q}}$. We will meet this space again in Chapter 4.

Remark 3.4.12. The Urysohn metric space, $\mathbb{U}$, as well as the rational Urysohn space $\mathbb{U}_{\mathbb{Q}}$, were introduced by Pavel S. Urysohn shortly before his death in 1924, and his results were published posthumously [168, 169]. With the exception of a 1945 paper by Sierpinski [159], the construction was completely forgotten by the mathematical world until the 1986 Katětov's work [96]. However, the real renaissance of the Urysohn space has begun at around 1990 beginning with the results by Uspenskij [172, 173], followed by Vershik [180, 181, 182], Gromov [76], Bogatyi [20, 21], and now others. In particular, the small book [53] by Gao and Kechris contains a good introduction to the Urysohn space.

Remark 3.4.13. There is still no known concerte realization of the Urysohn space, and finding such a model is one of the most interesting open problems of the theory, which was mentioned by such mathematicians as Fréchet [50], p. 100 and P.S. Alexandroff [170], and currently is being advertised by Vershik. The only bit of constructive knowledge about the structure of the Urysohn space is that it is homeomorphic to the Hilbert space (Uspenskij [175]).

There is however a hope that such a model can be found, because a simpler relative of the Urysohn space $\mathbb{U}$, the so-called random graph R, has one (in fact, at least four different models, cf. [25]). The random graph can be viewed as the universal Urysohn metric space whose metric only takes values $0, 1, 2$, and it will appear, too, in Chapter 4.

Lemma 3.4.14. *The Urysohn metric space $\mathbb{U}$ has the following, in apparence stronger, form of the one-point extension property: whenever $K \subseteq X$ is a compact metric subspace of X and $K' = K \cup \{b\}$ is an abstract one-point metric space extension of K, the embedding $K \hookrightarrow X$ extends to an isometric embedding $K' \hookrightarrow X$.*

Proof. Choose an increasing sequence of finite subsets F_n of K so that each F_n forms a 2^{-n}-net. Using the same amalgamation technique and Exercise 3.4.6, one can choose a Cauchy sequence (b_n) in X in such a way that the map from $F_n \cup \{b\}$ that is identity on F_n and takes $b \mapsto b_n$ is an isometric embedding. □

Proposition 3.4.15. *Every isometry between two compact subsets of the Urysohn metric space $\mathbb{U}$ extends to a global isometry of $\mathbb{U}$ with itself.*

Proof, sketch. The familiar back-and-forth argument as in the proof of Lemma 3.4.5, this time using the one-point extension property of Lemma 3.4.14. □

Now we will present another proof of the existence of the Urysohn space, belonging to Katětov [96], whose work had in effect opened the current chapter in theory of the Urysohn space. The following result belongs to folklore.

Lemma 3.4.16. *Let $f: X \to \mathbb{R}$ be a function on a metric space X. There exists a one-point metric extension $X \cup \{b\}$ of X such that f is the distance from b if and only if*

$$|f(x) - f(y)| \leq d_X(x, y) \leq f(x) + f(y) \tag{3.19}$$

for all $x, y \in X$.

Proof. The necessity of the condition is quite obvious, because the left-hand side inequality follows from the triangle inequality, while the right-hand side one simply means that f is 1-Lipschitz.

To prove sufficiency, notice first that if f vanishes at some point x_0, then for all $y \in X$, the inequalities (3.19) result in that $f(x) = d(x, x_0)$. Therefore, we can assume $f \neq 0$. Define a metric on the set $X \cup \{b\}$, $b \notin X$, by the condition $d(x, b) = d(b, x) = f(x)$ for all $x \in X$. The first two axioms of a metric being obvious, the triangle inequality is checked by considering two separate cases: $d(x, y) \leq d(x, b) + d(b, y)$ and $d(x, b) \leq d(x, y) + d(b, y)$, which correspond to the two sides of the double inequality (3.19). ☐

Definition 3.4.17 (Katětov). Say that a 1-Lipschitz real-valued function f on a metric space X is *controlled by a metric subspace* $Y \subseteq X$ if for every $x \in X$

$$f(x) = \inf\{\rho(x, y) + f(y) \colon y \in Y\}.$$

Put otherwise, f is the largest 1-Lipschitz function on X with the given restriction to Y.

Example 3.4.18. The distance function $x \mapsto \rho(x, x_0)$ from a point $x_0 \in X$ is controlled by a singleton, $\{x_0\}$.

Let X be an arbitrary metric space. Denote by $\mathscr{F}(X)$ the collection of all functions $f: X \to \mathbb{R}$ controlled by some finite subset of X (depending on the function) and satisfying the double inequality (3.19).

Exercise 3.4.19. Let f and g be two functions on a metric space satisfying Eq. (3.19). Verify that the value of the supremum metric

$$d(f, g) = \sup\{d(f(x), g(x)) \mid x \in X\}$$

is finite.

Exercise 3.4.20. Show that the *Kuratowski embedding*

$$X \ni x \mapsto [d_x \colon X \ni y \mapsto \rho(x,y) \in \mathbb{R}] \in \mathscr{F}(X) \tag{3.20}$$

is an isometry of X with a subspace of $\mathscr{F}(X)$.

Exercise 3.4.21. Show that the density character of the metric space $\mathscr{F}(X)$ is the same as that of X. In particular, if X is separable, then so is $\mathscr{F}(X)$.

Lemma 3.4.22. *Let A be a finite metric subspace of a metric space X and let $A' = A \cup \{b\}$ is an abstract one-point metric space extension of A. Then the Kuratowski embedding (3.20) of A into $\mathscr{F}(X)$ extends to an isometric embedding $A' \hookrightarrow \mathscr{F}(X)$.*

Proof. The function $f_b = d_{A'}(b,-)$ on A extends to a function $\tilde{f}_B$ on X by the Katětov formula

$$\tilde{f}_b(x) = \inf\{\rho(x,y) + f_b(a) \colon a \in A\}.$$

The function $\tilde{f}_b$ is controlled by A and satisfies the double inequality (3.19) because it is the distance function from the point b in the amalgamation of A' and X along A (cf. Exercise 3.4.6 and Lemma 3.4.16). Therefore, $\tilde{f}_b \in \mathscr{F}(X)$. It remains to verify that for all $a \in A$,

$$d(d_a, \tilde{f}_b) = d_{A'}(a,b),$$

which is left as another exercise. □

Starting with an arbitrary metric space X, one can form an increasing sequence of iterated extensions of the form

$$X, \mathscr{F}(X), \mathscr{F}^2 = \mathscr{F}(\mathscr{F}(X)), \ldots, \mathscr{F}^n(X) = \mathscr{F}(\mathscr{F}^{n-1}(X)), \ldots$$

Denote by $\mathscr{F}^\infty(X)$ the union:

$$\mathscr{F}^\infty(X) = \cup_{i=1}^\infty \mathscr{F}^n(X).$$

Then $\mathscr{F}^\infty(X)$ is, in a natural way, a metric space, containing an isometric copy of X.

Theorem 3.4.23. *Let X be an arbitrary separable metric space. The completion of the space $\mathscr{F}^\infty(X)$ is isometric to the universal Urysohn metric space.*

Proof. As a consequence of Lemma 3.4.22, the space $\mathscr{F}^\infty(X)$ has a one-point extension property, and Lemma 3.4.10 finishes the proof. □

Remark 3.4.24. Other well-known ultrahomogeneous metric spaces are the unit sphere $\mathbb{S}^\infty$ of the infinite-dimensional Hilbert space $\mathcal{H}$ and the infinite-dimensional Hilbert space $\mathcal{H}$ itself ([19], Ch. IV, §38). Cf. also a description of the *universal spherical metric space* in the open problem section 5.2, and the *universal ultrametric space* [133].

Remark 3.4.25. A long-standing open problem, known as the *Banach–Mazur problem* (cf. [6]), asks whether every infinite-dimensional separable Banach space E whose unit sphere is a homogeneous metric space is isometrically isomorphic to the Hilbert space ℓ^2. (In fact, this remains unknown even if one assumes that E is *isomorphic* to ℓ^2 as a topological vector space.)

Exercise 3.4.26. (Explained to me by Gilles Godefroy.) Let E be a Banach space whose unit sphere $\mathbb{S}_E$ is a 2-homogeneous metric space. Show that E is isometrically isomorphic to a Hilbert space. (*Hint:* use Dvoretzky's theorem.)

Remark 3.4.27. There are some obvious modifications of the concept of Urysohn metric space. We have already mentioned the rational Urysohn space $\mathbb{U}_Q$. More generally, one can consider Urysohn spaces in the class of all metric spaces whose distances take values in a specified subset D of $\mathbb{R}_+$. For example, there is an Urysohn metric space $\mathbb{U}_d$ of diameter not exceeding a given positive number d. Another possibility is to consider Urysohn metric spaces in the class of metric spaces whose metrics only take values in the lattice $\varepsilon\mathbb{Z}$, $\varepsilon > 0$.

Certainly, the above are not the only classes of metric spaces for which the Urysohn-type universal objects exist. For instance, the Urysohn metric space for the class of separable spherical metric spaces of a fixed diameter in the sense of Blumenthal [19] is the sphere $\mathbb{S}^\infty$. The Hilbert space ℓ^2 plays the role of Urysohn metric space for the class of separable metric spaces embeddable into Hilbert spaces.

Remark 3.4.28. An interesting approach to the Urysohn space that we do not consider here at all was proposed by Vershik who regards the Urysohn space as a generic, or random, metric space. Here is one of his results. Denote by M the set of all metrics on a countably infinite set ω. Let $P(M)$ denote the Polish space of all probability measures on M. Then, for a generic measure $\mu \in P(M)$ (in the sense of Baire category), the completion of the metric space (X, d) is isometric to the Urysohn space $\mathbb{U}$ μ-almost surely in $d \in M$. We refer the reader to a very interesting theory developed in [180] and especially [182]. Cf. also [183].

3.4.2 Isometry group

Remark 3.4.29. Let X be a metric space. Then the topology of simple convergence on the group $\mathrm{Iso}\,(X)$ of isometries of X with itself coincides with the compact-open topology and makes $\mathrm{Iso}\,(X)$ into a topological group.

Indeed, given a compact $K \subseteq X$ and an $\varepsilon > 0$, find a finite $\varepsilon/3$-net N for X. Then every mapping f contained in the basic neighbourhood of identity in the topology of simple converence of the form

$$V[N; \varepsilon/3] = \{f \in \mathrm{Iso}\,(X) \colon \forall x \in N,\ d(x, f(x)) < \varepsilon/3\}$$

has the property: for each $x \in K$, $d(x, f(x)) < \varepsilon$. This shows that the two topologies coincide. The rest is obvious.

If X is separable (and thus second-countable), then so is $\mathrm{Iso}\,(X)$.

Remark 3.4.30. In general, the action of $\mathrm{Iso}\,(X)$ by translations on the space of bounded uniformly continuous (or Lipschitz) functions on X, equipped with the supremum norm, is discontinuous.

However, the following is true. Notice that for an isometry $g \in \mathrm{Iso}\,(X)$ and a function f controlled by a finite subset $A \subseteq X$ the function

$${}^g f \colon x \mapsto f(g^{-1}x)$$

is controlled by the finite subset $g^{-1}(A)$. Therefore, left translations define an action of the isometry group $\mathrm{Iso}\,(X)$ in the metric space $\mathscr{F}(X)$, by isometries.

Lemma 3.4.31. *The action of the group* $\mathrm{Iso}\,(X)$ *in the metric space* $\mathscr{F}(X)$ *by left translations is continuous and moreover defines a topological group embedding* $\mathrm{Iso}\,(X) \hookrightarrow \mathrm{Iso}\,(\mathscr{F}(X))$.

Proof. To verify continuity of the action, it suffices to observe that if a function $f \in \mathscr{F}(X)$ is controlled by a finite $A \subseteq X$, then the translation $g \circ f$ does not differ from f by more than ε at any point of X, provided $g \in V[A;\varepsilon]$. Therefore, the orbit maps

$$G \ni g \mapsto {}^g f \in \mathscr{F}(X)$$

are all continuous, and since this is an action by isometries, it is continuous.

To check that the action determines a topological embedding, notice that the image $\tilde{X}$ of X in $\mathscr{F}(X)$ under the Kuratowski embedding $x \mapsto d_x = d(x,-)$ is invariant under the action, and the restriction of the action to $\tilde{X}$ is in fact isomorphic to the standard action of $\mathrm{Iso}\,(X)$ on X:

$$\begin{aligned} {}^g d_x(y) &= d(g^{-1}y, x) \\ &= d(gx, y) \\ &= d_{gx}(y). \end{aligned}$$

Therefore, the topology of simple convergence on $\mathscr{F}(X)$ is at least as fine as the the topology of simple convergence on X. $\square$

Thus, the group $G = \mathrm{Iso}\,(X)$ acts continuously by isometries on every iterated Katětov extension $\mathscr{F}^n(X)$, and the G-spaces $\mathscr{F}^n(X)$ form an increasing equivariant chain. This leads to a continuous isometric action of G on $\mathscr{F}^\infty(X)$, which in its turn extends to a continuous action of $\mathrm{Iso}\,(X)$ on the metric completion $\hat{\mathscr{F}}^\infty(X) \cong \mathbb{U}$.

Definition 3.4.32. Say that a metric subspace Y is g-embedded into a metric space X if there exists an embedding of topological groups $e\colon \mathrm{Iso}\,(Y) \hookrightarrow \mathrm{Iso}\,(X)$ with the property that for every $h \in \mathrm{Iso}\,(Y)$ the isometry $e(h)\colon X \to X$ is an extension of h:

$$e(h)|_X = h.$$

We have just established the following result.

Proposition 3.4.33. *Every separable metric space X can be g-embedded into the complete separable Urysohn metric space $\mathbb{U}$.* □

Now it is time to have the statement of Lemma 2.1.8 strengthened as follows.

Exercise 3.4.34. Let G be a topological group. The left regular representation λ_G of G in the space $\mathrm{RUCB}\,(G)$ of all right uniformly continuous bounded functions on G, given by

$$(g, f) \mapsto {}^g f,$$

determines a topological group embedding of G into $\mathrm{Iso}\,(\mathrm{RUCB}\,(G))$.

Corollary 3.4.35. *Every (second-countable) topological group G embeds into the isometry group of a suitable (separable) metric space as a topological subgroup.* □

We arrive at the following.

Theorem 3.4.36. *The topological group* $\mathrm{Iso}\,(\mathbb{U})$ *is a universal second-countable topological group. In other words, every second-countable topological group G embeds into* $\mathrm{Iso}\,(\mathbb{U})$ *as a topological subgroup.* □

Remark 3.4.37. Recently Julien Melleray has proved [110] that any Polish group G is isomorphic to the subgroup of all isometries of $\mathbb{U}$ that stabilize a suitable closed subset $F \subseteq \mathbb{U}$. Moreover, the group G is isomorphic to $\mathrm{Iso}\,(F)$, the subspace F is g-embedded into $\mathbb{U}$, and every isometry of F admits a unique extension to an isometry of $\mathbb{U}$.

For compact subspaces, Proposition 3.4.33 can be strengthened. Since every isometry between two compact subspaces of $\mathbb{U}$ can be extended to an isometry of $\mathbb{U}$ onto itself (Proposition 3.4.15), we obtain the following useful corollary of Proposition 3.4.33.

Proposition 3.4.38. *Each isometric embedding of a compact metric space into $\mathbb{U}$ is a g-embedding.* □

Remark 3.4.39. With the exception of the old folklore result contained in the Exercise 3.4.34, the remaining results of this subsection, as well as the concept of g-embedding, belong to Uspenskij [172, 173].

3.4.3 Approximation by finite groups

In this subsection we will show that the isometry group of the Urysohn space is a Lévy group, where the approximating compact subgroups can be chosen finite (so that in particular $\mathrm{Iso}\,(\mathbb{U})$ contains an everywhere dense locally finite subgroup).

Let $\Gamma = (V, E)$ be an (undirected) graph. A *weight* on Γ is an assignment of a non-negative real number to every edge, that is, a function $w\colon E \to \mathbb{R}_+$. The pair (Γ, w) forms a *weighted graph*. The *path pseudometric* on a connected weighted graph Γ is the maximal pseudometric on Γ with the property $d(x, y) = w(x, y)$ for any pair of adjacent vertices x, y. Equivalently, the value of $\rho(x, y)$ is given for each $x, y \in V$ by

$$\rho(x, y) = \inf \sum_{i=0}^{N-1} d(a_i, a_{i+1}), \tag{3.21}$$

where the infimum is taken over all natural N and all finite sequences of vertices $x = a_0, a_1, \ldots, a_{N-1}, a_N = b$, with the property that a_i and a_{i+1} are adjacent for all i. The reason why we call the function ρ a *pseudometric* rather than metric, is that it is symmetric and satisfies the triangle inequality, but it may happen that $d(x, y) = 0$ for $x \neq y$, in the case where the weight is allowed to take the value zero.

In particular, if every edge is assigned the weight one, the corresponding path pseudometric is a metric, called the *path metric* on Γ.

Let G be a group, and V a generating subset of G. Assume that V is symmetric ($V = V^{-1}$) and contains the identity, $e \in V$. The *Cayley graph* associated to the pair (G, V) has elements of the group G as vertices, and two of them, $x, y \in G$, are adjacent if and only if $x^{-1}y \in V$. This graph is connected. The corresponding path metric on G is called the *word distance* with regard to the generating set V, and denoted by d_w. The value of the word distance between e and an element x is called the *reduced length* of x with regard to the generating set V, and denoted $\ell_V(x)$. It is simply the smallest integer n such that x belongs to the n-th power V^n of V.

Lemma 3.4.40. *Let G be a group equipped with a left-invariant pseudometric, d. Let V be a finite symmetric generating subset of G containing the identity. Then there is the maximal psuedometric, $\rho = \widetilde{d|V}$, among all left-invariant pseudometrics on G, whose restriction to V is majorized by d. The restrictions of ρ and d to V coincide. If $d|_V$ is a metric on V, then ρ is a metric as well, and for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that $\ell_V(x) \geq N$ implies $\rho(e, x) \geq \varepsilon$.*

Proof. Denote $V^2 = V \cdot V$. Make the Cayley graph associated to the pair (G, V^2) into a weighted graph, by assigning to every edge (x, y) the value $d(x, y) \equiv d(x^{-1}y, e)$. Denote by ρ the corresponding path pseudometric on the weighted graph Γ. To prove left-invariance of ρ, let $x, y, z \in G$. Consider any sequence of elements of G,

$$x = a_0, a_1, \ldots, a_{N-1}, a_N = y, \tag{3.22}$$

where $N \in \mathbb{N}$ and $a_i^{-1}a_{i+1} \in V^2$, $i = 0, 1, \ldots, n-1$. Since for all i the elements za_i, za_{i+1} are adjacent in the Cayley graph ($(za_i)^{-1}za_{i+1} = a_i^{-1}a_{i+1} \in V^2$), one has

$$\begin{aligned} d(zx, zy) &\leq \sum_{i=0}^{n-1} d(za_i, za_{i+1}) \\ &= \sum_{i=0}^{n-1} d(a_i, a_{i+1}), \end{aligned}$$

and taking the infimum over all sequences as in Eq. (3.22) on both sides, one concludes $d(zx, zy) \leq d(x, y)$, which of course implies the equality.

From the definition of ρ, for every $x, y \in V$ one has $\rho(x, y) = \rho(x^{-1}y, e) \leq d(x^{-1}y, e) = d(x, y)$. Now let ς be any left-invariant pseudometric on G whose restriction to V is majorized by d. If $a, b \in G$ are such that $a^{-1}b \in V^2$, then for some $c, d \in V$ one has $a^{-1}b =$

$c^{-1}d$, and

$$\begin{aligned}
\varsigma(a,b) &= \varsigma(a^{-1}b,e)\\
&= \varsigma(c^{-1}d,e)\\
&= \varsigma(c,d)\\
&\leq d(c,d)\\
&= d(c^{-1}d,e)\\
&= d(a^{-1}b,e)\\
&= d(a,b).
\end{aligned}$$

For every sequence as in Eq. (3.22), one now has

$$\begin{aligned}
\varsigma(x,y) &\leq \sum_{i=0}^{n-1}\varsigma(a_i,a_{i+1})\\
&= \sum_{i=0}^{n-1}\varsigma(a_i^{-1}a_{i+1},e)\\
&\leq \sum_{i=0}^{n-1} d(a_i,a_{i+1}),
\end{aligned}$$

and by taking the infimum over all such finite sequences on both sides, one concludes that

$$\varsigma(x,y)\leq\rho(x,y),$$

as required. Applying this conclusion to $\varsigma=d$, we have $\rho|_V=d|_V$.

Assuming that $d|_V$ is a metric, all the weights on the Cayley graph assume strictly positive values only, and consequently ρ is a metric. In such a case, denote by δ the smallest value taken by d between pairs of distinct elements of V. Clearly, for every $x\in G$ one has $\rho(e,x)\geq\delta\ell_V(x)$, and the proof is finished. □

Lemma 3.4.41. *Let ρ be a left-invariant metric on a group G, and let $H\triangleleft G$ be a normal subgroup. The formula*

$$\begin{aligned}
\bar\rho(xH,yH) &:= \inf_{h_1,h_2\in H}\rho(xh_1,yh_2) \qquad (3.23)\\
&\equiv \inf_{h_1,h_2\in H}\rho(h_1x,h_2y)\\
&\equiv \inf_{h\in H}\rho(hx,y)
\end{aligned}$$

defines a left-invariant pseudometric on the factor-group G/H. This is the largest pseudometric on G/H with respect to which the quotient homomorphism $G \to G/H$ is 1-Lipschitz.

Proof. The triangle inequality follows from the fact that, for all $h' \in H$,

$$\begin{aligned}
\bar\rho(xH, yH) &= \inf_{h\in H} \rho(hx, y) \\
&\leq \inf_{h\in H} [\rho(hx, h'z) + \rho(h'z, y)] \\
&= \inf_{h\in H} \rho(hx, h'z) + \rho(h'z, y) \\
&= \inf_{h\in H} \rho(h'^{-1}hx, z) + \rho(h'z, y) \\
&= \bar\rho(xH, zH) + \rho(h'z, y),
\end{aligned}$$

and the infimum of the r.h.s. taken over all $h' \in H$ equals $\bar\rho(xH, zH) + \bar\rho(zH, yH)$. Left-invariance of $\bar\rho$ is obvious. If d is a pseudometric on G/H making the quotient homomorphism into a 1-Lipschitz map, then $d(xH, yH) \leq \rho(xh_1, yh_2)$ for all $x, y \in G$, $h_1, h_2 \in H$, and therefore $d(xH, yH) \leq \bar\rho(xH, yH)$. □

Remark 3.4.42. We will make a distinction between the notion of a *distance-preserving map* $f\colon X \to Y$ between two pseudometric spaces, which has the property $d_Y(fx, fy) = d_X(x, y)$ for all $x, y \in X$, and an *isometry*, that is, a distance-preserving bijection. A distance-preserving map need not be an injection: for instance, if d is a left-invariant pseudometric on a group G, then the natural map $G \to G/d$ is distance-preserving, onto, but not necessarily an isometry.

A group G is *residually finite* if it admits a separating family of homomorphisms into finite groups, or, equivalently, if for every $x \in G$, $x \neq e$, there exists a normal subgroup $H \triangleleft G$ of finite index. Every free group is residually finite, and the free product of two residually finite groups is residually finite. (Cf. e.g. [105] or [81].)

Lemma 3.4.43. *Let G be a residually finite group equipped with a left-invariant pseudometric d, and let $V \subseteq G$ be a finite symmetric subset containing the identity. Suppose the restriction $d|_V$ is a metric, and let $\rho = \widetilde{d|_V}$ be the maximal left-invariant metric with $\rho|_V = d|_V$.*

There exists a normal subgroup $H \triangleleft G$ of finite index with the property that the restriction of the quotient homomorphism $G \to G/H$ to V is an isometry with regard to the quotient-metric $\bar{\rho}$.

Proof. Let $\delta > 0$ be the smallest distance between any pair of distinct elements of V, and let $\Delta \geq \delta$ be the diameter of V. Let $N \in \mathbb{N}_+$ be so large that $(N-2)\delta > \Delta$. The subset $V^N \subseteq G$ is finite, and, since the intersection of finitely many subgroups of finite index has finite index (Poincaré's theorem), one can choose a normal subgroup $H \triangleleft G$ of finite index such that $H \cap V^N = \{e\}$. As a consequence, one has for every $x, y \in V$ and $h \in H$, $h \neq e$:

$$d(hx, y) \geq (N-2)\delta > \Delta,$$

that is, the distance $\bar{\rho}(xH, yH)$ is realized at x, y, and $\bar{\rho}(xH, yH) = \rho(x, y)$. □

Lemma 3.4.44. *Let X be a finite subset of the Urysohn space $\mathbb{U}$, and let a finite group G act on X by isometries and freely. Let f be an isometry of $\mathbb{U}$, and let $\varepsilon > 0$. There exist a finite group $\tilde{G}$ containing G as a subgroup, an element $\tilde{f} \in \tilde{G} \setminus G$, and a finite metric space Y, $X \subseteq Y \subset \mathbb{U}$, upon which $\tilde{G}$ acts by isometries and freely, extending the original action of G on X and so that for all $x \in X$ one has $d(\tilde{f}x, fx) < \varepsilon$.*

Proof. Since every compact set such as X is g-embedded into the Urysohn space (Proposition 3.4.38), one can extend the action of G by isometries from X to all of $\mathbb{U}$. Let $n = |X|$. Choose any element $\xi \in \mathbb{U}$ and isometries f_x, $x \in X$ such that $f_x(\xi) = x$. Let F_{n+1} be the free group on $n+1$ generators f, f_x, $x \in X$. Denote by $F = G * F_{n+1}$ the free product of two groups. There is a unique homomorhism $G * F_{n+1} \to \mathrm{Iso}\,(\mathbb{U})$, which sends all elements of G, as well as the generators of F_{n+1}, to the respective global isometries of $\mathbb{U}$. In this way, F acts on $\mathbb{U}$ by isometries.

The formula

$$d_\xi(g, h) := d_{\mathbb{U}}(g(\xi), h(\xi)), \quad g, h \in G * \mathbb{Z},$$

defines a left-invariant pseudometric d_ξ on the group $G * F_{n+1}$. The evaluation map $\phi \mapsto \phi(\xi)$ from F to $\mathbb{U}$ is not, in general, an injection, but it is distance-preserving.

From the beginning, without loss in generality, one could have assumed that the image $f(X)$ does not meet X, by replacing f, if necessary, with an isometry f' such that the image $f'(X)$ does not intersect X, and yet for every $x \in X$ one has $d_{\mathbb{U}}(f(x), f'(x)) < \varepsilon$. Consequently, the subspace $V = (G \cup \{f\}) \cdot \{f_x : x \in X\}$ of the pseudometric group F is in fact a metric subspace, and the mapping $\phi \mapsto \phi(\xi)$ establishes an isometry between V and $X \cup f(X)$. Let $\rho = \widetilde{d|_V}$ be the maximal left-invariant pseudometric such that $\rho|_V = d|_V$. (Lemma 3.4.40.)

Choose a normal subgroup $H \triangleleft F$ of finite index as in Lemma 3.4.43: if the finite factor-group F/H is equipped with the factor-pseudometric $\bar{\rho}$, then the restriction of the factor-homomorphism $\pi \colon F \to F/H$ to V is an isometry. In addition, one can clearly choose H so that $H \cap G = \{e\}$, and thus $\pi|G$ is a monomorphism. Denote $\tilde{f} = \pi(f)$, and let $\tilde{G}$ be the (finite) subgroup generated by $G \cup \{\tilde{f}\}$.

Let $Y = (G/H)/\bar{\rho}$ be the metric homogeneous space corresponding to the left-invariant pseudometric $\bar{\rho}$ on G/H. The group $\tilde{G}$ acts on it on the left by isometries, the space Y contains V as a metric subspace, and in particular the original metric G-space X. The metric space $X \cup f(X)$ isometrically embeds into Y in a natural way under the correspondence $f(x) \mapsto \tilde{f}(x)$. Extend the initial embedding of $X \cup f(X)$ into $\mathbb{U}$ to an embedding $Y \hookrightarrow \mathbb{U}$. Being finite, Y is g-embedded into $\mathbb{U}$, that is, the action of $\tilde{G}$ by isometries extends to a global action on $\mathbb{U}$. This action satisfies the required properties. □

We will be using again the group $L^0(X, \mu; G)$ of equivalence classes of measurable maps from a probability measure space (X, μ) to a Polish topological group G.

Exercise 3.4.45. Suppose that $Y = (Y, \rho)$ is a (separable) metric space. Show that for every value of $\lambda > 0$, the formula

$$\mathrm{me}_\lambda(f, g) = \inf\{\varepsilon > 0 \mid \mu\{x \in X : \rho(f(x), g(x)) > \varepsilon\} < \lambda\varepsilon\} \quad (3.24)$$

determines a metric on $L_0(X, \mu; Y)$, generating the topology of convergence in measure (so in particular such metrics for different $\lambda > 0$ are all equivalent).

Exercise 3.4.46. Suppose a finite group G act by isometries on a finite metric space Y. Then the topological group $L_0(X, \mu; G)$ acts continuously by isometries on the metric space $L(X,\mu;Y)$ equipped with any of the metrics me_λ as in Eq. (3.24), where the action is defined pointwise:

$$(g\cdot f)(x) := g(x)\cdot f(x),\ g\in L(X,\mu;G),\ f\in L(X,\mu;Y).$$

Remark 3.4.47. For infinite metric spaces Y the result requires an additional assumption of the action of G (boundedness) to remain true.

Theorem 3.4.48 (Vershik [184]). *The isometry group* $\mathrm{Iso}\,(\mathbb{U})$ *of the Urysohn space, equipped with the standard Polish topology, contains an everywhere dense locally finite subgroup.*

Proof. Choose an everywhere dense subset $F=\{f_i\colon i\in\mathbb{N}\}$ of $\mathrm{Iso}\,(\mathbb{U})$ and an everywhere dense subset $X=\{x_i\colon i\in\mathbb{N}\}$ of $\mathbb{U}$. Set $G_1=\{e\}$, along with a trivial action on $\mathbb{U}$.

Assume that for an $n\in\mathbb{N}_+$ a finite group G_n and an action σ_n by isometries on $\mathbb{U}$ have been chosen, so that the restriction of the action to the G_n-orbit of $\{x_1,x_2,\dots,x_n)$ is free.

Using Lemma 3.4.44, choose a finite group G_{n+1} containing (an isomorphic copy) of G_n, an element $\tilde f_n\in G_{n+1}$ such that G_{n+1} is generated by $G_n\cup\{\tilde f_n\}$, and an action σ_{n+1} of G_{n+1} on $\mathbb{U}$ by isometries such that for every $j=1,2,\dots,m$ and each $g\in G_n$ one has

$$\sigma_n(g)x_j=\sigma_{n+1}(g)x_j,$$

and

$$d_{\mathbb{U}}(f_i(x_j),\tilde f_n(x_j)) < 2^{-i}.$$

The group $G=\cup_{i=1}^{\infty}G_n$ is locally finite. Let $g\in G$. For every $i\in\mathbb{N}_+$, the value $g\cdot x_i$ is well-defined as the limit of an eventually constant sequence, and determines an isometry from an everywhere dense subset $X\subset\mathbb{U}$ into $\mathbb{U}$. Consequently, it extends uniquely to an isometry from $\mathbb{U}$ into itself. If $g,h\in G$, then the isometry determined by gh is the composition of isometries determined by g and h: every $x\in X$ has the property $(gh)(x)=g(h(x))$, once $x=x_i$, $i\le N$, and $g,h\in G_N$, and this property extends over all of $\mathbb{U}$. Thus, G acts on $\mathbb{U}$ by isometries (which are therefore onto).

Finally, notice that G is everywhere dense in $\mathrm{Iso}\,(\mathbb{U})$. It is enough to consider the basic open sets of the form

$$V[x_{i_1},\dots,x_{i_N},\varepsilon] = \{f \in \mathrm{Iso}\,(\mathbb{U})\colon d(f(x_i),x_{i_1}) < \varepsilon, \quad i=1,2,\dots,N\},$$

for arbitrary finite subcollections $x_{i_1},\dots,x_{i_N}$ of X and $\varepsilon > 0$. Since F is everywhere dense in $\mathrm{Iso}\,(\mathbb{U})$, one can find isometries $f_{k_1},\dots,f_{k_N} \in F$ such that $d(f_{k_i}(x_i),x_{i_1}) < \varepsilon$ for all $i = 1,2,\dots,N$. Let $M \in \mathbb{N}$ be the maximum of integers

$$i_1, i_2, \dots, i_N,\ k_1,\dots,k_N,\ \lfloor \log_2 \varepsilon \rfloor.$$

There are elements $\tilde f_{k_1},\dots,\tilde f_{k_N}$ in the finite group G_M whose effect on each element $x_{i_1},\dots,x_{i_N}$ is at a distance less than ε from that produced by the corresponding isometry f_{k_i}. This settles the claim. $\square$

A slight refinement of the argument leads to the following result.

Theorem 3.4.49 (Pestov [146]). *The isometry group* $\mathrm{Iso}\,(\mathbb{U})$ *of the Urysohn space, equipped with the standard Polish topology, is a Lévy group.*

Proof. Here we intersperse the construction steps in the previous Theorem 3.4.48 with that contained in Exercise 3.4.46. Choose F, X, and G_1 as before. Assume that for an $n \in \mathbb{N}_+$ a finite group G_n and an action σ_n by isometries on $\mathbb{U}$ have been chosen. Denote by X_n the G_n-orbit of the set $\{x_1, x_2, \dots, x_n\}$, and let d_n be the diameter of X_n. Choose $m_n \in \mathbb{N}$ so that

$$\frac{2^{m_n}}{16 d_i^2} \geq n. \tag{3.25}$$

Let $\mu^{\otimes m}$ denote the uniform probability measure on the Bernoulli space $\{0,1\}^m$. The metric space $L(\{0,1\}^{m_n}, \mu^{\otimes n}; X_n)$, equipped with (say) the me_1-metric, contains X_n as a subspace of constant functions, therefore one can embed $L(\{0,1\}^{m_n}, \mu^{\otimes n}; X_n)$ into $\mathbb{U}$ so as to extend the embedding $X_n \hookrightarrow \mathbb{U}$. The group $L(\{0,1\}^{m_n}, \mu^{\otimes n}; G_n)$ acts on $L(\{0,1\}^{m_n}, \mu^{\otimes n}; X_n)$ by isometries, and every embedding of a compact subspace into $\mathbb{U}$ is a g-embedding, so one can simultaneously extend $L(\{0,1\}^{m_n}, \mu^{\otimes n}; G_n)$ to a global group of isometries of

$\mathbb{U}$. Denote this group $\tilde{G}_n$, and the action by $\tilde{\sigma}_n$; the restriction $\tilde{\sigma}_n|_{G_n}$ coincides with σ_n. Now construct the group G_{n+1} and its action σ_{n+1} by isometries exactly as in the proof of Theorem 3.4.48, but beginning with $\tilde{G}_n$ instead of G_n. The union $G = \cup_{i=1}^{\infty} G_n = \cup_{i=1}^{\infty} \tilde{G}_n$ is an everywhere dense locally finite subgroup of $\mathrm{Iso}\,(\mathbb{U})$, and it only remains to show that the groups $\tilde{G}_n$, $n \in \mathbb{N}_+$, form a Lévy family with regard to the uniform structure inherited from $\mathrm{Iso}\,(\mathbb{U})$.

First of all, consider the groups $\tilde{G}_n = L(\{0,1\}^{m_n}, \mu^{\otimes n}; G_n) = (G_n)^{2^{m_n}}$, equipped with the Hamming distance, that is, the ℓ^1-metric formed with regard to the discrete $\{0,1\}$-valued metric on G_n. If V_ε is the ε-neighbourhood of the identity, then for every $g \in V_\varepsilon$ and each $x \in L(\{0,1\}^{m_n}, \mu^{\otimes n}; X_n)$ one has $d(g \cdot x, x) < \varepsilon \cdot d_n$, where $d_n = \mathrm{diam}\, X_n$. Consequently, if $g \in V_{\varepsilon/d_n}$, then $d(g \cdot x, x) < \varepsilon \cdot d_n$.

Let now $V[x_1, \dots, x_n; \varepsilon]$ be a standard neighbourhood of the identity in $\mathrm{Iso}\,(\mathbb{U})$, where one can assume that $x_i \in X$ are from the chosen dense subset of $\mathbb{U}$. For all $N \geq n$, if $A_N \subseteq \tilde{G}_n$ contains at least half of all elements, the set $V_{\varepsilon/d_n} A$ is of measure at least $1 - 2e^{-2^{m_n}\varepsilon^2/16d_i^2}$, according to Theorem 3.2.10. The set $V[x_1, \dots, x_n; \varepsilon] \cdot A_N$ has a measure that is at least as big. According to the choice of numbers m_n (Eq. 3.25), the family of groups $\tilde{G}_n$ is normal Lévy. $\square$

Remark 3.4.50. Extreme amenability of the Polish group $\mathrm{Iso}\,(\mathbb{U})$ was established by the present author in the 2002 paper [142], where it was shown that it forms a generalized Lévy group. Theorem 3.4.49 is a recent refinement, obtained by the author [146] after Theorem 3.4.48 was announced by Vershik [184]. Vershik's result was in its turn largely motivated by a will to extend the well-known result by Hrushovski [94] about extending partial isomorphisms of finite graphs to a result about extending partial isometries of finite metric spaces. This generalization was also obtained by Vershik in the same preprint [184] and, independently, by Solecki [160]; the result is closely related to the existence of a locally finite subgroup, but the two don't seem quite equivalent to each other. The proof of Theorem 3.4.48 proposed here is different from the original proof given in [184].

Remark 3.4.51. Extreme amenability of the Polish group $\mathrm{Iso}\,(\mathbb{U})$ distinguishes it from another example of a universal Polish group, also obtained by Uspenskij [171]: the homeomorphism group $\mathrm{Homeo}\,(\mathbb{I}^{\aleph_0})$

of the Hilbert cube. Indeed, the latter group acts minimally on a non-trivial compact space.

Remark 3.4.52. Here are some other properties of the isometry group $\mathrm{Iso}\,(\mathbb{U})$ of the Urysohn space. This group is topologically simple (that is, contains no proper closed normal subgroups) and admits no strictly coarser group topology other than the indiscrete one [173].

Modulo the existence of topological groups without non-trivial unitary representations (e.g. the exotic groups mentioned in the Introduction), it follows from these results that the group $\mathrm{Iso}\,(\mathbb{U})$ does not admit any non-trivial strongly continuous unitary representations. (Exercise.) Consequently, $\mathrm{Iso}\,(\mathbb{U})$ admits no nontrivial near-actions on measure spaces, even measure class preserving ones, because every such action gives rise to a non-trivial strongly continuous unitary representation (a quasi-regular one).

Cameron and Vershik have shown in [27] that the Urysohn space admits the structure of a monothetic Polish group, and moreover in the Polish space of all invariant metric on the additive group $\mathbb{Z}$ of integers those metrics whose completion gives a metric copy of the Urysohn metric are generic (form a dense G_δ-set).

3.5 Lévy groups and spatial actions

Let X be a Polish space, that is, a completely metrizable separable topological space, equipped with a Borel probability measure, and let G be a topological group. Assume now that G acts on X in a measure-preserving way. What does the last sentence mean? It can be interpreted in three different ways.

(1) G acts continuously on X, and the measure μ is G-invariant.

(2) The action of G on X is Borel, that is, the associated mapping

$$G \times X \ni (g, x) \mapsto gx \in X$$

is Borel measurable, and again the measure μ is G-invariant.

(3) The action of G on X is measurable, or a *near-action.* This means that for every $g \in G$ the motion $X \ni x \mapsto gx \in X$ is a bi-measurable map only defined μ-a.e., while for every measurable set $A \subseteq X$ the function $G \ni g \mapsto \mu(gA\Delta A) \in \mathbb{R}$ is continuous. In

addition, the identity $g(hx) = (gh)x$ holds for μ-a.e. $x \in X$ and every $g, h \in X$. Again, the G-invariance of the measure μ admits only one meaning: for every measurable subset $A \subseteq X$ and every $g \in G$, the set $g \cdot A$, defined up to a μ-null set, has measure $\mu(A)$.

Exercise 3.5.1. Measure-preserving near-actions of a topological group G on a measure space (X, μ) are in a natural one-to-one correspondence with continuous homomorphisms from G to the group $\mathrm{Aut}\,(X, \mu)$, equipped with the weak topology.

Obviously, (1) $\implies$ (2). Shortly we will see (Exercise 3.5.3) that (for Polish groups) (2) $\implies$ (3), so the class of measurable actions is the most general of all.

The difference between (1) and (2) is, surprisingly, not very considerable, according to the following classical result (Th. 5.2.1 in [11]).

Theorem 3.5.2 (Bekker and Kechris). *Let G be a Polish group acting in a Borel way on a Polish space X. Then there exist a continuous action of G on a compact space K and an embedding of X into K as a Borel subset and a Borel G-subspace.* $\square$

Of course, every invariant Borel measure μ on X can be then considered as an invariant Borel measure on K. Therefore, Borel actions are in a sense just as good as continuous actions on compact spaces. Actions in (2) are referred to as *spatial* actions.

Exercise 3.5.3. Let G be a Polish group G acting in a Borel fashion on a Polish space X, equipped with an invariant probability measure μ. Deduce from the theorem of Bekker and Kechris that the action of G on the measure space (X, μ) is weakly continuous, that is, the mapping

$$G \ni g \mapsto \mu(gA \Delta A) \in \mathbb{R}$$

is continuous for every Borel subset $A \subseteq X$.

(*Hint:* after having applied Th. 3.5.2, consider the continuous action of G on the compact space of all probability measures on X, paying particular attention to the normalized restriction of μ to A.)

The transition from near-actions to spatial actions is less straightforwad. Here one has a classical result of Mackey [104], Varadarajan [176], and Ramsay [150].

Theorem 3.5.4 (Mackey, Varadarajan, and Ramsay). *Every near-action of a locally compact second countable group admits a spatial model.* □

It was unknown for a while if the result can be extended to arbitrary (non locally compact) Polish groups. The answer turned out to be in the negative. First, in 1987 A.M. Vershik has proved [179] that for every near-action of the group $U(\ell^2)$ (with the strong topology), there is no set of full measure that is invariant under all $u \in U(\ell^2)$. Since the group $U(\ell^2)$ admits faithful near-actions (cf. Example 3.5.12 below), this result would give an example of a near-action without spatial model, though the authors of [63] believe the proof in [179] to be incomplete (Remark 1.9). Later, Howard Becker has circulated a handwritten note [10] in which he proved that there is a near action of the Polish group (in our notation) $L^0(X,\mu;\mathbb{Z}_2)$ that has no spatial model. (The author is grateful for A.S. Kechris for this reference.)

Notice that both of the above examples are Polish Lévy groups. As shown by Glasner, Tsirelson and Weiss [63], cf. also [62], this is no coincidence, moreover the answer to the above problem is very strongly in the negative.

Theorem 3.5.5 (Glasner, Tsirelson and Weiss [63, 62]). *Let G be a Polish Lévy group acting in a Borel way on a Polish space X. Then every invariant probability measure μ on X is supported on the set of G-fixed points.*

Let us call a measure-preserving near-action of a topological group G *nontrivial* if the image of G in the group $\mathrm{Aut}\,(X,\mu)$ under the homomorphism associated to the near-action is different from the trivial subgroup.

Corollary 3.5.6. *No non-trivial measure-preserving near-action of a Lévy group admits a spatil model.* □

This important result also demonstrates for the first same time how the behaviour of Lévy groups is different not only from that of locally compact groups, but from that of more general extremely amenable groups as well (cf. Example 3.5.17 and Remark 3.5.18 below).

The proof of Theorem 3.5.5 is based on the following new notion, introduced by the same three authors, which is a strengthened form of ergodicity suited for actions of "infinite-dimensional" groups.

Definition 3.5.7. Say that a measure-preserving near-action of a topological group G on a probability space (X, μ) is *whirly* if for every two measurable sets A, B of positive measure, every neighbourhood V of identity contains an element g such that $\mu(A \cap gB) > 0$.

Let G be a separable topological group acting measurably in a measure-preserving way on a probability space (X, μ), let V be an open subset in G, and let A be a measurable subset of X. Define the set $V \cdot A$ as

$$V \cdot A = \cup_{n=1}^{\infty} v_i \cdot A,$$

where $\{v_i \colon i \in \mathbb{N}_+\}$ is any everywhere dense subset in V.

Exercise 3.5.8. Verify that, up to a set of measure zero, the above definition does not depend on the choice of an everywhere dense subset of V.

Now the definition of a whirly action can be rephrased as follows.

Exercise 3.5.9. Show that a measure-preserving near-action of a separable group G on a probability space (X, μ) is whirly if and only if for every set $A \subseteq X$ of positive measure and every neighbourhood of identity V in G the set $V \cdot A$ has full measure.

The concept is devoid of content in the classical case, as the following shows.

Exercise 3.5.10. Prove that every whirly measure-preserving near-action of a locally compact group is trivial.

At the same time, whirly actions of infinite-dimensional groups certainly exist.

Exercise 3.5.11. Show that the standard near-action of the group $\mathrm{Aut}\,(X, \mu)$ with the uniform topology (consequently, with the weak topology as well) on a standard Borel space equipped with a non-atomic Borel probability measure (X, μ) is whirly.

Here is another, perhaps more interesting, example.

Example 3.5.12. Equip the countable power $\mathbb{R}^{\aleph_0}$ of the real line with the product topology and the resulting Borel structure, and with the infinite product of the Gaussian measures

$$\gamma^\infty = \gamma^{\otimes \aleph_0},$$

where γ is the standard normal distribution on the real line with mean zero and variance 1. The real Hilbert space ℓ^2 is densely embedded into $\mathbb{R}^{\aleph_0}$ in a usual way, and the standard action of the orthogonal group $O(\ell^2)$ (with the strong topology) on ℓ^2 extends to a near-action of $O(\ell^2)$ on the probability measure space $(\mathbb{R}^{\aleph_0}, \gamma^\infty)$. The action is measure-preserving and ergodic. Cf. [63] for details.

The result by Glasner, Tsirelson and Weiss means in particular that there is no spatial model for the above action of the infinite orthogonal group.

Here is the core observation.

Theorem 3.5.13 (Glasner and Weiss [62]). *Every ergodic measure-preserving near-action of a Lévy group G on a probability space (X, μ) is whirly.*

Proof. Let (K_n) be an approximating Lévy sequence of compact subgroups of G, where we can assume without loss in generality that $G = \cup_{n=1}^\infty K_n$. Choose a left-invariant compatible metric d on G, and let $\mathcal{O}_\varepsilon$ denote the ε-neighbourhood of e in G. Denote by α_n the concentration function of K_n equipped with the normalized Haar measure and the restriction of the metric d.

Let $n \in \mathbb{N}$. The K_n-space (X, μ) admits a spatial model by force of the Mackey–Varadarajan–Ramsay Theorem 3.5.4, and the partition of (the spatial model for) X into the K_n-orbits is measurable, because the orbit space X/K_n is Polish. For μ-a.e. $x \in X$, the conditional measure μ_x on the orbit $K_n \cdot x$ is easily seen to be K_n-invariant as a consequence of Rokhlin's theorem 3.2.24, and therefore must coincide with the push-forward measure of the normalized Haar measure on K_n under the orbit map $K_n \ni g \mapsto g \cdot x$. Consequently, for every $\varepsilon > 0$ and every $A \subseteq K_n \cdot x$, if $\mu_x(A) \geq 1/2$, then

$$\mu_x((\mathcal{O}_\varepsilon \cdot A) \cap (K_n \cdot x)) \geq 1 - \alpha_n(\varepsilon)$$

μ-a.e. in $x \in X$, where $\mathcal{O}_\varepsilon$ stands for the ε-neighbourhood of e in G.

Let $\varepsilon, \delta > 0$ be arbitrary. If $A \subseteq X$ is of positive measure then, because of ergodicity of the action of G, there is an $n \in \mathbb{N}$ such that for all $N \geq n$ the measure of the set of all K_N-orbits whose intersection with A is of measure $\geq \varepsilon$ is at least $1-\delta$. This property, jointly with concentration of measure, implies that for all N large enough, the set of all K_N-orbits whose intersection with $\mathcal{O}_\varepsilon \cdot A$ is of measure $\geq 1-\delta$ has measure (in X/K_N) at least $1-\delta$. In particular, again by Rokhlin's theorem, $\mu(\mathcal{O}_\varepsilon \cdot A) > (1-\delta)^2$. Since this is true for all $\delta > 0$, we condlude that

$$\mu(\mathcal{O}_\varepsilon \cdot A) = 1$$

for every $\varepsilon > 0$, that is, the action of G on (X, μ) is whirly. □

The following observation is practically evident.

Lemma 3.5.14. *Let a topological group G act in a measurable and whirly way on a measure space (X, μ). Then every measurable π-uniform function f on X is essentially constant (that is, constant μ-a.e.).*

Proof. For every $\varepsilon > 0$ there is a neighbourhood V_ε of the identity in G such that for μ-almost all $x \in X$ and all $g \in V_\varepsilon$, $|f(x) - f(gx)| < \varepsilon$. Let J denote an arbitrary open interval in $\mathbb{R}$ of length ε, and let $A_J = f^{-1}(J)$. Whenever $\mu(A_J) > 0$, one has for every $\varepsilon > 0$

$$f(V_\varepsilon \cdot A_J) \subseteq J_\varepsilon,$$

where J_ε is the ε-neighbourhood of J in $\mathbb{R}$. At the same time, the set $V_\varepsilon \cdot A_J$ has full measure. Thus, for every $\varepsilon > 0$ the oscillation of f on a set of full measure is less than 3ε, and so f is constant on a set of full measure. □

Lemma 3.5.15. *Let a Lévy group G act continuously on a compact space X, preserving a probability measure, so that the action is ergodic. Then the measure is supported at one point (which is of course G-fixed).*

Proof. Suppose a continuous action of a topological group on a compact space preserves a probability measure μ and is ergodic. If the support of the measure is non-trivial, there exist $x \in \operatorname{supp} \mu$ and $g \in G$ with $gx \neq x$. Choose a continuous function f on X such that $f(x) \neq f(gx)$. Since both x and gx belong to the support of μ, there are arbitrarily small disjoint neighbourhoods of x and of gx of strictly positive measure each. Thus, f cannot be essentially constant. At the same time, f is (measurable and) π-uniform. In view of Theorem 3.5.13 and Lemma 3.5.14, G is not a Lévy group. □

Let a Polish group G act in a Borel, measure-preserving and ergodic way on a Polish space X. By the Becker–Kechris theorem 3.5.2, one can assume without loss in generality that X is a dense invariant Borel subset contained in a compact space K, upon which G acts continuously. The extended action is also ergodic, and one can now apply Lemma 3.5.15 to deduce the following result.

Theorem 3.5.16. *Let a Borel action of a Polish Lévy group G on a Polish space X equipped with a probability measure μ be ergodic measure preserving. Then μ is supported at a G-fixed point.* □

Example 3.5.17. The above result is no longer true if G is a Lévy group (even metrizable) that is not necessarily Polish (completely metrizable).

Consider the following example. Let S_n denote, as in Subsection 3.3.2, the symmetric group of rank n acting on $[0,1]$ by interval exchange transformations (that is, mapping each dyadic interval of rank n onto a dyadic interval of rank n via a translation). The groups S_n, $n \in \mathbb{N}$ form an increasing sequence inside the group $\operatorname{Aut}(\mathbb{I}, \lambda)$. The union $G = \cup_{n=1}^{\infty} S_n$, equipped with the uniform (or weak) topology, is a metrizable Lévy group, which is clearly not Polish in both cases for Baire category reasons. The group G acts in a Borel way on the interval $\mathbb{I} = [0,1]$, preserving the Lebesgue measure, this action is ergodic and therefore whirly (which can also be easily seen directly), and at the same time spatial.

As a consequence, the action of G on the interval has no compact continuous model, stressing that the Polish property of the acting group G is essential for the Becker–Kechris theorem 3.5.2 as well.

It may seem that such a compact model is provided by the Cantor set C realized as the subspace of the unit interval and the action of G permuting the bits of the Cantor space between themselves, but this action is in fact discontinuous with regard to the uniform topology.

Remark 3.5.18. A Polish extremely amenable group that is not a Lévy group can act continuously and ergodically on a compact space equipped with an invariant probability measure. An example is the shift (regular) action of the group $\mathrm{Aut}\,(\mathbb{Q},\leq)$ on the Cantor space $\{0,1\}^{\mathbb{Q}}$ equipped with the product measure.

Proof of Theorem 3.5.5. By the Becker-Kechris theorem 3.5.2, one can assume without loss in generality that the Polish Lévy group G acts continuously on a compact space X, equipped with an invariant probability measure μ. Choose a countable dense subgroup $\tilde{G}$ of G.

Let $\{\mu_y : y \in Y\}$ be the ergodic decomposition of the measure μ with regard to the action of the countable group $\tilde{G}$, that is, a family of pairwise singular $\tilde{G}$-invariant, $\tilde{G}$-ergodic probability measures μ_y on X indexed by elements of a standard Borel space Y equipped with a probability measure ν so that for every measurable $A \subseteq X$, the mapping $y \mapsto \mu_y(A)$ is Borel on Y and $\mu(A) = \int_Y \mu_y(A)\, d\nu(y)$. (The existence of such a decomposition can be deduced from the Rokhlin theorem 3.2.24 applied to the countable orbit equivalence relation generated by the action of $\tilde{G}$. For a more general result, see [74].)

The continuous action of G on X extends to a continuous action of G on the compact space $P(X)$ of all probability measures on X by Proposition 2.7.2, and consequently every measure μ_y, $y \in Y$ remains invariant — and clearly ergodic — under the action of the entire Polish group G.

By Theorem 3.5.16, $\mathrm{supp}\,\mu_y$ is a singleton. Since $\mathrm{supp}\,\mu_y$ is G-invariant, its only element is G-fixed. Consequently, the support of the measure μ is the closure of $\cup_{y\in Y}\mathrm{supp}\,\mu_y$, every element of which set is a G-fixed point. □

The study of invariant means on a given topological (usually locally compact) group occupies an important place in abstract harmonic analysis. One of the few known results in the infinite-dimensional theory is the following corollary of the theorem of Glasner–Tsirelson–Weiss.

Theorem 3.5.19. *Let G be a Polish Lévy group. Then every left-invariant mean on the space* $\mathrm{RUCB}\,(G)$ *is contained in the weak* closed convex hull of the set of multiplicative left-invariant means.*

Idea behind the proof. Left-invariant means on $\mathrm{RUCB}\,(G)$ are exactly invariant probability measures on $X = \mathcal{S}(G)$. It is well-known that extreme points in the compact convex set of all invariant probability measures are ergodic measures. According to Theorem 3.5.5, those are Dirac measures, hence fixed points in $\mathcal{S}(G)$, hence the result follows. □

Remark 3.5.20. Strictly speaking, the above argument would only work under the (unrealistic) assumption that $\mathcal{S}(G)$ is metrizable. One needs therefore to decompose $\mathcal{S}(G)$ into the inverse limit of metrizable compact G-spaces before being able to apply Theorem 3.5.5. This is done using standard and well-established techniques.

The latter result was conjectured by Michael Cowling and the author back in 2000, but the necessary tools for the proof only became available after Theorem 3.5.5 has been established.

Remark 3.5.21. Concepts and results are mostly taken from the papers [63, 62], with the exception of Example 3.5.17. The original proof of Theorem 3.5.5 given in [63] was different, because Lemma 3.5.14 and Theorem 3.5.13 were only proved by Glasner and Weiss in a later paper [62]. The proofs proposed in this subsection somewhat differ from those in [62] either.

Chapter 4

Minimal flows

4.1 Universal minimal flow

By Zorn's lemma, every compact G-flow contains a minimal G-subflow. This applies of course to the greatest ambit $\mathcal{S}(G)$ of a topological group G. However, even a G-ambit may contain more than one minimal subflow.

Exercise 4.1.1. Give such an example. For instance, show that the greatest ambit of the group $\mathrm{Aut}\,(\mathbb{Q},\leq)$ with the standard Polish topology contains more than one fixed point.

Every minimal subflow, say M, of the greatest ambit $\mathcal{S}(G)$ has the universal property of the following kind: if X is a minimal compact G-space, there exists a morphism of G-spaces from M onto X. This is a consequence of the universal property of the greatest ambit itsel (Corollary 2.3.10). If M and M' are two minimal subflows of $\mathcal{S}(G)$, there are morphisms of G-spaces $f\colon M \to M'$ and $g\colon M' \to M$. However, apriori they need not be isomorphisms. Fortunately, it happens to be the case, in view of the following property.

Lemma 4.1.2. *Every morphism from a minimal subflow of $\mathcal{S}(G)$ to itself is an isomorphism.*

As a corollary, we obtain:

Theorem 4.1.3. *Every two minimal subflows of the greatest ambit $\mathcal{S}(G)$ are isomorphic as G-spaces.* □

The above result, established by R. Ellis, follows from a more abstract result of his (Th. 4.1.4 below), in combination with the fact that the greatest ambit $\mathcal{S}(G)$ of a topological group G possesses a richer structure than just that of a compact G-space: $\mathcal{S}(G)$ is, in a natural way, what is called a *left semitopological semigroup* (with unit). Here *left semitopological* means that all the *right* translations

$$R_a \colon \mathcal{S}(G) \ni s \mapsto sa \in \mathcal{S}(G), \quad a \in \mathcal{S}(G),$$

are continuous maps.

Here is how a semigroup structure comes into being. Let $a, s \in \mathcal{S}(G)$. By Corollary 2.3.10, there is a unique morphism of G-spaces from $\mathcal{S}(G)$ to itself taking the distinguished point e to a. Denote this map R_a. Now one can introduce a binary operation on the greatest ambit by postulating that the continuous map R_a be a genuine right translation by a for each $a \in \mathcal{S}(G)$. Namely,

$$\mathcal{S}(G) \times \mathcal{S}(G) \ni (x, y) \mapsto xy \stackrel{def}{=} R_y(x) \in \mathcal{S}(G).$$

As a consequence of the uniqueness of the morphism in Corollary 2.3.10, $R_{ab} \equiv R_{R_b(a)} = R_b R_a$, and therefore the law is associative. Since $R_e = \mathrm{I}d$ and $R_a(e) = a$, our law possesses two-sided unit, e.

Since R_x is a morphism of G-spaces, one has for every $g \in G$ and $x \in \mathcal{S}(G)$,

$$gx = R_x(g) = R_x(ge) = gR_x(e) = gx.$$

This means that the semigroup law on $\mathcal{S}(G)$ extends the action of G on $\mathcal{S}(G)$.

Consequently, every closed G-subspace X of $\mathcal{S}(G)$ is a right ideal: for an $a \in X$ the image $R_a(G) = \{ga \colon g \in G\}$ is contained in X, and since R_a is continuous, $R_a(\mathcal{S}(G))$ is contained in the closure of $R_a(G) \subseteq X$. In particular, X is a subsemigroup of $\mathcal{S}(G)$.

Theorem 4.1.4 (Ellis theorem). *Every non-empty compact left semitopological semigroup contains an idempotent.*

Proof. Let S be such a semigroup as in the assumptions of the theorem. It contains a minimal closed non-empty subsemigroup, T, by Zorn's lemma. We claim that any element $a \in T$ is an idempotent (that is, T is a trivial semigroup). Indeed, $Ta = T$ by minimality of T, consequently the closed subsemigroup $\{t \in T : ta = a\}$ is non-empty and must coincide with T, whence $a^2 = a$. □

Proof of Lemma 4.1.2. Let M be a minimal subflow of $\mathcal{S}(G)$. Being a closed subsemigroup of the greatest ambit, M contains an idempotent, p, by Ellis theorem 4.1.4. By minimality of M, one has $Mp = M$ and therefore

$$R_p(x) = R_p(yp) = yp^2 = yp = x$$

for all $x \in M$. If $f : M \to M$ is a morphism of G-spaces, then $f \circ R_p$ is a morphism from $\mathcal{S}(G)$ to $M \subseteq \mathcal{S}(G)$, and hence $f \circ R_p = R_b$ for $b = f(p) = f(R_p(e))$ (uniqueness of morphisms). Since the restriction of $f \circ R_p$ to M coincides with f, one concludes that $f(x) = xb$ for all $x \in M$. Again we notice that $Mb = M$, and so for some $c \in M$, $cb = p$ (the idempotent). The morphism $g = R_c : M \to M$ is a right inverse to f, because

$$fg(x) = xcb = xp = x.$$

It is a well-known exercise in group theory to conclude that the semigroup of all endomorphisms of M is in fact a group. □

Theorem 4.1.3 leads to the following concept.

Definition 4.1.5. A minimal compact G-flow isomorphic to a minimal subflow of the greatest ambit $\mathcal{S}(G)$, is called the *universal minimal flow* of the topological group G and denoted by $\mathscr{M}(G)$.

Here is a useful – and obvious – reformulation of the concept.

Lemma 4.1.6. *A minimal G-flow X is universal if and only if it admits a morphism to any compact G-flow.*

Proof. Necessity is clear. Let X have the property above. There are morphisms $f : \mathscr{M}(G) \to X$ and $g : X \to \mathscr{M}(G)$, whose composition $g \circ f$ is an isomorphism by Lemma 4.1.2. Thus, each one of them is an isomorphism. □

Example 4.1.7. A topological group G is extremely amenable if and only if the universal minimal flow of G is trivial:

$$\mathscr{M}(G) = \{*\}.$$

Example 4.1.8. If a topological group G is compact, then the universal minimal flow $\mathscr{M}(G)$ is G itself, with the action by left translations. Slightly more generally, if G is precompact (that is, totally bounded), then $\mathscr{M}(G)$ is the compact group completion of G.

However, if a group G is locally compact non-compact, the universal minimal flow $\mathscr{M}(G)$ is a very large and highly complicated compact space, in particular it is never metrizable. Quite remarkably, for some non-extremely amenable "large" groups the universal minimal flows are now computed explicitely, and they are metrizable compacta.

Remark 4.1.9. The standard reference here is Ellis's Lectures on Topological Dynamics [43]. Cf. also books by Auslander [5] and de Vries [186]. Our presentation here follows [174].

4.2 Group of circle homeomorphisms

4.2.1 Example

As the first example (also historically), consider the Polish group $\mathrm{Homeo}_+(\mathbb{S}^1)$ of all orientation-preserving homeomorphisms of the circle, equipped with the usual compact-open topology.

Theorem 4.2.1. *The circle $\mathbb{S}^1$, equipped with the standard action of* $\mathrm{Homeo}_+(\mathbb{S}^1)$*, forms the universal minimal flow for the latter topological group.*

Proof. The proof is based on the existence of a "large" extremely amenable subgroup in the topological group $G = \mathrm{Homeo}_+(\mathbb{S}^1)$. Select a distinguished point $z \in \mathbb{S}^1$. The isotropy subgroup

$$\mathrm{St}_z = \{h \in G \colon h(z) = z\}$$

is isomorphic, as a topological group, to $\mathrm{Homeo}_+[0,1]$ and is therefore extremely amenable. Denote $H = \mathrm{St}_z$. The homogeneous factor-space G/H is canonically isomorphic, as a G-space, to $\mathbb{S}^1$ under the map

$$G/H \ni hH \mapsto h(z) \in \mathbb{S}^1.$$

Suppose now G acts on compact space X. The extremely amenable subgroup H has a fixed point, ξ, in X. According to Lemma 2.1.5, the orbit map

$$G \ni g \mapsto g(\xi) \in X \tag{4.1}$$

is right uniformly continuous. Further, the orbit mapping factors through the homogeneous factor-space $\mathbb{S}^1 = G/H$, because if $g, f \in G$ belong to the same left H-coset, then $g(\xi) = f(\xi)$, that is, the mapping

$$G/H \ni gH \mapsto g(\xi) \in X \tag{4.2}$$

is well-defined. By the definition of the factor-topology on G/H, and taking into account the continuity of mapping (4.1), the mapping (4.2) is continuous. Clearly, (4.2) is G-equivariant as well. We have verified that $\mathbb{S}^1$ admits a morphism to an arbitrary compact G-space, and since $\mathbb{S}^1$ is of course minimal, it is a universal minimal G-flow by Lemma 4.1.6. □

That's all concerning the example. However, while we still remember the simple argument above, it makes sense to extend it to a more general situation, just because it will apply to a score of other examples later in the chapter.

Remark 4.2.2. The result was obtained in [138].

4.2.2 Large extremely amenable subgroups

Let a topological group G act continuously on a compact space X, and let $H = \mathrm{St}_z$ be the isotropy subgroup of a point $z \in X$. Just as before, the right uniformly continuous (Lemma 2.1.5) orbit mapping

$$\mathrm{o}rb_z \colon G \ni g \mapsto gz \in X$$

factors through the homogeneous factor-space G/H, that is, $\mathrm{o}rb_z$ is the composition of the canonical factor-map $\pi\colon G \to G/H$ and the mapping

$$\delta_z\colon G/H \ni gH \mapsto g(z) \in X.$$

It follows that δ_z is uniformly continuous with regard to the finest uniformity on G/H making the quotient map π right uniformly continuous.

Definition 4.2.3. Let H be a subgroup of a topological group G. The *right uniform structure* on G/H, also called the *standard uniform structure,* is the finest uniformity $\mathcal{U}$ under which the canonical factor-map $\pi\colon G \to G/H$ is right uniformly continuous.

We have already established:

Lemma 4.2.4. *Let a topological group G act continuously on a compact space X, and let $H = \mathrm{St}_z$ be the isotropy subgroup of a point $z \in X$. Then the mapping*

$$\delta_z\colon G/H \ni gH \mapsto g(z) \in X$$

is right uniformly continuous. Besides, it is G-equivariant. □

The right uniform structure on the factor-space behaves rather well.

Lemma 4.2.5. *A basis of the right uniformity on G/H is formed by all images of the elements of the right uniform structure on G under the map $\pi \times \pi\colon G \times G \to (G/H) \times (G/H)$, in other words, by all entourages*

$$\tilde{V}_R = \{(xH, yH)\colon xy^{-1} \in V\}.$$

Proof. It is easy to see that the sets as above form a basis of some uniform structure on G/H (not necessarily separated). The map π is clearly right uniformly continuous with regard to this structure, and this structure ought to be finer than any other uniformity possessing this property. This finishes the proof. □

Remark 4.2.6. Interestingy, for the left uniform structure on the homogeneous factor-space G/H the analogue of Lemma 4.2.5 is no longer true.

Here is another useful observation.

Lemma 4.2.7. *The right uniform structure on the factor-space G/H is always compatible with the factor-topology on G/H.*

Proof. Because of homogeneity, it is enough to verify the statement locally at one point, for instance, the equivalence class of unity in G/H. If V is a neighbourhood of e_G in the group, then $\tilde{V}_R[H] = VH$, and the statement follows, because sets of the form VH form a neighbourhood basis at H. ☐

Example 4.2.8. If $G = \text{Homeo}_+(\mathbb{S}^1)$ and $H = \text{St}_z$, then the right uniform structure on the homogeneous factor-space $G/H = \mathbb{S}^1$ is the standard compatible uniformity on the circle. Indeed, by Lemma 4.2.7, the right uniformity is compatible with the quotient topology, which is easily checked to coincide with the topology of the circle.

By contrast, the left uniformity on $G/H = \mathbb{S}^1$ is trivial (indiscrete).

Example 4.2.9. Here $G = U(\ell^2)$ is the infinite unitary group with the strong operator topology. Fix a point $\xi \in \mathbb{S}^\infty$ ('north pole'), denote $H = \text{St}_\xi$, the isotropy subgroup of ξ:

$$\text{St}_\xi = \{u \in U(\ell_2)\colon u(\xi) = \xi\}.$$

This is a closed subgroup, isomorphic to $U(\ell_2)$ itself. There is a natural identification

$$U(\ell_2)/\text{St}_\xi \ni u\text{St}_\xi \mapsto u(\xi) \in \mathbb{S}^\infty,$$

as topological G-spaces.

The left uniform structure on $\mathbb{S}^\infty$ viewed as a factor-space of the unitary group is the norm uniformity. In other words, basic entourages of diagonal in $\mathcal{U}_L(\mathbb{S}^\infty)$ are of the form

$$V_\varepsilon = \{(\xi,\zeta)\colon \ \|\xi - \zeta\| < \varepsilon\}.$$

The right uniform structure on the sphere $\mathbb{S}^\infty$ is the additive weak uniformity. That is to say, basic entourages of diagonal in $\mathcal{U}_R(\mathbb{S}^\infty)$ are of the form

$$\{(\xi,\zeta)\colon \forall i = 1,2,\dots,n, \quad \langle \xi - \zeta, \eta_i\rangle < \varepsilon\},$$

where $\eta_1, \eta_2, \ldots, \eta_n \in \mathbb{S}^\infty$. In particular, the uniformity $\mathcal{U}_R(\mathbb{S}^\infty)$ is totally bounded, and the completion of the sphere with regard to this uniformity is the closed uint ball $\mathcal{B}^\infty$ with the weak topology.

Let us get back to our situation of a topological group G acting on a compact space X with the isotropy group $H = \mathrm{St}_z$, $z \in X$. The mapping $\delta_z : G/H \to X$, being uniformly continuous and equivariant, determines an equivariant compactification of G/H. There is a morphism f_z of G-spaces from the universal equivariant compactification $\alpha(G/H)$ to X such that $\delta_z = f_z \circ i$, where $i : G/H \to \alpha_G(G/H)$ is the canonical compactification map. Every minimal subflow of X is, by a simple application of Zorn's lemma, the image under f_z of a minimal subflow of $\alpha(G/H)$. Lemma 4.1.6 leads us to formulate the following result.

Theorem 4.2.10 (Pestov [144]). *Let G be a topological group, and let H be a closed subgroup. Suppose the topological group H is extremely amenable. Then any minimal compact G-subspace, $\mathcal{M}$, of the compact G-space $\alpha(G/H)$ is a universal minimal compact G-space.* □

Here is another consequence of Theorem 4.2.10, showing that the class of extremely amenable group is closed under extensions, similarly to the class of amenable groups. This result can be of course proved directly, as was first done during author's discussion with Thierry Giordano and Pierre de la Harpe in April 1999.

Corollary 4.2.11. *Let H be a closed normal subgroup of a topological group G. If topological groups H and G/H are extremely amenable, then so is G.*

Proof. In this case, the universal equivariant compactification $\alpha(G/H)$ is just the greatest ambit $\mathcal{S}(G/H)$ of the extremely amenable factor-group G/H, which contains a fixed point. □

Remark 4.2.12. Results of this subsection are based on the paper [144].

4.3 Infinite symmetric group

As we have seen in subsection 2.2.3, the infinite symmetric group S_∞ (considered as a Polish group) is not extremely amenable, and consequently the universal minimal flow $\mathscr{M}(S_\infty)$ is non-trivial. Theorem 4.2.10 allows to give an explicit description of this flow. This is a result by Glasner and Weiss [60], which was originally obtained by slightly different means, while we here in the rest of the section follow the proof subsequently proposed in [144].

Recall the compact minimal S_∞-space LO introduced by Glasner and Weiss (Example 2.2.18): it consists of characteristic functions of all linear orders on the set ω, equipped with the topology induced from the zero-dimensional compact space $\{0,1\}^{\omega\times\omega}$ and the action of S_∞ by double permutations:

$$(x \ {}^{\sigma}\!\!\prec y) \Leftrightarrow (\sigma^{-1}x \prec \sigma^{-1}y).$$

Theorem 4.3.1 (Glasner and Weiss [60]). *The compact space* LO *forms a universal minimal S_∞-space.*

Before proceeding to the proof, recall the following concept.

Definition 4.3.2 (Hausdorff). A dense linear order on a countable set without the first and the last elements is said to have *type η_ω*.

Lemma 4.3.3. *Every two ordered sets, X and Y, of order type η_ω are order-isomorphic between themselves, and therefore to the set $\mathbb{Q}$ of rationals with the usual order.*

Proof. The proof is simple but instructive, and a similar way of reasoning will be used again later in a different context. This is again the *back-and-forth* (or *shuttle*) argument, identical to that used in the proof of Lemma 3.4.5.

Enumerate $X = \{x_i\}_{i=0}^\infty$ and $Y = \{y_i\}_{i=0}^\infty$, and let $f_0(x_0) = y_0$, as the basis of recursion. The outcome of n-th recursive step will be an order-preserving bijection f_n with finite domain such that $\{x_0, x_1, \dots, x_n\} \subseteq \operatorname{dom} f_n$ and $\{y_0, y_1, \dots, y_n\} \subseteq \operatorname{im} f_n$. Clearly, once this is achieved, the "limit" mapping f, defined by $f(x_n) = f_n(x_n)$ for all $n \in \mathbb{N}$, is an order-preserving bijection between X and Y.

To perform a step of recursion, suppose an order-preserving bijection f_n has been constructed in such a way that the domain $\operatorname{dom} f_n$

is finite and contains the set $\{x_0, x_1, \dots, x_n\}$, while the image $\mathrm{im}\, f_n$ contains $\{y_0, y_1, \dots, y_n\}$. We actually split the n-th step in two.

If $x_{n+1} \notin \mathrm{dom}\, f_n$, then, using density of the linear order on X and finiteness of the image of f_n, one can select an $y \in Y \setminus \mathrm{im}\, f_n$ in such a way that the (clearly bijective) mapping $\tilde{f}_n$, defined by

$$\tilde{f}_n(a) = \begin{cases} f_n(a), & \text{if } a \in \mathrm{dom}\, f_n, \\ \tilde{f}_n(x_{n+1}), & \text{if } a = x_{n+1} \end{cases}$$

is order preserving. If $x_{n+1} \in \mathrm{dom}\, f_n$, then nothing happens and we set $\tilde{f}_n = f_n$.

If $y_{n+1} \notin \mathrm{im}\,(\tilde{f}_n)$ then, since the ordering is dense and the domain of $\tilde{f}_n$ is finite, one can find an $x \in X \setminus \mathrm{dom}\, \tilde{f}_n$ so that the bijection

$$f_{n+1}(a) = \begin{cases} \tilde{f}_n(a), & \text{if } a \in \mathrm{dom}\, f_n, \\ y_{n+1}, & \text{if } a = x \end{cases}$$

is order preserving. Again, in the case where $y_{n+1} \in \mathrm{im}\,(\tilde{f}_n)$, we simply put $f_{n+1} = \tilde{f}_n$.

Clearly, $\mathrm{dom}\, f_n$ is finite, and $\{x_0, x_1, \dots, x_{n+1}\} \subseteq \mathrm{dom}\, f_{n+1}$ and $\{y_0, y_1, \dots, y_{n+1}\} \subseteq \mathrm{im}\, f_{n+1}$, which finishes the proof. $\square$

Now fix an arbitrary order $\prec$ of type η_ω on $\mathbb{N}$. Let $H = \mathrm{Aut}\,(\prec)$ be the subgroup of S_∞ all permutations of $\mathbb{N}$ preserving the linear order $\prec$. This H is a closed topological subgroup of S_∞. It is isomorphic to $\mathrm{Aut}\,(\mathbb{Q}, \leq)$ and therefore extremely amenable by Theorem 2.2.8.

The left factor-space $G/H = S_\infty/\mathrm{Aut}\,(\prec)$ consists of all linear orders on $\mathbb{N}$ that can be obtained from $\prec$ by a permutation. According to Lemma 4.3.3, those are precisely all linear orders of type η_ω. Denote the set of such orders by $\mathrm{LO}(\eta_\omega)$. Thus, the S_∞-space $G/H \cong \mathrm{LO}(\eta_\omega)$ embeds into LO.

Lemma 4.3.4. *The uniform structure on* $\mathrm{LO}(\eta_\omega)$*, induced from the compact space* LO*, coincides with the right uniform structure* $\mathcal{U}_R(S_\infty/\mathrm{Aut}\,(\prec))$.

Proof. Standard basic entourages of the diagonal for the uniformity of LO are of the form

$$W_F := \{(<_1, <_2) \in \mathrm{LO} \times \mathrm{LO} \colon \ <_1|_F = <_2|_F\},$$

where $F = \{x_1, \ldots, x_n\} \subset \mathbb{N}$ is an arbitrary finite subset of natural numbers. Denote by St_F the subgroup of S_∞ stabilizing every $x_i \in F$. This is an open subgroup of S_∞, and such subgroups form a (standard) neighbourhood basis of the identity for the Polish topology on S_∞. The open subgroup St_F determines an element of the right uniformity $\mathcal{U}_R(S_\infty)$:

$$V_F := \{(\sigma, \tau) \in S_\infty \times S_\infty \mid \sigma\tau^{-1} \in \mathrm{St}_F\}.$$

In other words, $(\sigma, \tau) \in V_F$ iff for all $i = 1, 2, \ldots, n$ one has $\tau^{-1}x_i = \sigma^{-1}x_i$.

Denote by $\tilde{V}_F$ the image of V_F under the square of the quotient map $S_\infty \to \mathrm{LO}(\eta_\omega)$. A pair $(<_1, <_2)$ of orders of type η_ω belongs to $\tilde{V}_F$ if and only if for some $(\sigma, \tau) \in V_F$ one has $<_1 = {}^{\sigma}\!\prec$ and $<_2 = {}^{\tau}\!\prec$. Therefore, for every $i, j = 1, 2, \ldots, n$ one has

$$\begin{aligned} x_i \ {}^{\sigma}\!\prec x_j \quad &\Leftrightarrow \quad \sigma^{-1}x_i \prec \sigma^{-1}x_j \\ &\Leftrightarrow \quad \tau^{-1}x_i \prec \tau^{-1}x_j \\ &\Leftrightarrow \quad x_i \ {}^{\tau}\!\prec x_j, \end{aligned}$$

meaning that the restrictions of $<_1$ and $<_2$ to F coincide, that is, $(<_1, <_2) \in W_F$. We have shown that $\tilde{V}_F = W_F$. But according to Lemma 4.2.5, a basis of the right uniformity on G/H is formed by all images of the elements of the right uniform structure on G under the Cartesian square of the quotient map $G \to G/H$. This finishes the proof. □

Lemma 4.3.5. *The universal compactification* $\alpha(S_\infty/Aut(\prec))$ *is isomorphic to* LO.

Proof. By Lemma 4.3.4, $S_\infty/\mathrm{Aut}\,(\prec)$ with the right uniformity embeds into LO as a uniform S_∞-subspace. In particular, the uniformity $\mathcal{U}_R(S_\infty/\mathrm{Aut}\,(\prec))$ is precompact. Clearly, $\mathrm{LO}(\eta_\omega)$ is everywhere dense in LO. As a consequence, the Samuel compactification of the factor-space in question coincides with its completion, that is, LO. □

Proof of Theorem 4.3.1. Follows from Theorem 4.2.10 together with Lemma 4.3.5 modulo the fact that the topological subgroup $H = \mathrm{Aut}\,(\prec)$ is extremely amenable (Theorem 2.2.8) and the G-space LO is clearly minimal. □

Remark 4.3.6. An advantage of the proof of the theorem by Glasner and Weiss presented above is that it remains true for the groups of permutations of an arbitrary infinite set X, not necessarily a countable one. As η, one needs to choose an ultrahomogeneous linear order on X. It is easy to show that such an order always exists, e.g. by finding a linearly ordered field of the same cardinality as X.

The group S_∞ contains, as a dense subgroup, the union of the directed family of permutation subgroups of finite rank, and consequently it is amenable. As a result, there is an invariant probability measure on the compact set LO.

Theorem 4.3.7 ([60]). *An invariant probability measure on* LO *is unique, that is, the action of the Polish group S_∞ on its universal minimal flow is uniquely ergodic.*

Proof. The argument can be made quite elementary as follows. Let μ be a S_∞-invariant probability measure on LO. If $F \subset X$ is a finite subset, then every linear order, $<$, on F determines a cylindrical subset

$$C_< := \{\prec \in \mathrm{LO}\colon \ \prec|_F = <\} \subset \mathrm{LO}.$$

Every two sets of this form, corresponding to different orders on F, are disjoint and can be taken to each other by a suitable permutation. As there are $n!$ of such sets, where $n = |F|$, the μ-measure of each of them must equal $1/n!$. Consequently, the functional $\int d\mu$ is uniquely defined on the characteristic functions of cylinder sets $C_<$, which functions are continuous and separate points, because sets $C_<$ are open and closed and form a basis of open subsets of LO. Now the Stone–Weierstrass theorem implies uniqueness of $\int d\mu$ on all of $C(\mathrm{LO})$. ☐

4.4 Maximal chains and Uspenskij's theorem

The circle is of course the only closed manifold in dimention one, so in view of the example in the previous chapter, it is natural to ask: given a finite-dimensional compact manifold without boundary, X, it is true that X forms the universal minimal flow for the group G of all (orientation preserving, if X is orientable) homeomorphisms of X, equipped with the compact-open topology? Of course X is a minimal G-flow, the question is, is it the largest one?

The answer to this question is in the negative, and is given by the following result [174].

Theorem 4.4.1 (V.V. Uspenskij). *Let G be a topological group whose universal minimal flow $\mathscr{M}(G)$ contains at least 3 points. Then the action of G on $\mathscr{M}(G)$ is not 3-transitive.*

Corollary 4.4.2. *If X is either a closed manifold in dimension ≥ 2, or the Hilbert cube Q, then X is not the universal minimal flow for its group of (orientation preserving) homeomorphisms.* □

Of course the same applies, for example, to the full diffeomorphism groups.

The proof of Theorem 4.4.1 is based on the *maximal chain construction* that in itself has proved fruitful.

Definition 4.4.3. Given a topological space X, denote by $\exp X$ the collection of all non-empty closed subsets of X, and equip it with the *Vietoris topology.* An open basis for this topology consists of all sets of the form

$$[\gamma] = \{F \in \exp X \colon F \subseteq \cup\gamma \text{ and } \forall\, V \in \gamma, \quad F \cap V \neq \emptyset\},$$

as γ runs over all finite collections of open subsets of X. (One can consult any standard textbook in set-theoretic topology, such as Engelking [45].)

Exercise 4.4.4. If X is compact, then so is $\exp X$.

Remark 4.4.5. If X is compact metrizable, the Vietoris topology is generated by the *Hausdorff distance* on $\exp X$:

$$d(F_1, F_2) = \inf\{\varepsilon > 0 \colon F_1 \subseteq (F_2)_\varepsilon, \quad F_2 \subseteq (F_1)_\varepsilon\}.$$

Exercise 4.4.6. If a topological group G acts continuously on a compact space X, then the induced action on $\exp X$ is also continuous.

By a *chain* (of closed subsets of X) we mean a subfamily $\mathscr{C} \subset \exp X$ that is totally ordered by inclusion.

Lemma 4.4.7. *The closure of a chain $\mathscr{C}$ in $\exp X$ is a chain. In particular, every maximal chain of closed subsets of X is a closed subset of* $\exp X$.

Proof. Let $A, B \in \mathrm{cl}\,\mathscr{C}$. Assume A and B are incomparable, that is, $A \setminus B$ and $B \setminus A$ are both non-empty. One can find open sets U, V, W such that $A \subset U \cup V$, $B \subset W \cup V$, $U \cap B = \emptyset$, $W \cap A = \emptyset$, and $A \cap B \subseteq V$, while $A \nsubseteq V$, $B \nsubseteq V$. The sets $\tilde{U} = [U, V]$ and $\tilde{W} = [W, V]$ are open neighbourhoods in $\exp X$ of A and B, respectively. Any two closed subsets, F_1 and F_2, belonging to $\tilde{U}$ and $\tilde{W}$ respectively, are incomparable as subsets of X, which is a contradiction, because both $\tilde{U}$ and $\tilde{W}$ must contain elements of the chain $\mathscr{C}$. $\square$

We will call a chain $\mathscr{C}$ that is closed as a subset of $\exp X$ a *closed chain*. As we have just seen, every maximal chain of closed subsets of X is closed.

Exercise 4.4.8. Show that every closed chain $\mathscr{C}$ is closed under forming intersections and unions of its arbitrary subfamilies, that is, $\mathscr{C}$ is a complete ordered set.

Denote by $\Phi(X)$ the subset of $\exp(\exp X)$ consisting of all maximal chains of non-empty closed subsets of X. By Zorn's lemma, $\Phi(X) \neq \emptyset$.

Lemma 4.4.9. $\Phi(X)$ *is a closed subset of* $\exp(\exp X)$.

Proof. First of all, let us notice that the set of all closed chains is closed in $\exp(\exp X)$. Indeed, if $\mathscr{F} \subseteq \exp(\exp X)$ is not a chain, there are incomparable $A, B \in \mathscr{F}$. Let $\tilde{U}$ and $\tilde{W}$ be constructed as in the proof of Lemma 4.4.7. The set $\mathscr{O} = [\exp X, \tilde{U}, \tilde{W}]$ is an open neighbourhood of the family $\mathscr{F}$ in $\exp(\exp X)$. Every element $\mathscr{G}$ of $\mathscr{O}$ contains F_1, F_2 such that $F_1 \in \tilde{U}$ and $F_2 \in \tilde{W}$, that is, $\mathscr{G}$ is not a chain, proving out claim.

It remains to notice that the set of maximal chains is closed inside the set of closed chains. Let a closed chain $\mathscr{C}$ be non-maximal, that is, there is an $F \in \exp X$, $F \notin \mathscr{C}$ such that $\mathscr{C} \cup F$ is a (closed) chain. According to Exercise 4.4.8, $\mathscr{C}$ contains a gap around F, that is, the set-theoretic difference between the elements of $\mathscr{C}$

$$F_1 = \cup\{K \in \mathscr{C} \colon K \subseteq F\}$$

and

$$F_2 = \cap\{K \in \mathscr{C} : K \supseteq F\}$$

contains at least two elements. (In the case where $F = X$, we of course put $F_2 = X$.) Now one can construct an open neighbourhood $\mathscr{O}$ of $\mathscr{C}$, much in the same way as in the previous paragraph, so that no closed chain contained in $\mathscr{O}$ is maximal because it contains a gap of the same type as $\mathscr{C}$. □

It is also clear that $\Phi(X)$ is G-invariant, and therefore a compact G-space.

Let us make the following simple observation.

Lemma 4.4.10. *Let G be a topological group, and let $z \in \mathscr{M}(G)$ be arbitrary. Then the isotropy subgroup $H = \mathrm{St}_z$ is* relatively extremely amenable *in G in the following sense: every compact G-space X contains a H-fixed point.*

Proof. There is a morphism f of G-spaces from $\mathscr{M}(G)$ onto X, and the point $f(z) \in X$ is H-fixed. □

Proof of Theorem 4.4.1. Let G be a topological group. Choose an arbitrary $z \in \mathscr{M}(G)$. As in Lemma 4.4.10, the isotropy subgroup $H = \mathrm{St}_z$ has a fixed point in the space of maximal chains $\Phi(\mathscr{M}(G))$, that is, H stabilizes a maximal chain, $\mathscr{C}$.

The chain $\mathscr{C}$ contains a singleton, which is clearly invariant under the action of H. Without loss in generality, we can assume that the action of G on $\mathscr{M}(G)$ is 2-transitive, for otherwise there is nothing to prove. Consequently, the stabilizer of z cannot leave fixed any other point, and we conclude: $\{z\} \in \mathscr{C}$.

The chain $\mathscr{C}$ induces on X a pre-order, where $x \prec y$ if and only if for some $F \in \mathscr{C}$ one has $x \in F \not\ni y$. The automorphism group of a chain preserves the preorder $\prec$.

Since $|\mathscr{M}(G)| \geq 3$, there must be at least one element of $\mathscr{C}$ strictly in between the singleton $\{z\}$ and the entire space. Therefore, there exist two points a, b in $\mathscr{M}(G)$ different from z and such that $a \prec b$. No element of H can interchange a and b between themselves, because such a transformation would not preserve the preorder. Therefore, the triple (z, a, b) cannot be mapped on the triple (z, b, a) by a transformation from G, and the action of G on $\mathscr{M}(G)$ is not 3-transitive. □

Remark 4.4.11. The action of $\mathrm{Homeo}_+(\mathbb{S}^1)$ on the circle $\mathbb{S}^1$ is 2-transitive, but not 3-transitive, and thus the chain construction applied to $\mathbb{S}^1$ leads to no problem. There are precisely two maximal chains of closed subsets of the sphere that are invariant under the action of the stabilizer St_1 of the unit element: one consists of all arcs beginning at 1, the other consists of all arcs ending at 1.

Remark 4.4.12. The concepts and results of this section come from Uspenskij's paper [174].

4.5 Automorphism groups of Fraïssé structures

4.5.1 Fraïssé theory

We have seen several objects possessing large and interesting groups of automorphisms: the Urysohn metric space $\mathbb{U}$, the unit sphere $\mathbb{S}^\infty$, the ordered set of rationals $(\mathbb{Q}, \leq)$. What they all have in common, is *ultrahomogeneity:* every isomorphism between finite subobjects extends to a global automorphism of the object. In the case where structures are countable, there is a well-developed general theory of such objects using tools of model theory and logic: *Fraïssé theory* [49].

In particular, it serves as a rich source of "infinite-dimensional" groups of automorphisms, where the methods of our theory can be successfully applied. Besides, Fraïssé theory provides a tool for understanding closed subgroups of the Polish group S_∞.

Definition 4.5.1. A *signature*, L, is a countable family of symbols R_i, $i \in I$ for relations and F_j, $j \in J$ for functions, to each of which there is associated *arity*, which is just a natural number (positive for relations and non-negative for functions).

Definition 4.5.2. A *structure*, $\mathbf{A}$, in a given signature L (or an L-*structure*), is a set A equipped with a family of relations $R_i^A \subseteq A^{n(i)}$, one for each $i \in I$, where $n(i)$ is the arity of R_i, and functions

$f_i^A\colon A^{m(j)} \to A$, $j \in J$, where $m(j)$ is an arity of f_j. The set A is called the *universe* of $\mathbf{A}$.

A structure is called *relational* if the signature L does not contain function symbols.

To every signature there is associated a *first-order language,* which includes all relational and function symbols contained in the signature L, as well as symbols for variables $(x, y, \dots)$, propositional connectives ($\vee$, $\wedge$, $\neg$, $\implies$), quantifiers ($\forall$ and $\exists$), and a binary relation symbol $=$ for equality. In addition, the language is required to contain constant symbols, used to denote relations, functions, and elements of structures in a class that we want to study. (Notice that constants can be identified with functions of arity 0.)

A *sentence* is a well-formed formula in the language L where every occurence of a variable is bound, that is, each variable x is preceded by either $\forall x$ or $\exists x$. Given a structure $\mathbf{A}$ and a sentence ϕ in the language associated to $\mathbf{A}$, one and only one of the following holds: either ϕ is true in $\mathbf{A}$, or it is false.

Here are some examples of structures.

Example 4.5.3. Structures in the empty signature $L = \emptyset$ are just sets. △

Example 4.5.4. Let L consist of a single relational symbol $<$ of arity 2. Every linearly ordered set is an example of an L-structure. So is every partially ordered set of course. Linearly ordered sets are specified by four usual axioms, which are sentences in the language associated to L. △

Example 4.5.5. A *graph* is a pair (V, E), where V is a set whose elements are called *vertices,* and E is a binary relation of *adjacency* on V, which is irreflexive and symmetric. △

Example 4.5.6. A *simplicial complex* C is a structure whose signature consists of relational symbols R_i of arity i each, $i \in \mathbb{N}_+$, which are symmetric under permutations and have the following property: whenever $(x_1, x_2, \dots, x_n) \in R_n$, for all $i \leq n$ each collection of i elements in $\{x_1, x_2, \dots, x_n\}$ satisfies R_i. Also, $R_1(x)$ for every x. A collection $(x_1, x_2, \dots, x_n)$ satisfying the relation R_n is called a *simplex* of C. The supremum of arities of R_i minus one is the *dimension* of C. In particular, graphs are essentially 1-dimensional simpicial complexes. △

Example 4.5.7. Let $S \subseteq \mathbb{R}_+$ be a countable subset of non-negative reals including 0. An *S-valued metric space* is a structure in the signature which has binary relation symbols D_s for every $s \in S$, where $(x, y) \in D_s$ is interpreted as the fact that $d(x, y) = s$. In addition, we require the usual three axioms of a metric to hold. For example, $\mathbb{U}_{\mathbb{Q}}$ is an example of a rational metric space. △

Example 4.5.8. A vector space over a countable field $\mathbb{F}$ can be treated as a structure whose signature includes the functional (constant) symbols 0, binary functional symbol $+$, and a unary function symbol for each scalar $\lambda \in \mathscr{F}$ interpreted as multiplication by λ. △

Example 4.5.9. A *Boolean algebra* is another example of a structure in a signature L that contains two binary function symbols $\vee$ and $\wedge$, an unary function symbol $-$ for the Boolean complementation, and two constant symbols 0 and 1. △

Now we need the notion of a substructure.

Definition 4.5.10. Given two structures, $\mathbf{A}$ and $\mathbf{B}$, in the same signature L, a map $\pi \colon \mathbf{A} \to \mathbf{B}$ is called a *homomorphism* if for every relation symbol R_i and every function symbol f_j one has

$$R_i^B \circ \pi^{n(i)} \cap A^{n(i)} = R_i^A \text{ and } f_j^B \circ \pi^{m(j)}|_B = f_j^B.$$

If in addition π is an injection, it is called a *monomorphism*, and one says that $\mathbf{A}$ is a *substructure* of $\mathbf{B}$. (Notation: $\mathbf{A} \leq \mathbf{B}$.) If π is bijective, it is an *isomorphism*. △

As the intersection of a family of substructures is again a substructure, every subset X of a structure $\mathbf{A}$ is contained in the smallest substructure of $\mathbf{A}$, *generated by X*.

Definition 4.5.11. A structure $\mathbf{F}$ is *locally finite* if every finitely generated substructure is finite. Equivalently: every finite subset of $\mathbf{F}$ is contained in a finite substructure. △

In particular, every relational structure is locally finite.

All the structures in the above examples are locally finite, with a possible exception of Example 4.5.8: a vector space is locally finite if and only if the field $\mathbb{F}$ is finite. From now on, we will only consider locally finite structures.

The following concept is already familiar to us in the context of the ordered set $(\mathbb{Q}, \leq)$ and the Urysohn metric space.

Definition 4.5.12. A locally finite structure $\mathbf{F}$ is called *ultrahomogeneous* if every isomorphism between two finitely generated substructures $\mathbf{B}$, $\mathbf{C}$ of $\mathbf{F}$ extends to an automorphism of $\mathbf{F}$. △

Definition 4.5.13. Given a structure $\mathbf{A}$ in a signature L, the *age* of $\mathbf{A}$, denoted $\mathrm{Age}(\mathbf{A})$, is the class of all finite structures isomorphic to substructures of $\mathbf{A}$. △

The following two results are established by the shuttle argument.

Proposition 4.5.14. *A countable locally finite structure $\mathbf{A}$ is ultrahomogeneous if and only if it satisfies the following* finite extension property: *if $\mathbf{B}, \mathbf{C} \in \mathrm{Age}(\mathbf{A})$, and $\mathbf{B}$ is a substructure of $\mathbf{C}$, then every embedding $\mathbf{B} \leq \mathbf{A}$ extends to an embedding $\mathbf{C} \leq \mathbf{A}$.* □

Theorem 4.5.15. *Every two countable ultrahomogeneous structures in the same signature L having the same age are isomorphic.* □

Now we are going to describe classes of finite structures serving as ages of ultrahomogeneous locally finite structures.

Definition 4.5.16. A structure $\mathbf{F}$ is called a *Fraïssé structure* if it is countably infinite, ultrahomogeneous, and locally finite.

Theorem 4.5.17 (Fraïssé). *A non-empty class $\mathscr{C}$ of finite structures in a signature L is the age of a Fraïssé structure if and only if it satisfies the following conditions:*

1. *$\mathscr{C}$ is closed under isomorphisms;*
2. *$\mathscr{C}$ is* heretidary, *that is, closed under substructures;*
3. *$\mathscr{C}$ contains structures of arbitrarily high finite cardinality;*
4. *$\mathscr{C}$ satisfies the* joint embedding property, *that is, if $\mathbf{A}$, $\mathbf{B}$ are in $\mathscr{C}$, then there is $\mathbf{D} \in \mathscr{C}$ containing $\mathbf{A}$ and $\mathbf{B}$ as substructures;*

5. *$\mathscr{C}$ satisfies the* amalgamation property, *that is, whenever $f_i\colon \mathbf{A} \to \mathbf{B}_i$, $i = 1, 2$ are embeddings of finite structures in $\mathscr{C}$, there is a $\mathbf{D} \in \mathscr{C}$ and embeddings $g_i\colon \mathbf{B}_i \to \mathbf{D}$ such that $g_1 \circ f_1 = g_2 \circ f_2$.*

In this case, there is a unique up to an isomorphism Fraïssé structure $\mathbf{F} = \mathrm{Flim}\,(\mathscr{C})$, called the Fraïssé limit *of the class $\mathscr{C}$, having the property*

$$\mathrm{Age}\,(\mathrm{Flim}\,(\mathscr{C})) = \mathscr{C}.$$

Proof, sketch. The necessity of the conditions is a more or less obvious consequence of the definition of a Fraïssé structure. The uniqueness of the Fraïssé limit of $\mathscr{C}$ will follow from Theorem 4.5.15.

The existence of the Fraïssé limit is modeled on the proof of Theorem 3.4.2, however with one modification. It is no longer enough to consider one-point extensions of a finite structure: at every step of the induction process, one needs to construct a list of all possible *finite* extensions of a given structure, each of which appears on the list infinitely many times.

Once the Fraïssé limit has been constructed, the ultrahomogeneity is verified by means of the back-and-forth argument. $\square$

Definition 4.5.18. A class $\mathscr{C}$ of finite structures in a given signature L is called a *Fraïssé class* if it satisfies the conditions listed in Theorem 4.5.17.

Example 4.5.19. Finite sets form a Fraïssé class, whose Fraïssé limit is just a countably infinite set, ω.

Exercise 4.5.20. Show that the finite totally ordered sets form a Fraïssé class, whose Fraïssé limit is the ordered set order-isomorphic to the set $(\mathbb{Q}, \leq)$ of rational numbers.

Exercise 4.5.21. Show that finite graphs form a Fraïssé class. Its Fraïssé limit is the *random,* or *universal*, graph Γ [149]. Show that it is determined by the following property: for every finite set A of vertices, there are infinitely many vertices adjacent to all vertices in A and infinitely many vertices non-adjacent to any vertex in A.

Exercise 4.5.22. Let V be a countably infinite set of vertices. Construct a graph Γ probabilistically, in the following manner: for every pair x, y of vertices, flip the coin (even a biased one will do, as long as both probabilities are non-zero), and say that x and y form an edge if and only if you get heads. Prove that with probability 1, the resulting graph will be isomorphic to the random graph Γ.

Exercise 4.5.23. Show that the finite-dimensional vector spaces over a finite field $\mathbb{F}$ form a Fraïssé class, whose Fraïssé limit is a countably infinite-dimensional vector space over $\mathbb{F}$.

Example 4.5.24. Finite Boolean algebras form a Fraïssé class. Its Fraïssé limit is the countable atomless Boolean algebra B_∞, which can be also realized as formed by all open-and-closed subsets of the Cantor set.

Example 4.5.25. The Fraïssé limit of the class of all finite rational spaces is the rational Urysohn space $\mathbb{U}_\mathbb{Q}$ (as follows directly from its properties), in particular the finite rational spaces form a Fraïssé class.

Remark 4.5.26. One of the modern standard references to Fraïssé theory is Hodges' book [93].

Now we will turn our attention to the automorphism groups of countable Fraïssé structures.

4.5.2 Extremely amenable subgroups of S_∞

As before, we regard the infinite symmetric group S_∞ as the group of all permutations of a countably infinite set ω, and S_∞ is equipped with the Polish topology of simple convergence on ω considered as a discrete topological space.

Lemma 4.5.27. *Let* $\mathbf{F}$ *be a structure on an infinite set* ω. *The group* $\mathrm{Aut}\,(\mathbf{F})$ *of all automorphisms of* $\mathbf{F}$ *is a closed subgroup of* S_∞.

Proof. Let us show that the complement to $\mathrm{Aut}\,(\mathbf{F})$ is open. If $\sigma \in S_\infty \setminus \mathrm{Aut}\,(\mathbf{F})$, there are a number $n \in \mathbb{N}$ and a finite collection $x = (x_1, x_2, \dots, x_n) \in \omega^n$ such that $\sigma x = (\sigma x_1, \sigma x_2, \dots, \sigma x_n)$ does not belong to the G-orbit of x. The subset $V = \{\tau \in S_\infty : \forall i =$

$1, 2, \ldots, n, \quad \tau x_i = \sigma x_i\}$ is an open neighbourhood of σ, disjoint with $\mathrm{Aut}\,(\mathbf{F})$. □

Definition 4.5.28. Let G be a subgroup of S_∞, where the latter group is viewed as the group of all permutations of an infinite set ω. Put otherwise, G is a group effectively acting on the set ω. One can associate a Fraïssé relational structure $\mathbf{F}_G$ to G as follows.

For each $n \in \mathbb{N}$, let ω^n/G denote the family of all distinct G-orbits of ω^n. The signature L_G will consist of all relation symbols $R_{n,o}$, where $n \in \mathbb{N}$, $o \in \omega^n/G$, where the arity of $R_{n,o}$ is n. The universe of the structure $\mathbf{F}_G$ is ω. An n-tuple $(x_1, x_2, \ldots, x_n) \in \omega^n$ belongs to the relation $R_{n,o}^{\mathbf{F}_G}$ if and only $(x_1, x_2, \ldots, x_n) \in o$ or, equivalently,

$$o = G \cdot (x_1, x_2, \ldots, x_n).$$

Remark 4.5.29. Let $g \in G$. The g-motion

$$A \ni x \mapsto g \cdot x \in A$$

determines an automorphism of the structure L_G. Indeed, for every $n \in \mathbb{N}$ and each n-tuple $x = (x_1, x_2, \ldots, x_n) \in \omega^n$ the G-orbits of x and of $g \cdot x$ coincide, consequently x and $g \cdot x$ either satisfy or fail every relation $R_{n,o}$, $o \in \omega^n/G$ simultaneously.

Thus, one can regard G as a subgroup of $\mathrm{Aut}\,(\mathbf{F}_G)$. (Clearly, if $g \neq e$, then the corresponding motion is a non-trivial automorphism of the structure $\mathbf{F}_G$.)

Lemma 4.5.30. *The automorphism group of the structure $\mathbf{F}_G$ is the closure of G in the natural Polish topology on S_∞.*

Proof. In view of Lemma 4.5.27 and Remark 4.5.29, it is enough to verify that G is dense in $\mathrm{Aut}\,(\mathbf{F}_G)$. Let σ be an automorphism of the structure $\mathbf{F}_G$, and let

$$V[x] = \{\tau \in S_\infty \colon \tau x_i = \sigma x_i \quad \forall i = 1, 2, \ldots, n\}$$

be a standard open neighbourhood of σ in S_∞ determined by a finite collection $x = (x_1, x_2, \ldots, x_n) \in \omega^n$.

Since $\sigma x \in o = G \cdot x$, there is a $g \in G$ such that

$$gx \equiv (gx_1, gx_2, \ldots, gx_n) = (\sigma x_1, \sigma x_2, \ldots, \sigma x_n) \equiv \sigma x.$$

This means that g belongs to V, and the proof is finished. □

Exercise 4.5.31. Using Remark 4.5.29, verify that the structure $\mathbf{F}_G$ is ultrahomogeneous. Consequently, being countably infinite and relational, $\mathbf{F}_G$ is a Fraïssé structure.

The following is an immediate consequence of Lemmas 4.5.27 and 4.5.30 and Exercise 4.5.31.

Theorem 4.5.32. *Closed subgroups of S_∞ are exactly automorphism groups of Fraïssé structures on ω.* □

Remark 4.5.33. The Fraïssé structure $\mathbf{F}_G$ is known as the *canonical structure* associated to a subgroup $G < S_\infty$, cf. 4.1.4. in [93].

Let now $\mathbf{A}$ and $\mathbf{B}$ be structures in a signature L. Denote by

- $\mathrm{Emb}(\mathbf{B}, \mathbf{A})$ the set of all embeddings of $\mathbf{B}$ into $\mathbf{A}$ as a substructure, and by

- $\binom{\mathbf{A}}{\mathbf{B}}$ the set of all pariwise distinct substructures of $\mathbf{A}$ isomorphic to $\mathbf{B}$.

There is a canonical surjection $\mathrm{Emb}(\mathbf{B}, \mathbf{A}) \to \binom{\mathbf{A}}{\mathbf{B}}$. In fact, the latter set is the quotient set $\mathrm{Emb}(\mathbf{B}, \mathbf{A})/\mathrm{Aut}\,(\mathbf{B})$ modulo the action of the group $\mathrm{Aut}\,(\mathbf{B})$, defined by

$$(\sigma, i) \mapsto i \circ \sigma^{-1} \in \mathrm{Aut}\,(\mathbf{B}),$$

where $i \colon \mathbf{B} \to \mathbf{A}$, $\sigma \in \mathrm{Aut}\,(\mathbf{B})$.

Exercise 4.5.34. Let $\mathbf{F}$ be a Fraïssé structure, and let $\mathbf{B} \leq \mathbf{F}$ be a finite substructure. Then the discrete metric space $\mathrm{Emb}(\mathbf{B}, \mathbf{F})$ is a homogeneous factor-space of the automorphism group $\mathrm{Aut}\,(\mathbf{F})$ modulo an open subgroup. Moreover, the topology on the group $\mathrm{Aut}\,(\mathbf{F})$ is the weakest topology making all factor-maps from $\mathrm{Aut}\,(\mathbf{F})$ to all homogeneous spaces of this form continuous.

Remark 4.5.35. If the substructure $\mathbf{B}$ is *rigid,* that is, admits no non-trivial automorphisms, then $\mathrm{Emb}(\mathbf{B}, \mathbf{A})$ coincides with $\binom{\mathbf{A}}{\mathbf{B}}$.

Definition 4.5.36. A structure $\mathbf{A}$ is called an *order structure* if the signature L contains the binary symbol $<$, interpreted as a linear order on $\mathbf{A}$.

Remark 4.5.37. If $\mathbf{A}$ and $\mathbf{B}$ are two order structures in the same signature L and $\mathbf{B}$ is finite, then the sets $\mathrm{Emb}(\mathbf{B},\mathbf{A})$ and $\binom{\mathbf{A}}{\mathbf{B}}$ are identical.

Remark 4.5.38. Let $A \leq B \leq C$ be structures in the same signature L. Using the traditional notation from Ramsey theory, we put

$$\mathbf{C} \to (\mathbf{B})^{\mathbf{A}}_k,$$

if for every colouring

$$c\colon \binom{\mathbf{C}}{\mathbf{A}} \to [k]$$

of the set $\binom{\mathbf{C}}{\mathbf{A}}$ with k colours, there exists an isomorphic copy $\mathbf{B}' \leq \mathbf{C}$ of B such that all elements of $\binom{\mathbf{B}'}{\mathbf{A}}$ have the same colour.

Definition 4.5.39. One says that a Fraíssé class $\mathscr{C}$ has *Ramsey property* if for every finite structure $\mathbf{B} \in \mathscr{C}$ and every substructure $\mathbf{A} \leq \mathbf{B}$ there is a finite structure $\mathbf{C} \in \mathscr{C}$ with the property that for every finite $k \in \mathbb{N}$

$$\mathbf{C} \to (\mathbf{B})^{\mathbf{A}}_k.$$

Of course it is enough to put $k = 2$ here.

The Ramsey property can be immediately restated in the language of finite oscillation stability.

Lemma 4.5.40. *Let $\mathscr{C}$ be a Fraíssé class with the Fraïssé limit $\mathbf{F} = \mathrm{Flim}\,\mathscr{C}$. The following are equivalent:*

1. *The Fraíssé class $\mathscr{C}$ has Ramsey property.*
2. *For every finite structure $\mathbf{A} \in \mathscr{C}$, the set $\binom{\mathbf{F}}{\mathbf{A}}$, equipped with the discrete uniform structure and the natural action of the group* $\mathrm{Aut}\,(\mathbf{F})$*, is finitely oscillation stable.*

Proof. It is enough to consider the bichromatic colourings of $\binom{\mathbf{C}}{\mathbf{A}}$ only. We will use Theorem 1.1.17 with $X = \binom{\mathbf{F}}{\mathbf{A}}$, $G = \mathrm{Aut}\,(\mathbf{F})$, and $V = \Delta$ (the diagonal of X). Item (1) states the finite oscilaltion stability of the pair (X, G). The equivalent item (10) is the Ramsey property of the class $\mathscr{C}$. Indeed, apply (10) with $F = \binom{\mathbf{B}}{\mathbf{A}} \subset X$, to obtain a finite subset $K \subset X$, such that for every bichromatic colouring of K there is a $g \in G$ with $gF \subseteq K$ and gF being monochromatic. Now let $\mathbf{C}$ be a finite structure generated by $\cup\{D \colon \mathbf{D} \in K\}$. Now Definition 4.5.39 of Ramsey property is verified. $\square$

Now we can characterize the extremely amenable groups of automorphisms of Fraïssé structures, as well as extremely amenable closed subgroups of S_∞. First of all, a simple observation.

Lemma 4.5.41. *If a topological subgroup G of S_∞ is extremely amenable, then it preserves a linear order on ω.*

Proof. The action of S_∞ on the compact space LO of all linear orders on ω is continuous (see Sec. 4.3), and consequently LO contains a G-fixed point. $\square$

Theorem 4.5.42. *Let $\mathbf{F}$ be a Fraïssé structure with age $\mathscr{C} = \mathrm{Age}\,(\mathbf{F})$. The automorphism group $\mathrm{Aut}\,(\mathbf{F})$, equipped with its natural Polish topology induced from S_∞, is extremely amenable if and only if the class $\mathscr{C}$ has Ramsey property and all the structures in $\mathscr{C}$ are rigid.*

Proof. $\Rightarrow$: assume that $\mathrm{Aut}\,(\mathbf{F})$ is extremely amenable in its natural Polish topology. Let $\mathbf{A}$ be a finite substructure of $\mathbf{F}$. The action by $\mathrm{Aut}\,(\mathbf{F})$ on the discrete metric space $\binom{\mathbf{F}}{\mathbf{A}}$ is continuous and transitive, and (trivially) by isometries. By Theorem 2.1.10(3), this action is finitely oscillation stable, and by Lemma 4.5.40, the class $\mathscr{C}$ has Ramsey property. Besides, by Lemma 4.5.41, $\mathrm{Aut}\,(\mathbf{F})$ preserves a linear order $\prec$ on the universe of the structure $\mathbf{F}$. It follows that no finite substructure of $\mathbf{F}$ admits non-trivial automorphisms, because an extension of such an automorphism to a global automorphism of $\mathbf{F}$ by ultrahomogeneity would not preserve the order $\prec$.

$\Leftarrow$: if all the finite substructures of $\mathbf{F}$ are rigid, it follows that for every finite substructure $\mathbf{B}$ of $\mathbf{F}$, the sets $\mathrm{Emb}(\mathbf{B},\mathbf{F})$ and $\binom{\mathbf{F}}{\mathbf{B}}$ coincide. (Remark 4.5.35.) By Lemma 4.5.40, the action of the automorphism group $\mathrm{Aut}\,(\mathbf{F})$ on every discrete metric space of the form $\mathrm{Emb}(\mathbf{B},\mathbf{F})$, where $\mathbf{B} \leq \mathbf{F}$ is a finite substructure, is finitely oscillation stable. Using Exercise 4.5.34 and Theorem 2.1.10(4), we conclude that $\mathrm{Aut}\,(\mathbf{F})$ is extremely amenable. $\square$

Corollary 4.5.43. *For a closed subgroup G of S_∞ the following are equivalent.*

1. *G is extremely amenable.*

2. *G preserves a linear order on the universe ω and is the automorphism group of a Fraïssé structure having Ramsey property.*

3. *G is the automorphism group of an order Fraïssé structure having Ramsey property.*

Proof. (1) $\implies$ (2): if G is extremely amenable, then it preserves a linear order (Lemma 4.5.41) and is the group of automorphisms of a Frïssé structure (Theorem 4.5.32), which must have the Ramsey property by Theorem 4.5.42.

(2) $\implies$ (3): let G preserve a linear order and serve as the automorphism group of a Fraïssé structure $\mathbf{F}_0$ having Ramsey property. Then one can extend the signature L_0 of $\mathbf{F}_0$ if necessary, by adding a symbol $<$ interpreted as a linear order on $\mathbf{F}_0$ and on every finite substructure. The resulting structure, which we will denote $\mathbf{F}$, is obviously still a Fraïssé structure, and it still possesses the Ramsey property, and besides $G = \mathrm{Aut}\,(\mathbf{F})$.

(3) $\implies$ (1): if $G = \mathrm{Aut}\,(\mathbf{F})$, where $\mathbf{F}$ is an order Fraïssé structure having Ramsey property, then finite substructures of $\mathbf{F}$ are rigid, and Theorem 4.5.42 applies. □

Example 4.5.44. The Fraïssé limit of the class of finite totally ordered sets is the ordered set η_ω having the order type of the rationals. The Ramsey property of the class in question is just the classical Ramsey theorem. Consequently, as we already know, $\mathrm{Aut}\,(\mathbb{Q}, \leq)$ with the natural Polish topology is extremely amenable.

Example 4.5.45. An *ordered graph* is a graph together with a linear order on the set of vertices. It is not difficult to verify that the set of all finite ordered graphs forms a Fraïssé class. The Fraïssé limit of this class is called the *random ordered graph*. One can further verify that as a graph, it is isomorphic to the random graph, and as an ordered set, to $\mathbb{Q}$. One can verify that the class of finite ordered graphs has the Ramsey property [131]. Therefore, the automorphism group of the random ordered graph is extremely amenable.

Remark 4.5.46. The presentation in this subsection and the next is based on a selection from the paper by Kechris, Pestov and Todorcevic [99], which contains, one should notice, a considerably larger wealth of results, concepts, and examples. For a general reference to structural Ramsey theory, we recommend the book [68] by Graham, Rothschild and Spencer, the book [132] by Nešetřil and Rödl, as well as the chapter [129] by Nešetřil.

4.5.3 Minimal flows of automorphism groups

The technique developed in Subsection 4.2.2 can be used with great efficiency to compute the universal minimal flows of automorphism groups of those Fraïssé structures $\mathbf{F}$ whose finite substructures are not necessarily rigid. Those universal minimal flows consist of linear orders on the universe of $\mathbf{F}$, compatible with the structure in a certain sense.

Definition 4.5.47. Let L be a countable signature, and let $\mathbf{A}$ be a structure in the signature L. For every subset $L_0 \subseteq L$, we denote by $\mathbf{A}_0 = \mathbf{A}|_{L_0}$ the *reduct* of $\mathbf{A}$ to L_0, that is, $\mathbf{A}$ only equipped with relations and functions indexed by elements of L_0. For any class $\mathscr{C}$ of structures in signature L, the *reduct of* $\mathscr{C}$ to L_0 is the class

$$\mathscr{C}_0 = \mathscr{C}|_{L_0} = \{\mathbf{A}_0 \colon \mathbf{A} \in \mathscr{C}\}.$$

Let now a signature L contain a binary symbol $<$, and let $L_0 = L \setminus \{<\}$. The following result states a necessary and sufficient condition for the reduct of a Fraïssé structure to be a Fraïssé structure in the signature L_0.

Theorem 4.5.48. *Let $\mathscr{C}$ be a Fraïssé order class in the signature L with the Fraïssé limit $\mathbf{F}$. Denote by $\mathscr{C}_0$ and $\mathbf{F}_0$ the corresponding reducts to the signature $L_0 = L \setminus \{<\}$. The following are equivalent.*

1. *$\mathscr{C}_0$ is a Fraïssé class with the Fraïssé limit $\mathbf{F}_0$.*
2. *Let $\mathbf{B}_0 \in \mathscr{C}_0$ and let $\mathbf{A}_0 \leq \mathbf{B}_0$ be a substructure. Every linear order $\prec$ on $\mathbf{A}_0$ such that $(\mathbf{A}_0, \prec) \in \mathscr{C}$ extends to a linear order $\prec'$ on $\mathbf{B}_0$ with $(\mathbf{B}_0, \prec') \in \mathscr{C}$.*
   (In this case, we say that the order class $\mathscr{C}$ is reasonable.*)*

Proof. (1) $\implies$ (2): embed $(\mathbf{A}_0, \prec)$ into $\mathbf{F}$ as a substructure. Since $\mathbf{F}_0$ is a Fraïssé structure, the embedding $\mathbf{A}_0 \leq \mathbf{B}_0$ extends to an embedding of $\mathbf{B}_0$ into $\mathbf{F}_0$ by the finite extension property. The restriction of the linear order on $\mathbf{F}$ to $\mathbf{B}_0$ is the desired order $\prec'$.

(2) $\implies$ (1): clearly, $\mathscr{C}_0 = \mathrm{Age}\,(\mathbf{F}_0)$. Let now $\mathbf{B}_0, \mathbf{C}_0 \in \mathscr{C}_0$, where $\mathbf{B}_0$ is a substructure of $\mathbf{C}_0$, and let $f\colon \mathbf{B}_0 \to \mathbf{F}_0$ be an embedding. The restriction of the order on $\mathbf{F}$ to $\mathbf{B}_0$ extends to an order $\prec$ on $\mathbf{C}_0$, and the order structure $(\mathbf{C}_0, \prec) \in \mathscr{C}$ embeds into $\mathbf{F}$ so as to extend the embedding f, because $\mathbf{F}$ satisfies the finite extension property (Prop. 4.5.14). It follows that $\mathbf{F}_0$ satisfies the finite extension property and is ultrahomogeneous by the same Proposition 4.5.14. Clearly, as a consequence, $\mathbf{F}_0$ is a Fraïssé structure, and consequently $\mathscr{C}_0 = \mathrm{Age}\,(\mathbf{F}_0)$ is a Fraïssé class (Theorem 4.5.17). According to Theorem 4.5.15 on the uniqueness of the Fraïssé limit, $\mathrm{Flim}\,\mathscr{C}_0 = \mathbf{F}_0$. $\square$

Example 4.5.49. Here is one simple way to construct reasonable Fraïssé order classes. Let L be a signature including the binary symbol $<$, and let $\mathscr{C}_0$ be a Fraïssé class in the signature $L_0 = L \setminus \{<\}$. Denote by $\mathscr{C}_0 * LO$ the class of all structures of the form $(\mathbf{A}_0, <)$, where $\mathbf{A} \in \mathscr{C}_0$ and $<$ is a linear order on $\mathbf{A}_0$. In this way, one obtains the class of all finite ordered sets, finite ordered graphs, ordered finite metric spaces with rational distances, etc. The order class $\mathscr{C}_0 * LO$ is obviously a reasonable Fraïssé order class.

Here is the class of orders that will serve as elements of the universal minimal flow of the automorphism groups.

Definition 4.5.50. Let $\mathbf{F}$ be a Fraïssé structure in signature $L = L_0 \cup \{<\}$, and let $\mathscr{C} = \mathrm{Age}\,(\mathbf{F})$. A linear order $\prec$ on the reduct $\mathbf{F}_0$ will be called *admissible*, or more precisely *$\mathscr{C}$-admissible*, if for every finite substructure $\mathbf{A}_0 \leq \mathbf{F}_0$ one has $(\mathbf{A}_0, \prec) \in \mathscr{C}$. In other words, $\prec$ is admissible if $\mathrm{Age}\,(\mathbf{F}_0, \prec) \subseteq \mathrm{Age}\,(\mathbf{F})$.

Let $\mathbf{F}$ be a Fraïssé structure in signature $L = L_0 \cup \{<\}$, and let $\mathbf{F}_0$ be, as usual, the reduct of $\mathbf{F}$ to the signature L_0. The automorphism group $\mathrm{Aut}\,(\mathbf{F}_0)$, being a closed topological subgroup of S_∞, acts continuously on the space LO of all linear orders on the universe of $\mathbf{F}$, which we identify with ω.

Exercise 4.5.51. Show that the subset $\mathrm{LO}(\mathbf{F})$ of all admissible orders on $\mathbf{F}_0$ is closed in LO and invariant under the action of $\mathrm{Aut}\,(\mathbf{F}_0)$.

Next we will explore the issue of minimality of $\mathrm{LO}(\mathbf{F})$.

Lemma 4.5.52. *Let $\mathbf{F}$ be the Fraïssé limit of a reasonable Fraïssé order class in signature $L = L_0 \cup \{<\}$, and let $\mathbf{F}_0$ be the reduct of $\mathbf{F}$ to L_0. The compact* $\mathrm{Aut}\,(\mathbf{F}_0)$*-flow* $\mathrm{LO}(\mathbf{F})$ *is minimal if and only if for every admissible order $\prec$ on $\mathbf{F}_0$ one has*

$$\mathrm{Age}\,(\mathbf{F}_0, \prec) = \mathrm{Age}\,\mathbf{F}.$$

Proof. Minimality, that is, the fact that every $\mathrm{Aut}\,(\mathbf{F}_0)$-orbit is everywhere dense, means – if one recalls the definition of the topology on the space LO – that for every two admissible orders $\prec$ and $\prec'$ on $\mathbf{F}_0$ and every finite substructure $\mathbf{A}_0$ of $\mathbf{F}_0$, there is an automorphism $\tau \in \mathrm{Aut}\,(\mathbf{F}_0)$ such that the restrictions of ${}^{\tau}\prec$ and $\prec'$ to $\mathbf{A}_0$ coincide. Taking into account the definition of an admissible order, it is equivalent to require, for every admissible order $\prec$ and every finite substructure $\mathbf{A}_0 \leq \mathbf{F}_0$, the existence of a τ as above such that the restriction of ${}^{\tau}\prec$ to $\mathbf{A}_0$ coincides with the canonical order induced from $\mathbf{F}$. Since the structure $\mathbf{F}_0$ is ultrahomogeneous by Theorem 4.5.48, the condition of minimality becomes: for every finite substructure $\mathbf{A}$ of $\mathbf{F}$ and every admissible order $\prec$ on $\mathbf{F}_0$, there is an isomorphic copy $\mathbf{A}_0'$ of $\mathbf{A}_0$ in $\mathbf{F}_0$ which, equipped with the restriction of $\prec$, is isomoprhic to $\mathbf{A}$. ☐

Here is a concept allowing one to verify the above condition in practical terms.

Definition 4.5.53. A class $\mathscr{C}$ of structures in signature $L = L_0 \cup \{<\}$ satisfies the *ordering property* if for every structure $\mathbf{A}_0$ in the reduct $\mathscr{C}_0$ there exists a $\mathbf{B}_0 \in \mathscr{C}_0$ such that for every pair of linear orderings $\prec$ and $\prec'$ on $\mathbf{A}$ and $\mathbf{B}$ respectively, if $\mathbf{A} = (\mathbf{A}_0, \prec)$ and $\mathbf{B} = (\mathbf{B}_0, \prec')$ are in $\mathscr{C}$, then $\mathbf{A}$ embeds into of $\mathbf{B}$ as a substructure.

Theorem 4.5.54. *Let $\mathbf{F}$ be the Fraïssé limit of a reasonable Fraïssé order class $\mathscr{C}$ in signature $L = L_0 \cup \{<\}$, and let $\mathbf{F}_0$ be the reduct of $\mathbf{F}$ to L_0. The compact* $\mathrm{Aut}\,(\mathbf{F}_0)$*-flow* $\mathrm{LO}(\mathbf{F})$ *is minimal if and only if the class $\mathscr{C}$ satisfies the ordering property.*

Proof. $\Rightarrow$: Assume the minimality of the $\mathrm{Aut}\,(\mathbf{F}_0)$-flow $\mathrm{LO}(\mathbf{F})$. Let $\mathbf{A}_0$ be a finite substructure of $\mathbf{F}_0$. For each one of the (finitely many) distinct orders $\prec_i$ on $\mathbf{A}_0$ such that $(\mathbf{A}_0, \prec_i) \in \mathscr{C}$, one can find a copy $\mathbf{A}_{0,i}$ of $\mathbf{A}_0$ in $\mathbf{F}_0$ which, equipped with the restriction of the canonical order from $\mathbf{F}$, is isomorphic to $(\mathbf{A}_0, \prec_i)$. Denoting by $\mathbf{A}'$ the finite structure generated by $\cup_i \mathbf{A}_i$, it is now enough to find a $\mathbf{B}_0$ such that, whenever $\prec$ is a linear order on $\mathbf{B}_0$ with $(\mathbf{B}_0, \prec) \in \mathscr{C}$, $\mathbf{A}'$ is a substructure of $(\mathbf{B}_0, \prec)$.

For every finite substructure $\mathbf{C}_0$ of $\mathbf{F}_0$, isomorphic to $\mathbf{A}'_0$, denote by $\mathrm{LO}(\mathbf{C}_0)$ the set of all linear orders $\prec$ in $\mathrm{LO}(\mathbf{F})$ with the property that $\mathbf{C}_0$ with the restriction of this linear order is isomorphic to $\mathbf{A}'$. The cover of $\mathrm{LO}(\mathbf{F})$ with open subsets $\mathrm{LO}(\mathbf{C}_0)$ admits a finite subcover, say $\mathrm{LO}(\mathbf{C}_{0,j})$. Let $\mathbf{B}_0$ be a (finite) substructure of $\mathbf{F}_0$ generated by the union $\cup_j \mathbf{C}_{0,j}$. An arbitrary order $\prec'$ on $\mathbf{B}_0$, satisfying $(\mathbf{B}, \prec') \in \mathscr{C}$, extends to an admissible order on $\mathbf{F}_0$, in fact to a suitable translate of the canonical order by an automorphism of $\mathbf{F}$ (in consequence of the ultrahomogeneity of the latter structure). There is a j such that $\mathbf{C}_{0,j}$ with the induced order is isomorphic to $\mathbf{A}'_0$, and we are done.

$\Leftarrow$: let $\mathbf{A}$ be a finite substructure of $\mathbf{F}$, and $\mathbf{A}_0$ its reduct. Choose $\mathbf{B}_0$ as in the assumption. Now for every admissible order $\prec$ on $\mathbf{F}_0$, there is a copy of $\mathbf{A}_0$ inside $\mathbf{B}_0$ which, if equipped with the restriction of $\prec$, is isomorphic to $\mathbf{A}$. This assures minimality by Lemma 4.5.52. $\square$

Here is the main result of this Section.

Theorem 4.5.55. *Let $\mathscr{C}$ be a reasonable Fraïssé order class, satisfying the Ramsey and the ordering properties. Denote $\mathbf{F} = \mathrm{Flim}\,\mathscr{C}$, and let $\mathbf{F}_0$ be the reduct of $\mathbf{F}$ to the signature $L_0 = L \setminus \{<\}$. Equip the automorphism group* $\mathrm{Aut}\,(\mathbf{F}_0)$ *the natural Polish topology of simple convergence on the universe ω considered as a discrete topological space. Then the zero-dimensional compact space* $\mathrm{LO}(\mathbf{F})$ *of all $\mathscr{C}$-admissible linear orders on ω, equipped with the standard action of* $\mathrm{Aut}\,(\mathbf{F}_0)$*, forms the universal minimal flow for the Polish topological group* $\mathrm{Aut}\,(\mathbf{F}_0)$.

Proof. The automorphism group $\mathrm{Aut}\,(\mathbf{F})$, which forms a closed topological subgroup of $\mathrm{Aut}\,(\mathbf{F}_0)$, is extremely amenable by Corollary 4.5.43. The homogeneous factor-space $\mathrm{Aut}\,(\mathbf{F}_0)/\mathrm{Aut}\,(\mathbf{F})$ is naturally identified with an everywhere dense subset of the space $\mathrm{LO}(\mathbf{F})$ of admissible linear orders, consisting of all orders that are permutations of the canonical order $\prec$ on $\mathbf{F}$ under the automorphisms of the reduct. Just like in the proof of Theorem 4.3.1, one can verify, using Lemma 4.2.5, that the right uniform structure on the factor-space coincides with the restriction of the unique compatible uniform structure from the compact space $\mathrm{LO}(\mathbf{F})$. The uniform completion of $\mathrm{Aut}\,(\mathbf{F}_0)/\mathrm{Aut}\,(\mathbf{F})$ coincides with $\mathrm{LO}(\mathbf{F})$ and is minimal by Theorem 4.5.54. Now Theorem 4.2.10 finishes the proof. $\square$

Before listing some examples, let us point out that for some Fraïssé order classes $\mathscr{C}$ the Ramsey property of $\mathbf{F}$ is equivalent to that of its reduct $\mathscr{C}_0$.

Definition 4.5.56. Let us say that an Fraïssé order class $\mathscr{C}$ is *order forgetful* if any two finite structures $\mathbf{A}, \mathbf{B} \in \mathscr{C}$ are isomorphic as soon as their reducts $\mathbf{A}_0$, $\mathbf{B}_0$ are isomorphic.

For instance, every order class of the form $\mathbf{F}_0 * LO$ (finite ordered sets, finite ordered graphs, etc.) is order forgetful, but there are other examples as well.

Definition 4.5.57. Let $\mathbb{F}$ be a finite field, and let V be a vector space over $\mathbb{F}$. One says that a linear order $\prec$ on V is *natural* if there are

1. a basis X of V,
2. a linear order on X, and
3. a linear order $<$ on $\mathscr{F}$ such that 0 is the smallest element, followed by 1,

such that the order $\prec$ on V is lexicographic with respect to the above pair of orders on X and on $\mathscr{F}$.

Example 4.5.58. The class of all finite vector spaces over a given finite field $\mathbb{F}$ that are naturally ordered forms a Fraïssé order class that is order forgetful (Simon Thomas [165].)

Example 4.5.59. One can introduce in a similar way the notion of a natural order on a finite Boolean algebra as an order induced lexocographically from an ordering on the set of all atoms, and prove that finite Boolean algebras form a Fraïssé order class that is order forgetful.

The following is left as an exercise.

Theorem 4.5.60. *An order forgetful Fraïssé order class $\mathscr{C}$ satisfies the Ramsey property if and only if the reduct $\mathscr{C}_0$ does.* $\square$

Now we are ready to consider a few typical examples of computing universal minimal flows of automorphism groups of Fraïssé structures.

Example 4.5.61. The countably infinite set ω, viewed as the Fraïssé limit of the class of finite sets, is a reasonable Fraïssé order class, satisfying the Ramsey and ordering properties. Every linear order on ω is easily seen to be admissible. Consequently, we reprove the result by Glasner and Weiss: the space LO of all linear orders on ω serves as the universal minimal flow for the Polish group S_∞ of all automorphisms of the Fraïssé structure ω (in empty signature).

Example 4.5.62. Starting with the class of finite ordered graphs (Example 4.5.45), one concludes that the universal minimal flow of the group of automorphisms $\mathrm{Aut}\,(R)$ of the random graph is the space LO of all linear orders on the vertices of R.

Example 4.5.63. The Fraïssé limit of the class of all finite vector spaces over a given finite field $\mathbb{F}$ is the countably-dimensional vector space $V = \mathbb{F}^{\aleph_0}$. Its automorphism group is the general linear group $\mathrm{GL}(\mathbb{F}^{\aleph_0})$, equipped with the topology of simple convergence on V equipped with the discrete topology. The universal minimal flow of the Polish group $\mathrm{GL}(\mathbb{F}^{\aleph_0})$ is the zero-dimensional compact metric space of all linear orders on V whose restrictions to finite-dimensional vector subspaces are natural.

Example 4.5.64. Let B_∞ denote the countable atomless Boolean algebra. There is, up to an isomorphism, only one such algebra, isomorphic to the algebra of all open-and-closed subsets of the Cantor set C. This algebra is the Fraïssé limit of the class of all finite Boolean algebras. The universal minimal flow of the automorphism group $\mathrm{Aut}\,(B_\infty)$ is formed by all linear orders on B_∞ whose restriction to every finite Boolean subalgebra is natural, that is, is a lexicographic order given by a suitable linear ordering of the set of atoms.

The minimal flow $\mathscr{M}(\mathrm{Aut}\,(B_\infty))$ admits another description: it is the space $\Phi(C)$ of all maximal chains of closed subsets of the Cantor set C. The isomorphism between two flows can be seen directly as follows. Given a maximal chain $\mathscr{C}$ of closed subsets of C, let F_A denote the intersection of all elements of the chain $\mathscr{C}$ meeting A. It is easy to see that $A\cap F_A$ is a singleton, and that the correspondence $A \mapsto F_A$ is monotone in the sense of inclusion. If $A \cap B = \emptyset$, then $F_A \neq F_B$, and since $F_A, F_B \in \mathscr{C}$, this gives a linear order on the set of atoms of every finite Boolean subalgebra of B_∞. One can check that the order thus emerging is natural. Thus, we have a mapping from the space $\Phi(C)$ to the space of natural orders, and one can verify that this is an isomorphism of $\mathrm{Aut}\,(B_\infty)$-flows.

The description of the minimal flow $\mathscr{M}(\mathrm{Aut}\,(B_\infty))$ was originally obtained by Glasner and Weiss [61] namely in terms of maximal chains. The description in terms of natural orders was later deduced in [99] from a general approach.

Example 4.5.65. Using the *ordered rational Urysohn metric space* (the Fraïssé limit of the class of finite ordered metric spaces with rational distances) and the fact that the class of finite rational metric spaces has the Ramsey property established by Nešetřil [130], one deduces that the universal minimal flow $\mathscr{M}(\mathrm{Iso}\,(\mathbb{U}_Q))$ of the rational Urysohn space, equipped with the topology of simple convergence on $\mathbb{U}_\mathbb{Q}$ regarded as discrete, is formed by all linear orders on $\mathbb{U}_\mathbb{Q}$. This leads to a new interesting proof of extreme amenability of the isometry group of the Urysohn space.

Every isometry of $\mathbb{U}_\mathbb{Q}$ admits a unique extension to the Urysohn space $\mathbb{U}$, which is the metric completion of $\mathbb{U}_\mathbb{Q}$. In this way, one obtains a group monomorphism $\mathrm{Iso}\,(\mathbb{U}_\mathbb{Q}) \to \mathrm{Iso}\,(\mathbb{U})$. It is continuous, but of course not a topological isomorphism, because the topologies on two groups are defined in different ways. At the same time, the image of $\mathrm{Iso}\,(\mathbb{U}_\mathbb{Q})$ is easily seen to be everywhere dense in $\mathrm{Iso}\,(\mathbb{U})$.

Because of this, there is a morphism of $\mathrm{Iso}\,(\mathbb{U}_\mathbb{Q})$-flows from the universal minimal flow $\mathscr{M}(\mathrm{Iso}\,(\mathbb{U}_\mathbb{Q})) = \mathrm{LO}$ onto $\mathscr{M}(\mathrm{Iso}\,(\mathbb{U}))$. One can prove now that for every neighbourhood of identity V in $\mathrm{Iso}\,(\mathbb{U})$ and every linear order $\prec$ on $\mathbb{U}_\mathbb{Q}$, the set $\{{}^g{\prec} \colon g \in V \cap \mathrm{Iso}\,(\mathbb{U}_\mathbb{Q})\}$ is dense in LO. This implies that the only factor of the flow LO upon which the group Iso$(\mathbb{U})$ acts continuously is the trivial one-point space, and consequently Iso$(\mathbb{U})$ is extremely amenable. (See [99] for details.)

The group Iso$(\mathbb{U})$ remains to the day the only group whose extreme amenability can be established by both major methods, that is, through concentration of measure as well as Ramsey theory. Notice, however, that the proof using concentration of measure gives a stronger result: Iso$(\mathbb{U})$ is a Lévy group rather than just an extremely amenable one.

The above examples, whose details we leave out, only form the tip of an iceberg. A wealth of examples, as well as proofs and references, can be found in the paper [99]. Cf. also the recent results by Nguyen Van The [133].

Chapter 5

Emerging developments

5.1 Oscillation stability and Hjorth theorem

The following concept was introduced by Vitali Milman at about the same time as the concept of finite oscillation stability [112, 113].

Definition 5.1.1. A scalar-valued function f on the unit sphere $\mathbb{S}^\infty$ of the Hilbert space ℓ^2 is called *oscillation stable* if for every $\varepsilon > 0$ there exists an infinite-dimensional subspace $\mathscr{K}$ of ℓ^2 such that

$$\operatorname{osc}(f|\mathscr{K} \cap \mathbb{S}^\infty) < \varepsilon.$$

Remark 5.1.2. Milman's concept was given in terms of non-emptiness of the *infinite spectrum,* $\gamma_\infty(f)$, of a function f, while the terminology we are using was introduced later, cf. [134]. See also Remark 5.1.13 below where we remind the notion of the infinite spectrum.

This is obviously a stronger property than the familiar finite oscillation stability. The following question by Vitali Milman has dominated research in geometric functional analysis for three decades: *is every uniformly continuous function on the sphere $\mathbb{S}^\infty$ of ℓ^2 oscillation stable?*

One of the reasons for such prominence was the equivalence of the question to the *distortion problem* for the Hilbert space. In 1994, the problem was solved by Odell and Schlumprecht [134], who have shown that the answer is in the negative.

The solution to the distortion problem has been one of the most important developments in the area. The proof is indirect in the sense that it is not based on the intrinsic geometry of the unit sphere, but rather uses the distortion property of another Banach space (either the Tsirelson space or its later modification, the Schlumprecht space) and then uses a sophisticated technique to "transfer" the result to the sphere $\mathbb{S}^\infty$. (Apart from the original proof in [134], the distortion property can now be found in the monograph form, including Chapter 13 in [16].)

Vitali Milman had expressed on a number of occasions a belief that, just like the finite oscillation stability, the concept of oscillation stability can be transferred to a setting of topological groups of transformations. Such a transfer came to fruition during discussions while the paper by Kechris, Pestov and Todorcevic [99] was being written.

It is arguably the right approach, because it fits well with a number of examples coming from discrete combinatorics which are clearly manifestations of the same phenomenon. At the same time, one should stress that the intersection of the approach to oscillation stability/distortion adopted in geometric functional analysis and that proposed for topological groups of transformations consists of the Hilbert space ℓ^2 only, and in particular it appears that Banach spaces with non-homogeneous unit spheres do not belong within this framework.

There are so far more questions in this area than answers, but we already have a fundamental result by Greg Hjorth about distortion in Polish groups. There is a hope that by studying oscillation stability in this general dynamical context, one can in particular achieve a deeper understanding of the phenomenon for the unit sphere $\mathbb{S}^\infty$ as well.

We will outline the topological transformation group approach to oscillation stability in this section. It is based on the concept of the one-sided completion of a topological group.

Denote the completion of the uniform space $(G,\mathcal{U}_R(G))$ by $\widehat{G}^R$, and that of $(G,\mathcal{U}_L(G))$ by $\widehat{G}^L$. Each one of them is a topological semigroup with jointly continuous multiplication ([151], Prop. 10.2(a)), as we will prove shortly, but not necessarily a topological group. It was first noted by Dieudonné [36]. In fact, this is typically the case for infinite-dimensional groups G.

Example 5.1.3. The left completion of $U(\ell^2)$ is the semigroup of all linear isometries $\ell^2 \hookrightarrow \ell^2$ with the strong topology.

Example 5.1.4. The left completion of $\mathrm{Aut}\,(\mathbb{Q},\leq)$ is the semigroup of all order-preserving injections $\mathbb{Q} \hookrightarrow \mathbb{Q}$, with the topology of simple convergence on $\mathbb{Q}$ considered as a discrete space.

Example 5.1.5. The *right* completion of $\mathrm{Homeo}_+[0,1]$ is the semigroup of all continuous order-preserving surjections from $[0,1]$ to itself ([151], Prop. 10.2(a)), and so the left completion is formed by all relations inverse to such surjections.

For a metric space X equip the group $\mathrm{Iso}\,(X)$ of all isometries of X, as usual, with the topology of simple convergence (which coincides with the compact-open topology). Denote by $\mathrm{Emb}(X)$ the semigroup of all isometric embeddings $X \hookrightarrow X$, equipped with the topology of simple convergence. Recall that a metric space is *ultrahomogeneous* if every isometry between two finite subspaces of X extends to an isometry of X with itself.

Proposition 5.1.6. *If X is a complete metric space, then the left completion of $\mathrm{Iso}\,(X)$ is topologically isomorphic to a subsemigroup of $\mathrm{Emb}(X)$. If in addition X is ultrahomogeneous, then $\widehat{\mathrm{Iso}\,(X)}^L$ is isomorphic to $\mathrm{Emb}(X)$.*

Proof. The uniform structure on $\mathrm{Emb}(X)$ is given by entourages

$$\{(\sigma,\tau)\colon \forall x\in F, d(\sigma(x),\tau(x))<\varepsilon\},$$

where $\varepsilon>0$ and $F\subset X$ is finite. It is compatible with the topology, invariant under left translations, and its restriction to G is $\mathcal{U}_L(G)$. With regard to this uniformity, $\mathrm{Emb}(X)$ is complete.
[If $(\tau_{F,\varepsilon})$, where $|F|<\infty$ and $\varepsilon>0$, is a Cauchy net, then for each $x\in X$ the sequence $\tau_{\{x\},1/n}(x)$, $n\in\mathbb{N}_+$ is Cauchy in X. Denote its limit by $\tau(x)$. Then τ is an isometry of X into itself, and $\tau_{F,\varepsilon}\to\tau$.]

Consequently, the closure of $\mathrm{Iso}\,(X)$ in $\mathrm{Emb}(X)$ is isomorphic, both as a semigroup and as a uniform space, to $\widehat{\mathrm{Iso}\,(X)}^L$.

Assuming X is ultrahomogeneous, it is easy to see that $\mathrm{Iso}\,(X)$ is in addition everywhere dense in $\mathrm{Emb}(X)$. $\square$

Example 5.1.7. The left completion of the group $\mathrm{Iso}\,(\mathbb{U})$ of isometries of the Urysohn metric space, with the standard Polish topology, is the unital semigroup $\mathrm{Emb}(\mathbb{U})$ of all isometric embeddings $\mathbb{U} \hookrightarrow \mathbb{U}$.

Exercise 5.1.8. Verify that the semigroup operation in $\mathrm{Emb}(X)$, equipped with the topology of simple convergence on X, is jointly continuous, that is, continuous as a mapping $\mathrm{Emb}(X) \times \mathrm{Emb}(X) \to \mathrm{Emb}(X)$.

Theorem 5.1.9. *The left completion of a topological group G is a topological semigroup (with jointly continuous multiplication).*

This can be verified directly, cf. e.g. [151], Prop. 10.2(a), or [47]. For second-countable groups one can give a nice proof based on the Urysohn space.

Proof of Theorem 5.1.9 for second-countable groups. If G is second-countable, then G is isomorphic with a topological subgroup of $\mathrm{Iso}\,(\mathbb{U})$ (Theorem 3.4.36). The closure of G in $\mathrm{Iso}\,(\mathbb{U})$ is consequently a sub-semigroup of $\mathrm{Emb}(\mathbb{U})$, with a jointly continuous multiplication. But the right completion $\widehat{G}^R$ is isomorphic, as a topological semigroup, with the closure of G in $\mathrm{Iso}\,(\mathbb{U})$. $\square$

Remark 5.1.10. Given a topological group G acting continuously by isometries on a complete metric space X, we will henceforth associate to each $\tau \in \hat{G}^L$ an isometric embedding $X \hookrightarrow X$, denoted by the same letter τ.

Recall that if H is a subgroup of G, then the left (right) uniform structure on G/H is, by definition, the finest uniform structure making the map $G \to G/H$ uniformly continuous with regard to $\mathcal{U}_L(G)$, respectively $\mathcal{U}_R(G)$.

Example 5.1.11. For a natural k, the set $[\mathbb{Q}]^k$ of k-subsets of the rationals can be identified with a factor-space of $\mathrm{Aut}\,(\mathbb{Q}, \leq)$ by the isotropy subgroup stabilizing a chosen k-subset. The left uniformity on $[\mathbb{Q}]^k$ is discrete (contains the diagonal).

The following definition, proposed in [99], is modelled on Milman's concept [113], though an extension to topological transformation groups is less evident here than it was for finite oscillation stability. Again, we adopt a later terminology [134].

Definition 5.1.12. Let G be a topological group, and let $f\colon G \to Y$ be a uniformly continuous function with values in a metric space. Say that f is *oscillation stable* if for every $\varepsilon > 0$ there is a right ideal $\mathcal{I}$ of the semigroup $\widehat{G}^L$ with the property

$$\operatorname{osc}(\hat{f} \mid \mathcal{I}) < \varepsilon,$$

where $\hat{f}$ is a unique extension by uniform continuity of f to the completion $\hat{G}^L$.

Remark 5.1.13. Here is a closer analogue of the original Milman's concept. Define the *infinite spectrum* of a function f as above as the set, $\gamma_\infty(f)$, of all $a \in Y$ such that for every $\varepsilon > 0$ there is a right ideal $\mathcal{I}$ of the semigroup $\widehat{G}_L$ with the property $d(\hat{f}(x), a) < \varepsilon$ for all $x \in \mathcal{I}$. If now the image $f(G)$ is relatively compact in Y, then clearly $\gamma_\infty(f) \neq \emptyset$ if and only if f is oscillation stable. Cf. [121, 134].

Exercise 5.1.14. Show that in the above remark the ideal $\mathcal{I}$ can be assumed closed without loss in generality.

Lemma 5.1.15. *For a topological group G, the following are equivalent.*

1. *Every bounded left uniformly continuous function f on G is oscillation stable.*
2. *For every bounded left uniformly continuous function f on G, for every $\varepsilon > 0$, each right ideal $\mathcal{I}$ of $\hat{G}^L$ contains a right ideal $\mathcal{J}$ with* $\operatorname{osc}(\hat{f} \mid \mathcal{J}) < \varepsilon$.

Proof. Only the implication (1) $\implies$ (2) needs verifying. Let $x \in \mathcal{I}$. The bounded function f_1 on G given by $f_1(g) = f(xg)$ is easily seen to be left uniformly continuous, so there is a $y \in \hat{G}^L$ with $\operatorname{osc}(\widehat{f_1}|y\hat{G}^L) < \varepsilon$, therefore $\operatorname{osc}(\hat{f}|xy\hat{G}^L) < \varepsilon$ as well. □

Here is the concept of central importance.

Definition 5.1.16. Let a topological group G act continuously on a topological space X. Say that the topological transformation group (G, X) is *oscillation stable* if for every bounded continuous real-valued function f on X and every $x_0 \in X$, if the function $G \ni g \mapsto f(gx_0)$ is left uniformly continuous, then it is oscillation stable. If (G, X) is not oscillation stable, we say that it has *distortion*.

Remark 5.1.17. A homogeneous space G/H is oscillation stable if and only if for every bounded left uniformly continuous function f on G/H the composition $f \circ \pi$ with the factor-map $\pi\colon G \to G/H$ is oscillation stable on G.

Lemma 5.1.18. *Let a topological group G act continuously by isometries on a complete metric space X. Associate to each element of $\hat{G}^L$ an isometry from* $\mathrm{Emb}(X)$ *(Prop. 5.1.6). Then a function $f\colon X \to Y$ is oscillation stable if and only if for every $\varepsilon > 0$ there is a $\tau \in \widehat{\mathrm{Iso}\,(G)}^L$ such that* $\mathrm{osc}\,(f|\tau(X)) < \varepsilon$.

Proof. The closure of the set of values of f on $\tau(X)$ is the same as the closure of set of values of the extension of the LUC function $\hat{G}^L \ni \sigma \mapsto f(\sigma x_0)$ on the right ideal τG, no matter what $x_0 \in X$ is. □

Remark 5.1.19. Let a topological group G act continuously and transitively by isometries on a metric space X. For a uniformly continuous function $f\colon X \to Y$ define the infinite spectrum $\gamma_\infty(f)$ as the infinite spectrum of the composition $\tilde{f}(g) = f(gx_0)$ of f with the orbit map, where $x_0 \in X$. It is easy to see that the definition does not depend on the choice of x_0, and that $\gamma_\infty(f) \subseteq \gamma(f)$, so that in particular if f is oscillation stable, then it is finitely oscillation stable.

Theorem 5.1.20. *Let a topological group G act on a complete metric space X continuously and transitively by isometries. The following are equivalent.*

1. *The pair (X, G) is oscillation stable.*
2. *Every bounded real-valued* 1*-Lipschitz function f on X is oscillation stable.*

3. *Every bounded* 1*-Lipschitz function* f *on* X *with values in a finite-dimensional normed space is oscillation stable.*

4. *For every finite cover* γ *of* X *and every* $\varepsilon > 0$*, there are* $g \in \hat{G}^L$ *and* $A \in \gamma$ *such that* $g(X)$ *is contained in the* ε*-neighbourhood* A_ε *of* A*.*

Proof. (1) $\Longrightarrow$ (2): trivial.

(2) $\Longrightarrow$ (3): let $f = (f_1, f_2, \ldots, f_n)$, where we can assume without loss in generality each f_i to be 1-Lipschitz. Recursively choose elements $g_1, g_2, \ldots, g_n \in \hat{G}^L$ so that at each step i the oscillation on $g_1g_2\ldots g_i(X)$ of the function f_i is $< \varepsilon$. (Equivalently, the oscillation of the 1-Lipschitz function $f \circ (g_1g_2\ldots g_{i-1})$ on $g_i(X)$ is $< \varepsilon$.) Set $g = g_1g_2\ldots g_n$ and notice that $\operatorname{osc}(f_i|g(X)) < \varepsilon$ for all i.

(3) $\Longrightarrow$ (4): an application of (3) to the collection of distance functions $f_A(x) = d(A, x)$, $A \in \gamma$ gives the existence of a $g \in \hat{G}^L$ with each f_A being ε-constant on $g(X)$, and as $g(X) \cap A_0 \neq \emptyset$ for at least one $A_0 \in \gamma$, one has $g(X) \subset (A_0)_\varepsilon$.

(4) $\Longrightarrow$ (1): let $\varepsilon > 0$ and let $f\colon X \to \mathbb{R}$ be bounded uniformly continuous. Fix a $\delta > 0$ with $|f(x) - f(y)| < \varepsilon/2$ whenever $d(x, y) < \delta$, and cover the range of f with finitely many intervals J_i of length $< \varepsilon/2$ each. Now apply (4) to the cover of X with sets $f^{-1}(J_i)$ and to the number δ. $\square$

Corollary 5.1.21. *For a complete ultrahomogeneous metric space* X *the following are equivalent.*

1. *Acted upon by the isometry group* $\operatorname{Iso}(X)$ *with the topology of simple convergence,* X *is oscillation stable.*

2. *For every bounded* 1*-Lipschitz function* $f\colon X \to \mathbb{R}$ *and every* $\varepsilon > 0$*, there is an isometric copy* $X' \subset X$ *of* X *such that* $\operatorname{osc}(f|X') < \varepsilon$*.*

3. *For every finite cover* γ *of* X *and every* $\varepsilon > 0$*, the* ε*-neighbourhood of some* $A \in \gamma$ *contains an isometric copy of* X*.* $\square$

Here is a reformulation in terms of the negation of the finite oscillation stability, the distortion property.

Corollary 5.1.22. *For a complete ultrahomogeneous metric space X the following are equivalent.*

1. *Acted upon by the isometry group* $\mathrm{Iso}\,(X)$ *with the topology of simple convergence, X has distortion.*
2. *There is a bounded uniformly continuous function $f\colon X \to \mathbb{R}$ whose oscillation on every isometric copy $X' \subset X$ of X is at least* 1.
3. *There are a finite cover γ of X and an $\varepsilon > 0$ such that no set A_ε, $A \in \gamma$ contains an isometric copy of X.*

□

Definition 5.1.23. Let X be a complete ultrahomogeneous metric space. If X satisfies any of the equivalent conditions listed in Corollary 5.1.21, we will say that X is *oscillation stable,* while if X satisfies any of the conditions of Corollary 5.1.22, we will say that X *has distortion property.*

Below is the complete list examples of G-spaces for which the situation is clear, presently known to the author.

Example 5.1.24 (Distortion property of the Hilbert space). The unit sphere $\mathbb{S}^\infty$, considered as a metric space with the action of the unitary group $U(\ell^2)$, has distortion in the above sense, which is equivalent to the classical concept [134] by Corol. 5.1.21.

Our next main example is based on the following folklore result, which is one of the many known consequences of the 1933 Sierpiński's partition argument.

Theorem 5.1.25. *Let k be a natural number. Every finite colouring of the set $[\mathbb{Q}]^k$ of k-subsets of the rationals admits a homogeneous subset $A \subseteq \mathbb{Q}$ order-isomorphic to $\mathbb{Q}$ if and only if $k = 1$.*

Proof. For $k = 1$ the statement is obvious. Else, the case $k = 2$ suffices, and is established by applying Sierpiński's partition argument (cf. proof of Theorem 1.3.13) to the usual order on $\mathbb{Q}$ and an order of type ω. □

Remark 5.1.26. A much stronger result is known as *Devlin's theorem*, [35], see also [166]: For every $k \geq 1$ there is a colouring c_k of all k-subsets of the rationals with t_k colours (where t_k is the k-th *tangent number*, $t_1 = 1, t_2 = 2, t_3 = 16, t_4 = 272, \ldots;\ t_k \uparrow \infty$) such that for every $X \subseteq \mathbb{Q}$ order-isomorphic to $\mathbb{Q}$ and every $1 \leq t \leq t_k$ there exists $\{x_1, \ldots, x_k\} \subseteq X$ such that $c_k(x_1, \ldots, x_k) = t$. The bound t_k is exact. □

Example 5.1.27. As a direct consequence of theorem 5.1.25, the Aut $(\mathbb{Q}, \leq)$-space $[\mathbb{Q}]^n$ with a discrete metric has distortion property for $n \geq 2$, while $\mathbb{Q} = [\mathbb{Q}]^1$ is oscillation stable.

Notice also that here we do not deal with a metric space, but rather with a (discrete) metric space equipped with a group action.

Example 5.1.28. The infinite discrete metric space is of course oscillation stable.

Example 5.1.29. The random graph R, viewed as a discrete metric space and equipped with the action of its own automorphism group Aut (R) (in its usual Polish topology), is oscillation stable, as follows from the following property of R, known as the *indecomposability* property: for every partition of the set of vertices of R into two subsets, A and B, the induced subgraph on either A or B is isomorphic to R itself. (The proof of this is left as an exercise.)

This example, in contrast to the Example 5.1.27, can be considered purely within the metric space framework. If equipped with the path metric, the random graph R is isometric to the universal Urysohn metric space whose distance only takes the values in $\{0, 1, 2\}$. (Exercise. Cf. also [26].)

This metric space, $\mathbb{U}_{\{0,1,2\}}$ and the discrete metric space are the only two infinite ultrahomogeneous metric space which are finitely oscillation stable, known to the author.

Example 5.1.30. The pair $(\text{Homeo}_+(\mathbb{I}), \mathbb{I})$ is oscillation stable (easily checked directly).

Exercise 5.1.31. Show that the universal Urysohn space $\mathbb{U}$ has distortion for trivial reasons, namely because of its unboundedness. Indeed, let $x_0 \in \mathbb{U}$ be any point. Prove that the function

$$\mathbb{U} \ni x \mapsto \cos(d(x, x_0)) \in \mathbb{R}$$

is not oscillation stable. (For instance, have a look at the title of the paper [175].)

Remark 5.1.32. The above fact was noticed at about the same time by at least three mathematicians (Julien Melleray, personal communication; Greg Hjorth [92]; and the present author). Of course, the proof is particularly simple for the Urysohn space, but it can be refined and extended to a much larger class of metric spaces, suggesting that the concept of oscillation stability/distortion (if understood in the sense of Definition 5.1.23, that is, in the case where the acting group is the full group of isometries) only makes sense for ultrahomogeneous metric spaces of *bounded diameter.* In particular, the same applies to the rational Urysohn space $\mathbb{U}_{\mathbb{Q}}$.

The main open question in this direction, in author's view, is presently this: is the universal Urysohn metric space $\mathbb{U}_1$ of diameter 1 oscillation stable?

The following two exercises will give the reader some feeling of the metric space $\mathbb{U}_1$.

Exercise 5.1.33. Prove that the unit sphere of radius 1/2 around any point of the Urysohn space $\mathbb{U}$ is isometric to the Urysohn space $\mathbb{U}_1$ of diameter one.

Exercise 5.1.34. Show that the Urysohn space $\mathbb{U}$, equipped with the metric

$$d(x,y) = \min\{1, d_{\mathbb{U}}(x,y)\},$$

is <u>not</u> isometric to the Urysohn space $\mathbb{U}_1$ of diameter one.

Exercise 5.1.35. The *two-sided uniform structure* on a topological group, $\mathcal{U}_\vee(G)$, is the supremum of the left and the right uniformities. In other words, the basic entourages of the diagonal for $\mathcal{U}_\vee(G)$ are of the form $V_L \cap V_R$, where V is a neighbourhood of the identity in G.

Show that the uniformity $\mathcal{U}_\vee(G)$ is compatible, and the completion of G with regard to the two-sided uniformity is always a topological group, containing G as a dense topological subgroup.

Example 5.1.36. Let a topological group $G \neq \{e\}$ satisfy the property that the restrictions of the left and the right uniform structures to a suitable neighbourhood of identity coincide. Then the left completion of G coincides with the two-sided completion, and therefore it is a topological group. Consequently, every right ideal in $\hat{G}^L$ is either $\{e\}$ or G itself, and the pair (G, G) with the left action of G upon itself has distortion. For example, this applies to all locally compact groups, SIN groups (including abelian topological groups), and Banach–Lie groups.

Up until recently, the entire theory of infinite oscillation stability in the context of topological groups of transformations consisted of a couple of definitions, a few motivating examples borrowed from geometric functional analysis and infinite combinatorics, and a few open problems. (Cf. a brief account in [145].) However, now things have changed and the theory can claim a substantial result of its own.

Theorem 5.1.37 (Hjorth [92]). *Let a group G act continuously by isometries on a complete metric space X. Assume that not all points of X are fixed under the action. Then there exist a pair $(x_0, x_1) \in X \times X$ and a uniformly continuous function f on the orbit closure $Y = \mathrm{cl}\,[G \cdot (x_0, x_1)]$ of (x_0, x_1) such that for every $g \in \hat{G}^L$ the oscillation of the function f on $g \cdot Y$ is 1.*

Corollary 5.1.38. *Every non-trivial Polish group G has distortion (that is, is not oscillation stable).* ☐

In particular, Hjorth's result leads to a direct proof of the distortion property of the Stiefel manifolds $\mathrm{St}_n(\infty)$, $n \geq 2$. Of course, this follows at once from the result of Odell and Schlumprecht for the sphere $\mathbb{S}^\infty = \mathrm{St}_1(\infty)$, but Hjorth's proof is, by contrast, very simple.

Example 5.1.39 (Hjorth). Consider the Stiefel manifold $\mathrm{St}_2(\infty)$ of orthonormal 2-frames in the space ℓ^2, equipped with the Hausdorff metric and the action of the unitary group $U(\ell^2)$ with the strong operator topology. Then the $U(\ell^2)$-space $\mathrm{St}_2(\infty)$ is not oscillation stable (has distortion).

Let $(z_n)_{n=1}^\infty$ be an everywhere dense subset of the sphere $\mathbb{S}^\infty$. Define a function $f \colon \mathbb{S}^\infty \times \mathbb{S}^\infty \to \mathbb{R}$ by

$$f(\xi, \eta) = \sup_{n=1}^{\infty} \left(|\langle \xi, z_n \rangle| - \max_{m \leq n} |\langle \eta, z_m \rangle| \right).$$

Since the function $\langle \xi, - \rangle$ is 1-Lipschitz, the function f is uniformly continuous (and bounded).

Let $\mathcal{H} \subseteq \ell^2$ be a closed subspace with $\dim \mathcal{H} = \infty$. We claim that

$$\operatorname{osc}\left(f|\operatorname{St}(\mathcal{H})\right) \geq 1. \tag{5.1}$$

Let $\varepsilon > 0$ be arbitrary. Find an N with $\|\operatorname{proj}_{\mathcal{H}} z_N\| > 1 - \varepsilon$, and let ξ be the projection of z_N to $\mathcal{H}$ normalized to one. Choose η that is orthogonal to all the vectors $z_1, \ldots, z_M$ (this is where the infinite-dimensionality of $\mathcal{H}$ is being used). One has

$$\begin{aligned} f(\xi,\eta) &\geq |\langle \xi, z_N \rangle| - 0 \\ &> 1 - \varepsilon. \end{aligned}$$

Now, if $k \leq N$, then

$$|\langle \eta, z_k \rangle| - \max_{m \leq n} |\langle \xi, z_k \rangle| = - \max_{m \leq n} |\langle \xi, z_k \rangle| \leq 0,$$

while for $k \geq N$,

$$|\langle \eta, z_k \rangle| - \max_{m \leq n} |\langle \xi, z_k \rangle| \leq |\langle \eta, z_k \rangle| - |\langle \xi, z_M \rangle| < 1 - (1 - \varepsilon) = \varepsilon.$$

Thus, $f(\eta, \xi) < \varepsilon$, and

$$\operatorname{osc}\left(f|\operatorname{St}_2(\mathcal{H})\right) \geq 1 - 2\varepsilon$$

for all $\varepsilon > 0$. The claim in Eq. (5.1) follows, and the restriction of function f to the Stiefel manifold $\operatorname{St}_2(\infty)$ is not oscillation stable. $\triangle$

Hands-waving on the proof of Hjorth's theorem 5.1.37. Very roughly speaking, the result is established by considering three cases.

Case 1. There is a point $x \in X$ whose orbit closure $\operatorname{cl}[G \cdot x]$ is compact and non-trivial. In this case, there is a continuous group homomorphism from G to the group of isometries of the compact space $\operatorname{cl}[G \cdot x]$. As every such group is compact, one can use Example 5.1.36 to conclude that the G-space X has distortion.

Case 2. There is a point $x \in X$ whose stabilizer St_x in the group G possesses a compact orbit closure with regard to some point $y \in X$, that is, $\operatorname{cl}[\operatorname{St}_x \cdot y]$ is compact and moreover different from one point.

In this case, the $\hat{G}^L$-orbit of (x, y) in $X \times X$ looks like a "fibre bundle" with a subset of X as the base and with compact fibre. The action of the semigroup $\hat{G}^L$ in the direction of those fibres looks like an action of a compact group, and one can use this observation to construct a uniformly continuous function whose oscillation on each fibre is 1.

Case 3. If there are two points x, y such that neither $G \cdot x$ nor $\mathrm{St}_x \cdot y$ are precompact, there is in a sense "plenty of space" to move the objects in the phase space around — we are in an "infinite-dimensional" situation — and the argument is being accomplished in a way similar to that in Example 5.1.39.

The only case that "falls between the cracks" and does not belong to any of the above is that of a non-precompact group G whose one-sided completion is again a group (that is, coincides with the two-sided completion), but such a group always has distortion. $\square$

The proof of Hjorth at this stage looks highly technical, as they say, "hard." As it is being slowly digested by the mathematical community, there is no doubt that it will lead to new concepts and insights into the theory of topological transformation groups and eventually will come to be fully understood and made into a "soft" proof. In any case, we encourage the reader to read the original work [92].

5.2 Open questions

1. Glasner's problem

(Eli Glasner [59]). Does there exist a group topology on $\mathbb{Z}$ that is minimally almost periodic but admits a free action on a compact space? (An equivalent question is: does there exist a minimally almost periodic topology that is not extremely amenable?)

See a discussion of this problem at the end of Section 2.6, and also in [59] and [187].

It even remains unknown if there exists an abelian minimally almost periodic topological group acting freely on a compact space. (This question is not equivalent to Glasner's problem, because there are examples of minimally almost periodic abelian Polish groups whose every monothetic subgroup is discrete, such as $L^p(0,1)$ with $0 < p < 1$.)

2. Generic monothetic subgroups of $\mathrm{Iso}\,(\mathbb{U})$

(Glasner and Pestov, c. 2001) Is a generic monothetic subgroup (in the sense of Baire category) of the isometry group $\mathrm{Iso}\,(\mathbb{U})$ of the Urysohn metric space extremely amenable? minimally almost periodic?

3. Extending Veech theorem to k_ω groups

A topological space X is called a *k_ω-space* (or: a *hemicompact* space) if it admits a countable cover K_n, $n \in \mathbb{N}$ by compact subsets in such a way that an $A \subseteq X$ is closed if and only if $A \cap K_n$ is closed for all n. For example, every countable CW-complex and every second countable locally compact group are such.

Is it true that every topological group G that is a k_ω-space admits a free action on a compact space?

Same for abelian groups.

A positive answer would have answered in the affirmative Glasner's question in view of known examples of minimally almost periodic k_ω group topologies on the group $\mathbb{Z}$ of integers [189, 147].

4. Geometric criterion for extreme amenability

(Edmond Granirer). Is it true that a topological group G is extremely amenable if and only if, whenever G is represented continuously by isometries in a Banach space E, for every $\xi \in E$ one has

$$\mathrm{dist}\,(G \cdot \xi, 0) = \mathrm{dist}\,(\mathrm{conv}\,(G \cdot \xi), 0)?$$

Here $G \cdot \xi$ is the G-orbit of ξ, and $\mathrm{conv}\,(G \cdot \xi)$ its convex hull.

The answer is "yes" for discrete semigroups [70], unfortunately it reveals nothing about the group situation because a discrete group is never extremely amenable.

5. Re-reading an old question by Furstenberg

Some time in late 1970's Furstenberg had asked if the groups $U(\ell^2)$ and S_∞ had the fixed point on compacta property, and if yes, could it be used to deduce concentration of measure in unitary groups of finite rank and symmetric groups of finite rank, respectively. (The only mention of this question can be found in [78].)

This question has proved very fruitful: as we know now, $U(\ell^2)$ is extremely amenable (Gromov and Milman), while S_∞ is not (Pestov), its universal minimal flow is nevertheless computed (Glasner and Weiss), and concentration of measure in finite symmetric groups indeed leads to extreme amenability of various groups of transformations of spaces with measure (Giordano and Pestov), etc.

However, all of this only touches one aspect of the question, while the other remains entirely unexplored. Here is a reasonable general reading of it. Suppose G is an extremely amenable topological group containing a chain of compact subgroups (G_n) whose union is everywhere dense in G. Is G a Lévy group? If yes, is the family (G_n) a Lévy family?

6. $U(\infty)$

Is the group $SU(\infty)$, the inductive limit of unitary groups of finite rank, equipped with the finest topology inducing the given topology on each $SU(n)$, an extremely amenable group?

7. $U(\infty)_1$

If K is an operator of Schatten class 1 (trace class), then the operator of the form $\mathbb{I} + K$ has a well-defined determinant, and all such operators with determinant 1 form a Banach-Lie group with regard to the trace class metric. Is this group Lévy? Extremely amenable?

8. Diffuse submeasures

(Farah and Solecki) Assume ϕ is a diffuse submeasure. Is $L_0(\phi)$ extremely amenable?

From a message by Ilijas Farah: "This may actually be rather ambiguous. $L_0(\phi)$ needs to be defined carefully, and two options are (1) ϕ is a Borel submeasure on the reals, and we consider all Borel functions wrt convergence in phi-submeasure. (2) ϕ is a submeasure on a countable Boolean algebra A and $L_0(\phi)$ is the completion of $S(A)$, the space of simple functions, wrt ϕ-convergence. Christensen and Herrer worked with (2). (1) is not separable unless ϕ is continuous. We have worked mostly with (2), and at this point it even seems possible that the answers in these two cases differ."

(V.P.) In connection with Furstenberg's question, is the exotic group constructed by Herer and Christensen [90] a Lévy group?

9. Ramsey results from concentration

Can one use concentration of measure in the dynamical setting (i.e., some generalized form of a Lévy group notion, like, for instance, that of a generalized Lévy group from [142, 58]) in order to prove Ramsey-type results for discrete structures? (Recall that the automorphism group of a Fraïssé structure is never Lévy, because whenever it is extremely amenable, it preserves an order and so contains no non-trivial compact subgroups.)

10. Oscillation stability of $\mathbb{U}_1$

Is the Urysohn space $\mathbb{U}_1$ of diameter one oscillation stable? That is, let γ be a finite cover of $\mathbb{U}_1$, and let $\varepsilon > 0$; does there exist $A \in \gamma$ containing an isometric copy of $\mathbb{U}_1$ to within ε? If $\mathbb{U}_1$ is oscillation stable, why is not the sphere $\mathbb{S}^\infty$? The answer is unknown even for the universal Urysohn metric space $\mathbb{U}_{\{0,1,2,3\}}$ with the distance taking values $0, 1, 2, 3$.

It is hoped that this problem will help get some insight into the phenomenon of oscillation stability and maybe lead to a direct proof of distortion property of $\mathbb{S}^\infty$, based entirely on the intrinsic metric geometry of the sphere.

11. Indecomposability of $\mathbb{U}_{\mathbb{Q},1}$

The previous problem is obviously related to the following question by Nešetřil: if the rational Urysohn space of diameter one is partitioned into finitely many pieces, does one of them contain an isometric copy of the original space?

12. Model for the Urysohn space $\mathbb{U}$

(M. Fréchet [50], p. 100; P.S. Alexandroff [170]). Find a model for the Urysohn space $\mathbb{U}$, that is, a concrete realization.

13. Structure of fixed points of isometries

(John Clemens). Suppose the set of fixed points of an isometry of the Urysohn space $\mathbb{U}$ is non-empty. Is it then isometric to the space $\mathbb{U}$ itself?

14. Universal spherical metric space

Study the properties of the *universal spherical metric space* and its group of isometries.

This unbounded ultrahomogeneous metric space, universal for all separable spherical spaces in the sense of [19], is in the same relation with the Urysohn space $\mathbb{U}$ as the unit sphere $\mathbb{S}^\infty$ is with the Urysohn space of diameter one. For comparison, $\mathbb{S}^\infty$ is the universal spherical metric space of diameter 1, or else the unit sphere in the universal spherical metric space.

Here is a model for this space: it is a metric subspace of ℓ^2, formed by all points $x \in \ell^2$ with the property

$$\lim_{n\to\infty}\left(n^2 - \sum_{i=1}^{n}|x_i - 1|^2\right) = 0,$$

that is, it is a "sphere of infinitely large radius around a point outside of ℓ^2." (This realization reminds one of the construction of non-trivial cocycles in the theory of groups with property (T), so there may be a link there.)

15. Universal topological group of weight $\tau > \aleph_0$

(Uspenskij). Does there exist a universal topological group of a given uncountable weight τ? Of *any* uncountable weight?

16. Co-universal Polish group

(Kechris). Does there exist a co-universal Polish topological group G, that is, such that every other Polish group is a topological factor-group of G?

In the abelian case, the answer is in the positive [156].

17. Factors of subgroups of $U(\ell^2)$

(Kechris). Is every Polish topological group a topological factor-group of a subgroup of $U(\ell^2)$ with the strong topology?

Again, in the abelian case the answer is in the positive [54].

18. Topological subgroups of $U(\ell^2)$

(A.I. Shtern [157]). Describe intrinsically those topological groups embeddable into $U(\ell^2)$ (with the strong topology) as topological subgroups.

19. Fixed point on metric compacta property

Does the unitary group $U(\ell^2)$, viewed as an abstract group without topology, have the *fixed point on metric compacta property* [154]? The same question for the isometry group of the Urysohn space of diameter one.

20. Uniform topology on the general linear group

(Megrelishvili) Is the uniform operator topology on $\mathrm{GL}(\ell^2)$ the weakest group topology with regard to which the standard action on the space ℓ^2 (equivalently: on the projective space $\mathbb{P}\mathrm{r}_{\ell^2}$) is continuous? This requires a good understanding of the equivariant $\mathrm{GL}(\ell^2)$-compactification of the projective space $\mathbb{P}\mathrm{r}_{\ell^2}$.

21. Oscillation stability of spheres in Banach spaces

(Hjorth [92]). Let E be a separable Banach space and let $\mathbb{S}_E$ denote the unit sphere of E viewed as an $\mathrm{Iso}\,(E)$-space, where the latter group is equipped with the strong operator topology. Is it true that the $\mathrm{Iso}\,(E)$-space $\mathbb{S}_E$ is never oscillation stable, that is, always has distortion? (Note: this would not mean that E has distortion itself, as for Banach spaces whose spheres are not ultrahomogeneous — that is, non-Hilbert spaces — the two concepts of oscillation stability diverge.)

22. Non-singular actions of Lévy groups

(Glasner–Tsirelson–Weiss [63]). Can a Polish Lévy group act in a Borel fashion on a Polish space X equipped with a probability measure μ in such a way that the action on (X, μ) is measure class preserving, ergodic, and non-trivial ($\mathrm{supp}\,\mu$ is not a singleton)?

23. Whirly actions versus Lévy property

(Glasner–Weiss [62]). Suppose the natural action of a closed subgroup $G < \mathrm{Aut}\,(X, \mu)$ is whirly; is G necessarily a Lévy group?

24. Generic subgroups of $\mathrm{Aut}\,(X, \mu)$

(Glasner–Weiss [62]) .Is the generic Polish monothetic subgroup $\Lambda(T) = \mathrm{cl}\,\{T^n : n \in \mathbb{Z}\} < \mathrm{Aut}\,(X, \mu)$ Lévy?

25. Extreme amenability of groups of maps

(Giordano and Pestov [58]). Let G be an amenable Polish group (not necessarily locally compact). Is the group $L^0(X, \mu; G)$ of all (equivalence classes) of Borel measurable maps from the standard non-atomic Borel probability space (X, μ) to G, equipped with the topology of convergence in measure, extremely amenable? (For G locally compact, the answer is yes [142].)

26. Groups of gauge transformations

(A. Carey and H. Grundling [28]). Let X be a smooth compact manifold, and let G be a compact (simple) Lie group. Is the group $C^\infty(X, G)$ of all smooth maps from X to G, equipped with the pointwise operations and the C^∞ topology, amenable?

To begin with, is the group of all continuous maps $C([0, 1], SO(3))$ with the uniform convergence topology amenable?

27. $\mathrm{Aut}^*(X,\mu)$ with uniform topology

Is the group $\mathrm{Aut}^*(X,\mu)$ of all non-singular transformations of a standard non-atomic finite measure space (X,μ), equipped with the uniform topology, amenable? extremely amenable?

28. Universal minimal flows and manifolds

(Uspenskij). Give an explicit description of the universal minimal flow of the homeomorphism group $\mathrm{Homeo}\,(X)$ of a closed compact manifold X in dimension $\dim X > 1$ (with the compact-open topology).

29. Universal minimal flow for $\mathrm{Homeo}\,(\mathbb{I}^\omega)$

(Uspenskij). The same question for the group of homeomorphisms of the Hilbert cube $Q = \mathbb{I}^\omega$.

30. Pseudoarc as the universal minimal flow

(Uspenskij). Is the pseudoarc P the universal minimal flow for its own homeomorphism group?

31. Concentration to a nontrivial space

M. Gromov [76], section $3\frac{1}{2}.45$ (p. 200) has defined a metric on the space of equivalence classes of Polish mm-spaces in such a way that a sequence $X_n = (X_n, d_n, \mu_n)$ of mm-spaces forms a Lévy family if and only if it converges to the trivial mm-space $\{*\}$.

This approach allows one to talk of *concentration to a nontrivial mm-space.* According to Gromov, this type of concentration commonly occurs in statistical physics. At the same time, there are very few known non-trivial examples of this kind in the context of transformation groups.

Here is a question (the author, [143]). Suppose a family of subgroups or other sub-objects of a topological group G concentrate to a non-trivial space. When is this space the universal minimal flow of G?

A more concrete question: do the symmetric subgroups S_n, $n \in \mathbb{N}$, of S_∞ concentrate in the sense of Gromov to the universal minimal flow LO, the compact space of all linear orders on ω, equipped with the topology induced from $\{0,1\}^{\omega\times\omega}$?

32. Separable Fraïssé structures

Does there exist a "continuous" version of Fraïssé theory, where the condition of countability would be replaced by that of separability? Can concepts and methods of model theory be transferred to the context of complete separable metric spaces? Any such theory would include, as particular cases of "continuous Fraïssé structures," the Urysohn metric space, the unit sphere $\mathbb{S}^\infty$, and the Hilbert space ℓ^2.

33. Non-commutative concentration

(V. Milman). Non-commutative spaces of Alain Connes support appropriate "quantum" versions of both metric and measure [32]. It is therefore an interesting problem, to develop a theory of concentration of measure for such objects. Some initial results concerning concentration of measure in non-commutative tori have been obtained by Milman and Moscovici (unpublished). A further problem is to link non-commutative concentration and dynamics.

34. Co-Lévy groups

The interaction between dynamics and concentration as captured in the concept of a Lévy group is built around the increasing chains of subgroups of "growing finite dimension." Can a similar dynamical theory be developed around a descreasing chain of normal subgroups of "growing finite codimention?"

Two concrete situations to consider here are: (a) the group $\mathrm{SL}(n,\mathbb{Z})$ and its factor-groups $\mathrm{SL}(n,\mathbb{Z}_p)$, where p is a prime number, equipped with the word distance (used in a famous way to produce examples of superconcentrator graphs [106]), and (b) the Hilbert-Smith conjecture, reduced to the following question: can the compact group A_p of p-adic integers act freely (or effectively) on a finite-dimensional compact manifold? (cf. e.g. [188].)

35. Invariant Subspaces

Suppose the answer to the Invariant Subspace Problem is in the affirmative, that is, every bounded linear operator on ℓ^2 has a non-trivial closed invariant subspace. Then every such operator admits a maximal chain of closed invariant subspaces. Is the automorphism group of a "generic" chain extremely amenable if equipped with some natural group topology? (E.g. the uniform topology induced from $\mathrm{GL}(\ell^2)$, or possibly a coarser group topology.) If yes, then what?

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Index