

# Divisibility of countable metric spaces

Christian Delhommé

E.R.M.I.T.

Département de Mathématiques et d'Informatique

Université de La Réunion

15, avenue René Cassin, BP 71551

97715 Saint-Denis Messag. Cedex 9, La Réunion, France

`delhomme@univ-reunion.fr`

Claude Laflamme\*

University of Calgary

Department of Mathematics and Statistics

Calgary, Alberta, Canada T2N 1N4

`laf@math.ucalgary.ca`

Maurice Pouzet

PCS, Université Claude-Bernard Lyon1,

Domaine de Gerland -bât. Recherche [B], 50 avenue Tony-Garnier,

F69365 Lyon cedex 07, France

`pouzet@univ-lyon1.fr`

Norbert Sauer<sup>†</sup>

University of Calgary

Department of Mathematics and Statistics,

Calgary, Alberta, Canada T2N 1N4

`nsauer@math.ucalgary.ca`

---

\*Supported by NSERC of Canada Grant # 690404

<sup>†</sup>Supported by NSERC of Canada Grant # 691325

**Proposed running head:** Divisibility of countable metric spaces  
**Contact:** Claude Laflamme  
University of Calgary  
Department of Mathematics and Statistics  
Calgary, Alberta, Canada T2N 1N4  
laf@math.ucalgary.ca

## Abstract

Prompted by a recent question of G. Hjorth [6] as to whether a bounded Urysohn space is indivisible, that is to say has the property that any partition into finitely many pieces has one piece which contains an isometric copy of the space, we answer this question and more generally investigate partitions of countable metric spaces.

We show that an indivisible metric space must be totally Cantor disconnected, which implies in particular that every Urysohn space  $\mathbb{U}_V$  with  $V$  bounded or not but dense in some initial segment of  $\mathbb{R}_+$ , is divisible. On the other hand we also show that one can remove “large” pieces from a bounded Urysohn space with the remainder still inducing a copy of this space, providing a certain “measure” of the indivisibility. Associated with every totally Cantor disconnected space is an ultrametric space, and we go on to characterize the countable ultrametric spaces which are homogeneous and indivisible.

Keywords: Partition theory, metric spaces, homogeneous relational structures, Urysohn space, ultrametric spaces.

## 1 Introduction, presentation of the results and basic notions

A metric space  $\mathbb{M} := (M; d)$  is called *divisible* if there is a partition of  $M$  into two parts, none of which contains an isometric copy of  $\mathbb{M}$ . If  $\mathbb{M}$  is not divisible then it is called *indivisible*. Note that by repeated partition of  $M$  into two pieces one obtains that if  $\mathbb{M}$  is indivisible then for every partition of  $M$  into finitely many pieces there is one piece which contains an isometric copy of the whole space. Every finite metric space (with at least two elements) is divisible, so the interest lies in infinite metric spaces. The uncountable case is different as the indivisibility property may fail badly. For example, every uncountable separable metric space can be divided into two parts such that no part contains a copy of the space via a one-to-one continuous map. This result, based on the Bernstein property, 1908 (see [7] p.422), does not really involve the structure of metric spaces. In this paper we deal essentially with the countable case.

After the extension of the above result to uncountable subchains of the real line (Dushnik, Miller, 1940), the notion of indivisibility was considered for chains and then for relational structures (see for example [3] [11]). The notion we consider also falls under the framework of relational structures. Indeed, a metric space can be interpreted ‘in several ways’ to be a relational structure whose relations are binary and symmetric, the isometries being

the isomorphisms of the relational structure. Because of this connection, we will use some basic notions and results about relational structures, and what we need is listed in Section 1.1.

One of the most important notion in the theory of relation is the notion of *homogeneous structure*. We start looking at homogeneous metric spaces, that is metric spaces  $\mathbb{M}$  for which every isometry from a finite subset of  $\mathbb{M}$  onto an other one extends to an isometry from  $\mathbb{M}$  onto  $\mathbb{M}$ . A basic example is the famous Urysohn metric space [15]. This space is the Cauchy completion of  $\mathbb{U}_{\mathbb{Q}_+}$ , the countable homogeneous universal metric space with rational values (this space is universal in the sense that every countable metric space with rational values embeds isometrically into it), a space that we call *the Urysohn space with rational values*. Replacing the set of non-negative rational numbers by a subset  $V$  of  $\mathbb{R}_+$  containing  $\{0\}$ , we show that there is a countable homogeneous universal metric space with values in  $V$  if and only if  $V$  is countable and satisfies a "four-element-condition" (Theorem 1.4). When it exists, this space, that we denote by  $\mathbb{U}_V$  and call *the Urysohn space over  $V$* , is *age-indivisible* in the sense that for every partition of it into two parts, one of the parts embeds every finite metric space with values in  $V$  (Theorem 1.10). In general it is not indivisible, indeed, we show that every indivisible metric space has a bounded diameter (Theorem 3.14) and is totally Cantor disconnected (Corollary 3.7, Theorem 3.8). In particular, if  $V$  is unbounded or contains a dense initial interval then  $\mathbb{U}_V$  is divisible. An example is  $\mathbb{U}_{\mathbb{Q}_+, \leq 1}$ , the Urysohn space with diameter 1 (Theorem 4.5). On the other hand, we will show that  $\mathbb{U}_{\mathbb{Q}_+, \leq 1}$  is "almost" indivisible, in the sense that we can remove "almost" all of the elements of the space in various ways and the remainder still contains an isometric copy of the space (Theorem 4.14). On a totally Cantor disconnected space  $\mathbb{M}$  there is a natural ultrametric distance ([8]). We prove that if an ultrametric space is indivisible then it contains no infinite strictly increasing sequence of balls and its diameter is attained (Theorem 2.11). We prove that if  $\mathbb{M}$  is countable homogeneous and indivisible then the corresponding ultrametric space  $\mathbb{M}^*$  is homogeneous and indivisible (Theorem 3.12); that such a space is of the form  $\mathbb{U}lt_V$ , the countable homogeneous and universal ultrametric space with values in some dually well-founded countable subset  $V$  of  $\mathbb{R}_+$  containing  $\{0\}$  (Theorem 2.12), and we give a description of  $\mathbb{U}lt(V)$  in terms of valued-trees (a similar description was independently obtained by L. Nguyen Van The 2005[10]). We conclude this paper by pointing out some problems.

Our notations are fairly standard. We point out that we denote by  $\min(P)$  the set of minimal elements of a poset  $P$ . We denote by  $\mathbb{N}$  the set of natural integers. As it is customary, a set is *denumerable* if there is a

bijection from it onto  $\mathbb{N}$  and it is *countable* if it is either finite or denumerable.

### 1.1 Relational structures, homogeneous structures and their ages

A relational structure is a pair  $A := (A; \mathbf{R})$  where  $\mathbf{R} := (R_i)_{i \in I}$  is made of relations on the set  $A$ , the relation  $R_i$  being an  $n_i$ -ary relation identified with a subset of  $E^{n_i}$ . The family  $\mu := (n_i)_{i \in I}$  is the *signature* of  $A$ . To  $\mu := (n_i)_{i \in I}$ , one may attach a family  $\rho := (r_i)_{i \in I}$  of predicate symbols and one may see  $A$  as a realization of the languages whose non logical symbols are these predicate symbols. Let  $F$  be a subset of  $A$ , the induced substructure on  $A$  is denoted  $A \upharpoonright_F$ . Let  $A' := (A'; \mathbf{R}')$  having the same signature as  $A$ . A local isomorphism from  $A$  to  $A'$  is an isomorphism  $f$  from an induced substructure of  $A$  onto an induced substructure of  $A'$ ; if the domain of  $f$  is  $A$  then  $f$  is an *embedding* of  $A$  to  $A'$ . The image of an embedding of  $A$  in  $A'$  is called a *copy* of  $A$  in  $A'$ .

A relational structure  $A := (A; \mathbf{R})$  is *divisible* if there is a partition  $A = X \cup Y$  none of  $X$  and  $Y$  containing a copy of  $A$ . A relational structure which is not divisible is called *indivisible*. The *age* of a relational structure is the class of all finite relational structures which have an embedding into the structure.

We will use several properties of homogeneous structures (also called ultrahomogenous structures). Most are restatements or consequences of the Theorem of R. Fraïssé (Point 6 below). A more detailed account can be found in the book [3].

1. A countable relational structure  $H := (H, \mathbf{R})$  is *homogeneous* if every local isomorphism defined on a finite subset of  $H$  into  $H$  has an extension to an automorphism of  $H$ .
2. A countable relational structure  $H := (H, \mathbf{R})$  is homogeneous if and only if it satisfies the following *mapping extension property*:

*If  $F := (F; \mathbf{R})$  is an element of the age of  $H$  for which the substructure of  $H$  induced on  $H \cap F$  is equal to the substructure of  $F$  induced on  $H \cap F$  then there exists an embedding of  $F$  into  $H$  which is the identity on  $H \cap F$ .*

3. Two countable homogeneous structures with the same age are isomorphic.

4. A class  $\mathcal{D}$  of relational structures has the *amalgamation property* (in brief AP) if for every members  $A, B, C$  of  $\mathcal{D}$ , embeddings  $f : A \rightarrow B$ ,  $g : A \rightarrow C$ , there is some member  $A'$  of  $\mathcal{D}$  and embeddings  $f' : B \rightarrow A'$ ,  $g' : C \rightarrow A'$  such that  $f' \circ f = g' \circ g$ .
5. A homogeneous structure embeds any countable younger structure, i.e. any countable structure whose age is included in that of the homogeneous one.
6. A class  $\mathcal{D}$  of finite relational structures is the age of a countable homogeneous structure if and only if it is non-empty, is closed under embeddability and has the amalgamation property.
7. A subset  $S \neq \emptyset$  of  $H$  is an *orbit* of  $H$  if it is an orbit for the action of the automorphism group  $Aut(H)$  of  $H$  which fixes pointwise a finite subset of  $H$ . That is to say that there exists a finite subset  $F$  of  $H$ , called a *socket* of the orbit  $S$ , so that for some  $s \in H \setminus F$ :

$$S := \{f(s) : f \in Aut(H) \text{ and } f(y) = y \text{ for all } y \in F\}.$$

8. If  $H$  is a countable homogeneous structure, then a subset  $S \subseteq H$  is an orbit of  $H$  if there is an  $s \in H \setminus F$  and  $S$  is equal to the set of all elements  $t \in H$  so that the function which fixes the socket  $F$  pointwise and maps  $s$  to  $t$  is an isomorphism of the substructure of  $H$  induced on  $S \cup \{s\}$  on the substructure of  $H$  induced on  $S \cup \{t\}$ . That is, the orbit  $S$  is the set of all elements of  $H$  which are of the same “one-type” over  $F$ .
9. If  $H$  is a countable homogeneous structure, a subset  $X \subseteq H$  induces an isomorphic copy of  $H$  if and only if  $S \cap X \neq \emptyset$  for every orbit  $S$  of  $H$  with socket a subset of  $X$ .
10. Let  $\kappa$  be a cardinal and  $\mathcal{A}_\kappa$  (resp.  $\mathcal{A}_{\kappa, < \omega}$ ) be the collection of all (resp. finite) relational structures  $B := (B; \mathbf{R})$  where  $\mathbf{R} := (R_i)_{i < \kappa}$  is a sequence of irreflexive and symmetric binary relations symbols for which for all  $x, y \in B$  with  $x \neq y$  there exists exactly one  $i < \kappa$  with  $R_i(x, y)$ . The class  $\mathcal{A}_{\kappa, < \omega}$  has the amalgamation property. If  $\kappa \leq \omega$  then it is countable, therefore, this is the age of a countable homogeneous structure, that we denote  $H_\kappa$ . For example,  $H_2$  is the well-known *Random graph* or *Rado graph*. Each such  $H_k$  is indivisible.

## 1.2 Metric spaces and relational structures

Let us recall a few standard notions. Given two metric spaces  $\mathbb{M} := (M; d)$  and  $\mathbb{M}' := (M'; d')$ , a *local isometry* from  $\mathbb{M}$  to  $\mathbb{M}'$  is an isometry  $f$  from a subspace of  $\mathbb{M}$  onto a subspace of  $\mathbb{M}'$ , and this is an *isometric embedding* if the domain of  $f$  is  $M$ .

$\mathbb{M}$  is called *homogeneous* if every local isometry defined on  $M$  and with values in  $M$  extends to an isometry from  $\mathbb{M}$  onto itself.

The *age* of  $\mathbb{M}$  is the collection of finite metric spaces which embed into  $\mathbb{M}$ . Finally the *spectrum* of  $a \in M$  is the set  $\text{Spec}(\mathbb{M}, a) = \{d(a, x) \mid x \in M\}$  and the *spectrum* of  $(M; d)$  is the set  $\text{Spec}(\mathbb{M}) := \cup\{\text{Spec}(\mathbb{M}, a) : a \in M\} = \{d(x, y) \mid x, y \in M\}$ .

Metric spaces also fall under the realm of relational structures in various ways. To exemplify this association, consider a set  $I$ , a map  $f : I \rightarrow \mathbb{R}_+$  and set  $\mu := (n_i)_{i \in I}$ , where  $n_i := 2$  for all  $i \in I$ . To a metric space  $\mathbb{M} := (M; d)$  associate two relational structures, namely  $\mathbb{M}_{f, \leq} := (M; \mathbf{R})$  and  $\mathbb{M}_{f, =} := (M; \mathbf{S})$  where  $\mathbf{R} := (R_i)_{i \in I}$  and  $\mathbf{S} := (S_i)_{i \in I}$  are defined by:

$$(x, y) \in R_i \iff d(x, y) \leq f(i) \quad (1)$$

$$(x, y) \in S_i \iff 0 \neq d(x, y) = f(i) \quad (2)$$

Using the above notation, the following result summarizes the connections we will need, and the straightforward proof is left to the reader.

**Lemma 1.1.** *1. Every local isometry of  $\mathbb{M}$  is a local isomorphism of  $\mathbb{M}_{f, \leq}$  and of  $\mathbb{M}_{f, =}$ .*

*2. Every local isomorphism of  $\mathbb{M}_{f, \leq}$  (resp. of  $\mathbb{M}_{f, =}$ ) is a local isometry of  $\mathbb{M}$  if and only if :*

- (a) either the spectrum of  $\mathbb{M}$  contains at most a non-zero element,*
- (b) or the image of  $f$  separates the spectrum of  $\mathbb{M}$  in the sense that  $f(I) \cap [p, p'] \neq \emptyset$ , (resp in the sense that  $f(I) \cap \{p, p'\} \neq \emptyset$ ) for every  $p, p' \in \text{Spec}(\mathbb{M})$  such that  $0 < p < p'$ .*

Conversely, every binary relational structure  $\mathbf{B} := (B; \mathbf{R})$  can be viewed as a metric space, provided that the number of isomorphic types of induced substructures on two element subsets of  $B$  is not greater than the continuum. Indeed, let  $a \in \mathbb{R}_+ \setminus \{0\}$  be given, we may define a one-to-one map  $\varphi : [B]^2 \rightarrow [a, 2a]$  such that  $\varphi(\{x, y\}) = \varphi(\{x', y'\})$  if and only if  $\mathbf{B}_{\upharpoonright\{x, y\}}$  and  $\mathbf{B}_{\upharpoonright\{x', y'\}}$  are isomorphic. The map  $d : B \times B \rightarrow [a, 2a]$  defined by setting  $d(x, y) := \varphi(\{x, y\})$  if  $x \neq y$  and  $d(x, y) := 0$  if  $x = y$  is a distance. This is

particularly the case if  $B \in \mathcal{A}_\kappa$ . A map  $f : \kappa \rightarrow [a, 2a]$  be given, we will set  $m_f(B) := (B, d)$  where  $d(x, y) := f(i)$  if  $R_i(x, y)$  and  $d(x, y) := 0$  otherwise (if  $x = y$ ).

### 1.3 Homogeneous metric spaces

Let  $V$  be a set such that  $0 \in V \subseteq \mathbb{R}_+$ . Let  $\mathcal{M}_V$  (resp.  $\mathcal{M}_{V, < \omega}$ ) be the collection of metric spaces (resp. finite metric spaces)  $\mathbb{M}$  whose spectrum is included into  $V$ . We may note that any such  $V$  is in fact a spectrum, indeed  $V = \text{Spec}(\mathbb{M})$  where  $\mathbb{M} := (V; d)$  and  $d(x, y) := \max(\{x, y\})$ .

For  $u_1, u_2, u'_1, u'_2$  in  $V$ , let

$$\phi(u_1, u_2, u'_1, u'_2) := [\max\{|u_1 - u_2|, |u'_1 - u'_2|\}, \min\{u_1 + u_2, u'_1 + u'_2\}]$$

and set

$$\rho_V(u_1, u_2, u'_1, u'_2) \text{ if } \phi(u_1, u_2, u'_1, u'_2) \cap V \neq \emptyset$$

We say that  $V$  satisfies the *four-values condition* if

$$\forall u_1, u_2, u'_1, u'_2 \text{ in } V \quad (\rho_V(u_1, u_2, u'_1, u'_2) \Rightarrow \rho_V(u_1, u'_1, u_2, u'_2))$$

**Lemma 1.2.** *Let  $u_1, u_2, u'_1, u'_2 \in V$  such that  $\rho_V(u_1, u_2, u'_1, u'_2)$  holds.*

1. *If  $u_1, u_2, u'_1$  and  $u'_2$  are all non zero then  $\phi(u_1, u_2, u'_1, u'_2) \cap (V \setminus \{0\}) \neq \emptyset$ .*
2. *If some argument is zero, then  $\rho_V(u_1, u'_1, u_2, u'_2)$  holds. In particular the four-value condition is equivalent to its restriction to non zero arguments.*

*Proof.* 1. If  $\max\{|u_1 - u_2|, |u'_1 - u'_2|\} > 0$  then every element of  $\phi(u_1, u_2, u'_1, u'_2)$  is positive. If  $\max\{|u_1 - u_2|, |u'_1 - u'_2|\} = 0$  then  $\min\{u_1, u_2, u'_1, u'_2\} \in \phi(u_1, u_2, u'_1, u'_2) \cap (V \setminus \{0\})$ .

2. We may assume wlog that  $u_1 = 0$ . Since  $\rho_V(u_1, u_2, u'_1, u'_2)$  holds,  $\phi(u_1, u_2, u'_1, u'_2) = \{u_2\}$  and likewise  $\phi(u_1, u'_1, u_2, u'_2) = \{u'_1\}$ . □

**Proposition 1.3.** *Let  $V$  be a set such that  $0 \in V \subseteq \mathbb{R}_+$ . The class  $\mathcal{M}_{V, < \omega}$  is the age of a metric space whose spectrum is  $V$ . Furthermore the following are equivalent :*



1.  $\mathcal{M}_{V, < \omega}$  has the amalgamation property;
2.  $\mathcal{M}_{V, < \omega}$  has the disjoint amalgamation property : i.e. two members of  $\mathcal{M}_{V, < \omega}$  that coincide on their intersection admit (on their union) a common extension in  $\mathcal{M}_{V, < \omega}$ ;
3. For any two members  $(M_1, d_1)$  and  $(M_2, d_2)$  of  $\mathcal{M}_{V, < \omega}$  such that  $d_1$  and  $d_2$  coincide on  $M_1 \cap M_2$  and such that  $|M_1| = |M_2| = 3$  and  $|M_1 \cap M_2| = 2$ , there is a semi-distance on  $M_1 \cup M_2$  with spectrum included in  $V$  and whose restrictions to  $M_1$  and  $M_2$  are  $d_1$  and  $d_2$  respectively. (A semi-distance is symmetric and satisfies the triangular inequality but may fail to satisfy the separation condition.)
4.  $V$  satisfies the four-values condition.

*Proof of the proposition.* First, there is a family  $(\mathbb{M}_i)_{i \in I}$  of at most  $\kappa := |V| \cdot \aleph_0$  members of  $\mathcal{M}_{V, < \omega}$  such that every member of  $\mathcal{M}_{V, < \omega}$  embeds into one of the  $\mathbb{M}_i$ 's. Pick an element  $0_i \in M_i$  for each  $i \in I$ . Set  $M := \{\bar{x} := (x_i)_{i \in I} : x_i \in M_i \text{ for all } i \in I \text{ and } \{i \in I : x_i \neq 0_i\} \text{ is finite}\}$ , set  $d(\bar{x}, \bar{y}) := \max\{d(x_i, y_i) : i \in I\}$  and set  $\mathbb{M} := (M, d)$ . Then the age of  $\mathbb{M}$  is  $\mathcal{M}_{V, < \omega}$ . Since every subset of  $\mathbb{R}_+$  containing 0 is a spectrum, the spectrum of  $\mathbb{M}$  is  $V$ .

Next, the implications  $2 \Rightarrow 1 \Rightarrow 3$  are obvious. We prove  $3 \Rightarrow 4 \Rightarrow 2$ .

- We assume that Point 3 holds and we check the four-value condition : Consider  $u_1, u_2, u'_1$  and  $u'_2$  in  $V$  and assume that  $\rho_V(u_1, u_2, u'_1, u'_2)$  holds. Thanks to the second part of the Lemma, we can assume that  $u_1, u_2, u'_1$  and  $u'_2$  are all positive, and then thanks to the first part, we know that  $\phi(u_1, u_2, u'_1, u'_2)$  contains some non zero element  $v$  of  $V$ . Then given a four-element set  $\{x_1, x_2, y, y'\}$ , the following define distances  $d_1$  and  $d_2$  on  $M_1 := \{x_1, y, y'\}$  and on  $M_2 := \{x_2, y, y'\}$  that coincide on  $M_1 \cap M_2 := \{y, y'\}$  :

$$\begin{cases} d_1(x_1, y) = u_1, & d_1(x_1, y') = u'_1, & d_1(y, y') = v \\ d_2(x_2, y) = u_2, & d_2(x_2, y') = u'_2, & d_2(y, y') = v \end{cases}$$

Now by Point 3,  $d_1$  and  $d_2$  have a common extension to a semi-distance  $d$  on  $M_1 \cup M_2$  with values in  $V$ . Then  $d(x_1, x_2)$  belong to  $\phi(u_1, u_2, u'_1, u'_2) \cap V$ .

- Now we assume that the four-value condition holds and we show the disjoint amalgamation property : Let  $\mathbb{M}_1 := (M_1, d_1)$  and  $\mathbb{M}_2 := (M_2, d_2)$  be two members of  $\mathcal{M}_{V, < \omega}$  such that  $d_1$  and  $d_2$  coincide on  $M_1 \cap M_2$ . We prove by induction on the cardinality  $m$  of the symmetric difference  $M_1 \Delta M_2$  that there is a distance on  $M := M_1 \cup M_2$  extending both  $d_1$  and  $d_2$ . If

$m \leq 1$  then  $M_1 \subseteq M_2$  or  $M_2 \subseteq M_1$ , in which case there is nothing to prove. So assume that  $m > 1$  and neither  $M_1 \subseteq M_2$  nor  $M_2 \subseteq M_1$ . Pick any  $x_1 \in M_1 \setminus M_2$ ,  $x_2 \in M_2 \setminus M_1$ .

**Case 1**  $m = 2$

Observe that extending  $d_1$  and  $d_2$  by setting  $d(x_1, x_2) = d(x_2, x_1) = w$  yields a distance extending  $d_1$  and  $d_2$  if and only if  $w > 0$  and the triangular inequalities involving both  $x_1$  and  $x_2$  are satisfied (the other ones involve only one of  $d_1$  or  $d_2$ ). and that holds precisely when

$$\forall z \in M_1 \cap M_2 \quad |d_1(x_1, z) - d_2(x_2, z)| \leq w \leq d_1(x_1, z) + d_2(x_2, z)$$

Besides, it follows from the triangular inequality that  $a' := \max\{|d_1(x_1, z) - d_2(x_2, z)| : z \in M_1 \cap M_2\} \leq a := \min\{d_1(x_1, z) + d_2(x_2, z) : z \in M_1 \cap M_2\}$ . Then pick  $y$  and  $y'$  in  $M_1 \cap M_2$  such that  $|d_1(x_1, y') - d_2(x_2, y')| = a'$  and  $d_1(x_1, y) + d_2(x_2, y) = a$ , and let

$$\begin{cases} u_1 := d_1(x_1, y), & u'_1 := d_1(x_1, y') \\ u_2 := d_2(x_2, y), & u'_2 := d_2(x_2, y') \end{cases}$$

All those values are positive members of  $V$  and the distance between  $y$  and  $y'$  attests that  $\rho_V(u_1, u_2, u'_1, u'_2)$  holds, thus by the four-value condition and the first part of Lemma 1.2 and given that  $\phi(u_1, u'_1, u_2, u'_2) = [a, b]$ , the intersection  $[a', a] \cap V$  is non-empty. That concludes that case.

**Case 2**  $m > 2$ .

By induction assumption,  $\mathbb{M}_1$  and  $\mathbb{M}_2 \upharpoonright (M_2 \setminus \{x_2\})$  admit a common extension  $\mathbb{M}'_1$  on  $M_1 \cup (M_2 \setminus \{x_2\}) = M \setminus \{x_2\}$  and then, still by induction assumption,  $\mathbb{M}'_1$  and  $\mathbb{M}_2$  admit a common extension  $\mathbb{M}'_2$  on  $(M \setminus \{x_2\}) \cup M_2 = M \setminus \{x_1\}$ . Then by Case 1,  $\mathbb{M}'_1$  and  $\mathbb{M}'_2$ , which coincide on  $M \setminus \{x_1, x_2\}$ , admit a common extension on  $M$ , which extension then extends  $\mathbb{M}_1$  and  $\mathbb{M}_2$ .  $\square$

Fraïssé's theorem (Point 6 above) gives immediately:

**Theorem 1.4.** *If  $V$  is countable then it satisfies the four-values condition if and only if there is a countable homogeneous space  $\mathbb{U}_V$  whose age is  $\mathcal{M}_{V, < \omega}$ .*

We call the space  $\mathbb{U}_V$  the *Urysohn space with spectrum  $V$* . If  $V := \mathbb{Q}_+$  then  $\mathbb{U}_{\mathbb{Q}_+}$  is the homogeneous metric space whose age is the set of all finite metric spaces whose spectrum is a subset of the set of rationals. The Cauchy completion of  $\mathbb{U}_{\mathbb{Q}_+}$  is the famous space discovered by Urysohn [15].

**Example 1.5.** 1. Suppose that for some  $a \in \mathbb{R}_+ \setminus \{0\}$ ,  $V \setminus \{0\} \subseteq [a, 2a]$ . Then  $V$  satisfies the four-values condition and in fact  $\mathbb{U}_V = m_f(\mathbf{H}_\kappa)$  where  $\kappa := |V \setminus \{0\}|$  and  $f : \kappa \rightarrow V \setminus \{0\}$  is a bijective map (see Point 10 of Section 1.1).

2. An example of finite set  $V$  which does not satisfies the four point condition is  $V = \{0, 1, 3, 4, 5\}$ . Indeed,  $M_1 := \{y, y', x_1\}$  and  $M_2 := \{y, y', x_2\}$  with  $v := d(y, y') = 4$ ,  $u_1 := d(x_1, y) = 1$ ,  $u_2 := d(x_2, y) = 1$ ,  $u'_1 := d(x_1, y') = 5$  and  $u'_2 := d(x_2, y') = 3$ . Each "distance" in  $M_1$  (namely 1,4 and 3) or in  $M_2$  (1,4,5) is less than or equal to the sum of the other two, thus  $d$  is indeed a distance on  $M_1$  and on  $M_2$ . The only possible value for  $d(x_1, x_2)$  in order that  $d$  be a semi-distance on  $M_1 \cup M_2$  is 2 : Indeed  $2 = 5 - 3 = d(x_1, y') - d(x_2, y) \leq d(x_1, x_2) \leq d(x_1, y) + d(x_2, y) = 1 + 1 = 2$ . If  $V$  is infinite is not sufficient that  $V$  be dense in  $] \inf V, \sup V[$  to have the four-values conditions, since the example above also shows that  $\mathbb{R}_+ \setminus \{2\}$  fails to have it.

3. A sufficient condition for the four-value condition is

$$\forall u_1, u_2, u'_1, u'_2 \in V \quad u_1 - u'_1 \leq u_2 + u'_2 \Rightarrow \exists v \in V \quad u_1 - u'_1 \leq v \leq u_2 + u_2$$

For example, the set  $V$  of positive powers of  $\frac{1}{2}$  satisfies it. Indeed, if  $0 < \frac{1}{2^p} - \frac{1}{2^q} \leq \frac{1}{2^r} + \frac{1}{2^s}$ , then  $\frac{1}{2^p}$  lies in between.

This condition is not necessary : consider  $V := \{0, 1, 3, 5\}$ .

Notice that this sufficient condition holds whenever  $V$  is closed under sum or absolute value of the difference, or more generally when for all  $a$  and  $b$  in  $V$ , if  $a + b < \sup V$  then  $a + b$  belongs to  $V$

Examples, like  $\mathbb{N}$ ,  $\mathbb{Q}_+$ ,  $\{0, \dots, n\}$  and their corresponding Urysohn spaces are considered in [13].

**Lemma 1.6.** *If  $V$  satisfies the four-values condition then every initial segment  $V'$  of  $V$  satisfies this condition*

*Proof.* Let  $u_1, u_2, u_3, u_4 \in V'$  such that  $\rho_{V'}(u_1, u_2, u_3, u_4)$  holds. Let  $w \in \phi(u_1, u_2, u_3, u_4) \cap V'$  and let  $r := \max\{u_1, u_2, u_3, u_4, w\}$ . Case 1:  $\min\{u_2 + u_3, u_4 + u_1\} \leq r$ . In this case, since  $r \in V'$ ,  $\min\{u_2 + u_3, u_4 + u_1\} \in V'$ . Since  $\rho_V(u_2, u_3, u_4, u_1)$  holds,  $\rho_{V'}(u_2, u_3, u_4, u_1)$  holds too. Case 2:  $r < \min\{u_2 + u_3, u_4 + u_1\}$ . In this case  $r \in \phi(u_2, u_3, u_4, u_1) \cap V'$  thus  $\rho_{V'}(u_2, u_3, u_4, u_1)$  holds. □

**Remark 1.7.** For a more intuitive proof, based on the amalgamation property of  $\mathcal{M}_{V, < \omega}$ , let  $\mathbb{M}_1, \mathbb{M}_2 \in \mathcal{M}_{V, < \omega}$  and let  $\mathbb{M} := (M, d) \in \mathcal{M}_{V, < \omega}$  be a common extension. Set  $d' := d \wedge \delta$ , where  $\delta$  is the maximum of the diameters of  $\mathbb{M}_1$  and  $\mathbb{M}_2$ .

With this lemma and the above theorem follows that if there is an Urysohn space with spectrum  $V$  then for every  $\ell \in \mathbb{R}_+$  there is an Urysohn space with spectrum  $V \cap [0, \ell]$ . We denote this space  $\mathbb{U}_{V, \leq \ell}$ , eg  $\mathbb{U}_{\mathbb{Q}_+, \leq \ell}$  is the the homogeneous metric space whose age is the set of all finite metric spaces whose spectrum is a subset of the set of rationals in the interval  $[0, \ell]$ .

Let  $V$  be subset of  $\mathbb{R}_+$  containing 0; we say that  $V$  is *residuated* if for every  $x, y \in V$ , the set  $\{r \in V : y \leq x + r\}$  has a least element, denoted  $y \setminus x$ . This is the case if  $V$  is finite, if  $V$  is the positive part of a additive subgroup of  $\mathbb{R}$  or if  $V$  is *meet-closed* in the sense that for every non-empty subset of  $V$  its infimum in  $\mathbb{R}$  belongs to  $V$ .

The following proposition shows that the four-values condition is just what is needed in order to extend to metric spaces over  $V$  the most fundamental property of ordinary metric spaces.

**Proposition 1.8.** *Let  $V$  be a subset of  $\mathbb{R}_+$  containing 0. If  $V$  is residuated, then  $V$  satisfies the four-values condition if and only if the map:  $d_V : V \times V \rightarrow V$  defined by  $d_V(x, y) := \max\{y \setminus x, x \setminus y\}$  is a distance on  $V$ . When this condition is realized,  $d(x, y) = \text{Sup}\{d_V(d(x, z), d(y, z)) : z \in M\}$  for every  $x, y \in \mathbb{M} := (M, d) \in \mathcal{M}_V$ . In particular, if  $\mathbb{M}$  is finite the map  $\bar{d} : M \rightarrow V^M$  defined by setting  $\bar{d}(x)(y) := d(x, y)$  is an isometric embedding of  $\mathbb{M}$  into  $V^M$  equipped with the "Sup" distance.*

*Proof.* Clearly,  $d_V(x, y) = \text{Inf}\{r \in V : |x - y| \leq r\}$ . It follows that  $d_V$  is symmetric,  $d_V(x, y) = 0$  iff  $x = y$  and  $d_V(0, x) = x$  for every  $x \in V$ . Suppose that the four-values condition holds. Let  $x, y, z \in V$ ; set  $u_1 := x, u_2 := d_V(x, z), u_3 := d_V(z, y), u_4 := y$ . We have  $z \in \phi(u_1, u_2, u_3, u_4) \cap V$ , proving that  $\rho_V(u_1, u_2, u_3, u_4)$  holds. Since the four-values condition holds,  $\rho_V(u_2, u_3, u_4, u_5)$  holds, that is  $\max\{|u_2 - u_3|, |u_4 - u_1|\} \leq r \leq \min\{u_2 + u_3, u_4 + u_1\}$  for some  $r \in V$ . In particular  $|u_4 - u_1| \leq r \leq u_2 + u_3$  that is  $|x - y| \leq r \leq d_V(x, z) + d_V(z, y)$ . The triangular inequality  $d_V(x, y) \leq d_V(x, z) + d_V(z, y)$  follows. Conversely, suppose that  $d_V$  is a distance on  $V$ . Let  $u_1, u_2, u_3, u_4 \in V$  such that  $\rho_V(u_1, u_2, u_3, u_4)$  holds. Let  $r := \max\{d_V(u_2, u_3), d_V(u_4, u_1)\}$ . First, we have  $r \in V$ ; next, from the definition of  $d_V$ , we have  $|u_2 - u_3| \leq d_V(u_2, u_3)$  and  $|u_4 - u_1| \leq d_V(u_4, u_1)$  hence  $\max\{|u_2 - u_3|, |u_4 - u_1|\} \leq r$ ; finally the triangular inequality  $d_V(u_4, u_1) \leq$

$d_V(u_4, w) + d(w, u_1)$  applied first to  $w := 0$  gives  $d_V(u_4, u_1) \leq u_4 + u_1$  and applied to  $w \in \phi(u_1, u_2, u_3, u_4) \cap V$  gives  $d_V(u_4, u_1) \leq u_3 + u_2$  hence  $d_V(u_4, u_1) \leq \min\{u_4 + u_1, u_3 + u_2\}$ ; the same gives  $d_V(u_2, u_3) \leq \min\{u_4 + u_1, u_3 + u_2\}$ , hence  $r \leq \min\{u_4 + u_1, u_3 + u_2\}$ . This proves that  $r \in \phi(u_2, u_3, u_4, u_1) \cap V$ , hence  $\rho_V(u_2, u_3, u_4, u_1)$  holds. Thus the four-values condition holds. If  $\mathbb{M} := (M, d)$  is an arbitrary metric space with spectrum included in  $V$ , the equality  $d(x, y) = \text{Sup}\{d_V(d(x, z), d(y, z)) : z \in M\}$  is obvious. If  $M$  is finite, it expresses the fact that  $\bar{d}$  is an isometric embedding from  $\mathbb{M}$  into  $V^M$  equipped with the Sup-distance.  $\square$

**Remark 1.9.** As it is well-known every metric space embeds into some  $\ell_{\mathbb{R}}^{\infty}$  space equipped with the Sup distance. A similar result holds for members of  $\mathcal{M}_V$  provided that  $V$  is meet closed and satisfies the four-values condition.

As we shall see later on, some  $\mathbb{U}_V$ 's are divisible, still all metric spaces with age  $\mathcal{M}_{V, < \omega}$  satisfy a weaker version of the indivisibility property :

**Theorem 1.10.** *Let  $\mathbb{M}$  be a metric space with age  $\mathcal{M}_{V, < \omega}$ .*

1. *For every partition of  $M$  into two parts  $X$  and  $Y$  one of the induced metric spaces  $\mathbb{M}_{\upharpoonright X}$  and  $\mathbb{M}_{\upharpoonright Y}$  has the same age as  $\mathcal{M}_{V, < \omega}$ .*
2. *If  $V$  is countable and bounded then there is an indivisible metric space with age  $\mathcal{M}_{V, < \omega}$*

*Proof.* This result is due to the fact that  $\mathcal{M}_{V, < \omega}$  is closed under finite product. This is a special case of Corollary 1 of [12] built on [5]. The key ingredient is the Hales-Jewett 's theorem [4].

**Claim** For every  $\mathbb{F} \in \mathcal{M}_{V, < \omega}$  there is some  $\mathbb{G} \in \mathcal{M}_{V, < \omega}$  such that for every partition of  $G$  into two parts  $X$  and  $Y$ , one of the spaces induced by  $\mathbb{G}$  embeds  $\mathbb{F}$ . Recall that a *combinatorial line* of a finite cartesian power  $N^n$  of  $N$  is a set of the form  $L_{\bar{l}} := \{\bar{x} := (x_i)_{i < n} \in N^n : x_i = l_i \text{ for all } i \in K \text{ and } x_i = x_j \text{ for all } i, j \notin K\}$  where  $l := (l_i)_{i \in K}$  and  $K \subset n$ . According to Hales-Jewett 's theorem, if  $n$  is large enough then for every partition of  $N^n$  into two parts  $X, Y$  one of the parts contains a combinatorial line. Thus, if we equip  $F^n$  with the "sup-distance", the resulting space  $\mathbb{G}'$  satisfies the conclusion of the claim.

To prove part 1, let  $\mathbb{M}$  be metric space with age  $\mathcal{M}_{V, < \omega}$  and let  $X, Y$  be a partition of  $M$ . Assume for a contradiction that the ages  $\mathcal{A}$  of  $\mathbb{M}_{\upharpoonright X}$  and  $\mathcal{B}$  of  $\mathbb{M}_{\upharpoonright Y}$  are distinct from  $\mathcal{M}_{V, < \omega}$ , and thus let  $\mathbb{M}_X \in \mathcal{M}_{V, < \omega} \setminus \mathcal{A}$  and

$\mathbb{M}_Y \in \mathcal{M}_{V, < \omega} \setminus \mathcal{B}$ . Select  $A, B \subseteq M$  such that  $\mathbb{M}_{\upharpoonright A}$  and  $\mathbb{M}_{\upharpoonright B}$  are an isometric copy of  $\mathbb{M}_X$  and  $\mathbb{M}_Y$  respectively. For  $\mathbb{F} := \mathbb{M}_{\upharpoonright A \cup B}$  there is no  $\mathbb{G}$  satisfying the conclusion of the claim, a contradiction.

Now to prove 2, let  $a \in V$  such that  $2a$  is an upper-bound of  $V$ . Let  $(\mathbb{F}_n)_{n < \omega}$  be an enumeration of the members of  $\mathcal{M}_{V, < \omega}$ . According to the Claim above, there is a sequence  $(\mathbb{G}_n)_{n < \omega}$  such that  $\mathbb{G}_{n+1}$  contains an isometric copy of  $\mathbb{F}_{n+1}$  and for every partition of  $G_{n+1}$  into two parts one of the part contains an isometric copy of  $\mathbb{G}_n$ . Let  $G$  be the disjoint union of the  $G_n$ 's and  $d : G \times G \rightarrow V$  be defined by  $d(x, y) := d_n(x, y)$  if  $x, y \in G_n$  for some  $n$  and  $d(x, y) := a$ . Then  $\mathbb{G} := (G, d)$  is an indivisible metric space with age  $\mathcal{M}_{V, < \omega}$ .  $\square$

Theorem 3.14 below asserts that the condition that  $V$  is bounded is necessary.

## 1.4 Indivisibility of Urysohn spaces

Here is a short summary of indivisibility results regarding the Urysohn spaces.

- $\mathbb{U}_V$  is indivisible if  $V \subseteq \{0\} \cup [a, 2a]$  for some  $a > 0$ . Indeed in this case  $\mathbb{U}_V = m_f(\mathbb{H}_\kappa)$ ,  $\kappa \leq \omega$ . Since  $\mathbb{H}_\kappa$  is indivisible, it follows that  $\mathbb{U}_V$  is indivisible as well.
- Let  $R$  and  $S$  be two relational structures with the same signature. Write  $R \preceq S$  if there exists a partition of  $R$  into finitely many parts  $R_0, R_1, \dots, R_{n-1}$  so that for all  $i \in n$  there is an embedding of  $R_i$  into  $S$ . A necessary condition for a homogeneous structure to be indivisible is that any two orbits of it are related under  $\preceq$ , see [2]. Urysohn metric spaces satisfy this necessary condition according to Corollary 4.8 below. This then implies together with Item 1 of Theorem 1.10, that if a homogeneous metric space is indivisible then the ages of any two orbits are comparable under  $\subseteq$ .

It follows from results in [2] and [14] that homogeneous binary structures with finite signature whose age has free amalgamation are indivisible if and only if they satisfy that necessary condition above. (It seems to one of us, Sauer, that this result could be extended without too much of a problem to homogeneous structures with free amalgamation and infinite binary signature.)

Ages of homogeneous metric spaces satisfy the weaker notion of strong amalgamation. (See the appendix of [3] for the definitions of free and strong amalgamation.) The Urysohn space  $\mathbb{U}_{\mathbb{Q}_+, \leq 1}$ , and of course by a similar argument the Urysohn space  $\mathbb{U}_{\mathbb{Q}_+, \leq a}$  for every positive real  $a$ , is divisible

according to Theorem 4.5; providing a large number of examples of homogeneous structures which satisfy the necessary condition above and which have strong amalgamation but which are divisible.

If  $V := \{0, 1\}$  the metric space  $\mathbb{U}_V$  is indivisible. If  $V := \{0, 1, 2\}$  then the homogeneous metric space  $\mathbb{U}_V$  is just a cryptomorphic version of the Rado graph. Associate with every edge of the Rado graph distance 1 and with every non-edge distance 2. Hence it follows that  $\mathbb{U}_V$  is indivisible in this case.

Let  $V := \{0, 1, 2, 3\}$ . Then  $\mathbb{U}_V$  is a cryptomorphic version of the homogeneous graph  $H$  with two types of edges,  $E_1$  and  $E_3$ , which does not contain a triangle with two edges of type  $E_1$  and one edge of type  $E_3$ . Associate with every edge of type  $E_1$  distance 1 and with every edge of type  $E_3$  distance 3 and with every non-edge distance 2. The homogeneous structure  $H$  has free amalgamation and satisfies the chain condition. Hence it follows from Corollary 8.2 of [14] that  $H$  and therefore  $\mathbb{U}_V$  is indivisible.

We do not know if  $\mathbb{U}_V$  is indivisible for  $V := \{0, 1, 2, 3, 4\}$ .

More generally the situation is as follows. Let  $V$  with  $0 \in V$  be a finite set of non negative real numbers satisfying the conditions that there is a number  $0 \neq a \in V$  so that:

$$2 \cdot \min(V) \leq a \text{ and } \min(V) + a \leq \max(V). \quad (3)$$

It follows that in this case the conditions of Lemma 1.2 are satisfied. Hence the age  $\mathcal{M}_{V, < \omega}$  has amalgamation and there exists a homogeneous metric space  $\mathbb{U}_V$ .

This metric space  $\mathbb{U}_V$  is a cryptomorphic version of the homogeneous graph  $H$  with several types of edges  $E_i$  for  $i \in V \setminus \{0, a\}$ . If we associate in  $H$  with the edges  $E_i$  the distance  $i$  and with the non-edge between different vertices the distance  $a$  we will obtain the metric space  $\mathbb{U}_V$ . The graph  $H$  satisfies the chain condition because the metric space  $\mathbb{U}_V$  satisfies the chain condition. Hence, according to Corollary 8.2 of [14],  $H$  is indivisible and so is  $\mathbb{U}_V$ .

It follows from the definition of free amalgamation of relational structures that the age of the graph  $H$  obtained as above from the metric space  $\mathbb{U}_V$  has free amalgamation if and only if  $V$  satisfies the Inequalities 3.

- If  $\mathbb{U}_V$  is indivisible then for every  $a \in \mathbb{R}_+ \setminus \{0\}$ ,  $[0, a] \cap V$  is not dense in  $[0, a]$  (Theorem 4.1). In particular neither  $\mathbb{U}_{\mathbb{Q}_+}$  nor  $\mathbb{U}_{\mathbb{Q}_+, \leq \ell}$  is indivisible.
- If  $\mathbb{U}_V$  is indivisible then  $V$  must be bounded (Theorem 3.14). In this case let  $a := \text{Sup}V$ , must then  $V \cap [0, \frac{a}{2}]$  be dually well-founded?
- If  $\mathbb{U}_V$  is indivisible then there is a map  $u : V \rightarrow \mathbb{R}_+$  such that  $u(v) \leq v$

for every  $v \in V$  and  $d^* := u \circ d$  is an ultrametric distance on  $U$ . It follows that  $\mathbb{U}_V^* := (U, d^*)$  is homogeneous and indivisible (see Theorem 3.12).

## 2 Ultrametric spaces and homogeneous ultrametric spaces

A metric space is an *ultrametric space* if it satisfies the strong triangle inequality  $d(x, z) \leq \max\{d(x, y), d(y, z)\}$ . See [8] for example. Note that a space is an ultrametric space if and only if  $d(x, y) \geq d(y, z) \geq d(x, z)$  implies  $d(x, y) = d(y, z)$ . What we did in the previous section for general metric spaces work for ultrametric spaces.

Let  $V$  be a set such that  $0 \in V \subseteq \mathbb{R}_+$ . Let  $\mathcal{Mult}_V$  (resp.  $\mathcal{Mult}_{V, < \omega}$ ) be the collection of ultrametric metric spaces (resp. finite ultrametric spaces)  $\mathbb{M}$  whose spectrum is included into  $V$ . Then  $\mathcal{Mult}_{V, < \omega}$  is the age of a metric space whose spectrum is  $V$ ; it is closed under embeddability and has the amalgamation property. If  $V$  is countable then there is a countable homogeneous ultrametric space  $\mathbb{Uult}_V$  whose age is  $\mathcal{Mult}_{V, < \omega}$  and has spectrum  $V$ ; we call it the *Urysohn ultrametric space with spectrum  $V$* . We give a description of this space in Proposition 2.8.

For a given set  $V$ ,  $\mathbb{U}_V$  and  $\mathbb{Uult}_V$  are in general different, except if  $V = \{a_n : n \in D\}$  where  $D$  is an interval of the set  $\mathbb{Z}$  of integers and  $2a_{i+1} < a_i$  for all  $i, i+1 \in D$ .

Homogeneous ultrametric are easy to describe. In fact ultrametric spaces can be described by means of real-valued trees. An ordered set  $P$  is a *forest* if for every  $x \in P$  the set  $\downarrow x := \{y \in P : y \leq x\}$  is a chain; this is a *tree* if in addition every pair of elements of  $P$  has a lower bound. If every pair  $x, y \in P$  has an infimum, denoted  $x \wedge y$ , we will say that  $P$  is a *meet-tree*. We say that  $P$  is *ramified* if for every  $x, y \in P$  such that  $x < y$  there is some  $y' \in P$  such that  $x < y'$  and  $y'$  incomparable to  $y$ . In the sequel, we consider ramified meet-trees such that every element is below some maximal element. These posets are meet-semilattices generated by their coatoms. We will need the following property

**Lemma 2.1.** *Let  $P$  be a ramified meet-tree such that every element is below some maximal element. For every  $x \in P \setminus \max(P)$  there is a subset  $X \subseteq \max(P)$  of maximum cardinality such that  $x = a \wedge b$  for every pair of distinct elements  $a, b$  of  $X$*

*Proof.* For two elements  $a$  and  $b$  above an element  $x$ , set  $a \equiv b$  if  $x \neq a \wedge b$ .



Observe that this is an equivalence relation. A set  $X$  which meets each equivalence classe has maximum size.  $\square$

The cardinality of  $X$ , denoted  $d_P(x)$ , is the *degree* of  $x$ . For  $x \in \max(P)$  we set  $d_P(x) := 0$ . If  $P$  is finite or well-founded, this is the number of upper-covers of  $x$ , which is the ordinary notion of out-degree in the poset  $P$ . Two meet-trees  $P, P'$  are *isomorphic* if they are isomorphic as posets; in particular, an isomorphism  $f$  from  $P$  to  $P'$  preserves meets, that is  $f(x \wedge y) = f(x) \wedge f(y)$  for all  $x, y \in P$ . A *positive real-valued meet-tree*, valued meet-tree for short, is a pair  $(P, v)$  where  $P$  is a meet-tree and  $v$  a map from  $P$  to  $\mathbb{R}^*$ . Two valued meet-trees  $(P, v), (P', v')$  are *isomorphic* if there is an isomorphism  $f$  from  $P$  onto  $P'$  such that  $v' \circ f = v$ . A *subtree* of a meet-tree  $P$  is a subset  $P'$  of  $P$  such that the meet of two arbitrary elements of  $P'$  belongs to  $P'$ ; a *valued subtree* of a valued meet-tree  $(P, v)$  is a pair  $(P', v')$  where  $P'$  is a subtree and  $v' := v|_{P'}$ . The *age* of a valued meet-tree  $(P, v)$  is the collection of finite valued meet-trees which are isomorphic to some valued subtree of  $P$ .

Let  $\mathbb{M} = (M, d)$  be a metric space,  $r \in R_+$  and  $a \in M$ , the *closed ball of center  $a$ , radius  $s$*  is the set  $\mathcal{B}_a(s) := \{x \in M \mid d(a, x) \leq s\}$ . The *diameter* of a subset  $B$  of  $E$  is  $\delta(B) := \sup\{d(x, y) : x, y \in B\}$ . We denote by  $\mathcal{Ball}(\mathbb{M})$  be the collection of closed balls of  $M$  and by  $Nerv(\mathbb{M}) := \{\mathcal{B}_a(s) : a \in M, s \in Spec(\mathbb{M}, a)\}$ .

Notice that  $\delta(\mathcal{B}_a(s)) = s$  whenever  $s \in Spec(\mathbb{M}, a)$ , but more generally let us recall the following fact.

**Lemma 2.2.** *If  $M$  is an ultrametric space then for every  $B \in \mathcal{Ball}(\mathbb{M})$  and  $a \in B$ ,  $B = \mathcal{B}_a(s)$  where  $s := \delta(B)$ ,*

We give below a description of ultrametric spaces in terms of valued trees. A very close description is given by Lemin [9] (who instead of  $Nerv(\mathbb{M})$  considered  $\mathcal{Ball}(\mathbb{M})$ ).

**Theorem 2.3.** *1. Let  $\mathbb{M} := (M, d)$  be an ultrametric space, then the pair  $(P, v)$ , where  $P := (Nerv(\mathbb{M}), \supseteq)$ ,  $v$  is the diameter function, is a valued ramified meet-tree such that every element is below some maximal element and the map  $\delta : Nerv(\mathbb{M}) \rightarrow Spec(\mathbb{M})$  is strictly decreasing,  $\delta(X) = 0$  for every  $X \in M' := \max(P)$  and  $d(x, y) = \delta(\{x\} \wedge \{y\})$  for every  $x, y \in M$ .*

*2. Conversely, let  $(P, v)$  a valued ramified meet-tree such that every element is below some maximal element of  $P$  and the map  $v : P \rightarrow \mathbb{R}_+$  is strictly decreasing with  $v(x) := 0$  for each maximal element of  $P$ .*

Then the map  $d$  defined on  $M' := \max(P)$  by  $d(x, y) := v(x \wedge y)$  is an ultrametric distance and  $Nerv(\mathbb{M}') = up(P) \upharpoonright_{M'}$  where  $up(P) \upharpoonright_{M'} := \{M' \cap \uparrow x : x \in P\}$ .

3. The two correspondences are inverse of each other.

*Proof.* 1) According to Lemma 2.2, balls are disjoint or comparable w.r.t. inclusion, hence  $P$  is a tree. Since  $\{x\} \in P$  for every  $x \in M$ ,  $P$  is ramified and every element is below some maximal element. Let  $B, B' \in P$ . Pick  $a \in B$ ,  $a' \in B'$  and set  $r := d(a, a')$ . It is easy to see that  $\mathcal{B}_a(r) = B \wedge B'$ , hence  $P$  is a meet-tree. The properties of  $\delta$  follow from Lemma 2.2.

2) a)  $d$  is an ultrametric distance : Let  $x \in M'$ . We have  $d(x, x) := v(x \wedge x) = v(x) = 0$ . If  $x \neq y$  then, since  $v$  is strictly decreasing,  $d(x, y) := v(x \wedge y) > v(x) = 0$ . Clearly  $d(x, y) = d(y, x)$ . Let  $x, y, z \in M'$ . Since  $P$  is a tree,  $x \wedge z$  and  $y \wedge z$  are comparable. Suppose  $x \wedge z \leq y \wedge z$ . Then  $x \wedge z \leq x \wedge y$ . Since  $v$  is decreasing, we have  $d(x, y) \leq d(x, z) \leq \max\{d(x, z), d(y, z)\}$ .

b)  $Nerv(\mathbb{M}') = up(P) \upharpoonright_{M'}$  :

Let  $B := M' \cap \uparrow x \in up(P) \upharpoonright_{M'}$ ,  $r := v(x)$  and  $y \in B$ .

**Claim 1.**  $B = \mathcal{B}_y(r)$  and  $r \in Spec(\mathbb{M}', y)$ . Thus  $B \in Nerv(\mathbb{M}')$ .

Indeed, let  $z \in B(y, r)$ , that is  $v(y \wedge z) \leq r$ . Since  $x \leq y$  and  $y \wedge z \leq y$ ,  $x$  and  $y \wedge z$  are comparable, since  $v$  is strictly decreasing  $x \leq y \wedge z$  hence  $z \in B$ . Conversely, if  $z \in B$  then  $x \leq y \wedge z$  thus, since  $v$  is strictly decreasing,  $d(y, z) := v(y \wedge z) \leq v(x) = r$  proving  $z \in \mathcal{B}_y(r)$ . Thus  $B = \mathcal{B}_y(r)$  as claimed. Since  $P$  is ramified and every element of  $P$  is below some element of  $M'$ , there is some  $z \in M'$  such that  $x = y \wedge z$ . Clearly,  $z \in B$  and  $r = d(y, z)$  thus  $r \in Spec(\mathbb{M}', y)$ .

Let  $B := B(y, r) \in Nerv(\mathbb{M}')$  with  $r \in Spec(\mathbb{M}', y)$

**Claim 2.**  $B \in up(P) \upharpoonright_{M'}$ .

Indeed, since  $r \in Spec(\mathbb{M}', y)$  there is some  $z \in M'$  such that  $d(y, z) = r$ . Let  $x := y \wedge z$ . Since  $v(x) = r$  we get  $B = \uparrow x \cap M' \in up(P) \upharpoonright_{M'}$  from the previous claim.

3) We simply note that if  $P := (Nerv(\mathbb{M}), \supseteq)$  then, for  $M' := \max(P)$ ,  $P$  is isomorphic to  $(up(P) \upharpoonright_{M'}, \supseteq)$ ; moreover, if  $v : P \rightarrow \mathbb{R}_+$  is the diameter function associated to  $\mathbb{M}$ , then  $v(x) = \delta'(M' \cap \uparrow x)$  where  $\delta$  is the diameter function associated to the metric defined on  $M'$  in part 2.  $\square$

**Lemma 2.4.** *Two ultrametric spaces have the same age if and only if the corresponding valued trees have the same age.*

The verification is immediate.

The *reduced valued tree associated to* an ultrametric space  $\mathbb{M}$  is the pair  $(P', v')$  where  $P' := P \setminus \max(P)$  and  $v' := v|_{P'}$ . The age of the reduced valued tree does not determine the age of the tree, because the information about the degree, in  $P$ , of terminal nodes in  $P'$  is missing. With this information added, we have easily:

**Lemma 2.5.** *If two reduced valued trees are isomorphic via a map which preserves the degree of the original trees then the ultrametric spaces have the same age.*

Let  $\lambda$  be a chain and let  $\bar{a} := (a_\mu)_{\mu \in \lambda}$  such that  $2 \leq a_\mu \leq \omega$ . Set  $\omega^{[\bar{a}]} := \{\bar{b} := (b_\mu)_{\mu \in \lambda} : \mu \in \lambda \Rightarrow b_\mu < a_\mu \text{ and } \text{supp}(\bar{b}) := \{\mu < \omega : b_\mu \neq 0\} \text{ is finite}\}$ . If  $a_\mu = \omega$  for every  $\mu \in \lambda$ , the set  $\omega^{[\bar{a}]}$  is usually denoted  $\omega^{[\lambda]}$ . Add a largest element, denoted  $\infty$  to  $\lambda$ . Given  $\bar{b}, \bar{c} \in \omega^{[\bar{a}]}$ , set  $\Delta(\bar{b}, \bar{c}) := \infty$  if  $\bar{b} = \bar{c}$ , otherwise  $\Delta(\bar{b}, \bar{c}) := \mu$  where  $\mu$  is the least member of  $\lambda$  such that  $b_\mu \neq c_\mu$ .

Suppose  $\lambda$  be countable. Let  $w : \lambda \cup \{\infty\} \rightarrow \mathbb{R}_+$  be a strictly decreasing map such that  $w(\infty) = 0$ , let  $d_w := w \circ \Delta$  and let  $V$  be the image of  $w$ . For  $\mu \in \lambda \cup \{\infty\}$  set  $\downarrow^* \mu := \downarrow \mu \setminus \{\mu\}$ . Let  $P' := \{f_{\downarrow^* \mu} : f \in \omega^{[\bar{a}]}, \mu \in \lambda \cup \{\infty\}\}$  ordered by extension and let  $v'(f_{\downarrow^* \mu}) := w(\mu)$ .

We have the following property, which is easy to check.

**Lemma 2.6.** *The pair  $\mathbb{M} := (\omega^{[\bar{a}]}, d_w)$  is an ultrametric space,  $\text{Spec}(\mathbb{M}) = V$  and the valued tree associated to  $\mathbb{M}$  is isomorphic to  $(P', v')$ .*

We say that  $\mathbb{M}$  is *point-homogeneous* if the automorphism group of  $\mathbb{M}$  acts transitively on  $\mathbb{M}$ .

**Theorem 2.7.** *Let  $\mathbb{M}$  be a countable ultrametric space,  $P := (\text{Nerv}(\mathbb{M}), \supseteq)$ ,  $v : P \rightarrow \mathbb{R}_+$  where  $v(B) := \delta(B)$ ,  $M' := \max(P)$ . The following properties are equivalent:*

- (i)  $\mathbb{M}$  is isometric to some  $(\omega^{[\bar{a}]}, d_w)$ .
- (ii)  $\mathbb{M}$  is homogeneous;
- (iii)  $\mathbb{M}$  is point-homogeneous;
- (iv) (a)  $v(x) = v(y) \Rightarrow d_P(x) = d_P(y)$  for every  $x, y \in P$ ;  
 (b)  $v[\downarrow x] = v[\downarrow y]$  for every  $x, y \in \max(P)$ .

*Proof.* (i)  $\Rightarrow$  (iv) Let  $\mathbb{M} := (\omega^{[\bar{a}]}, d_w)$ . According to Lemma 2.6, the valued tree associated to  $\mathbb{M}$  is isomorphic to  $(P', v')$ . Condition (b)(iv) immediately follows. Let  $x := f_{\downarrow * \mu} \in P'$ ; if  $\mu = \infty$  then  $d_{P'}(x) = 0$ , otherwise  $d_{P'}(x) = a(\mu)$ . Thus Condition (a)(iv) holds too.

(ii)  $\Rightarrow$  (iii) Trivial

(iii)  $\Rightarrow$  (iv) Suppose  $\mathbb{M}$  point homogeneous. First, Condition (b)(iv) holds. Indeed, let  $x, y \in M' := \max(P)$ . Then  $x := \{x'\}$  and  $y := \{y'\}$ , with  $x', y' \in \mathbb{M}$ . Let  $f$  be an isometry from  $\mathbb{M}$  onto itself such that  $f(x') = y'$ . Then  $\text{Spec}(x', \mathbb{M}) = \text{Spec}(y', \mathbb{M})$  and the result follows. Next, Condition (a)(iv) holds. Let  $x := B \in P, y := C \in P$  and  $r := v(x) = v(y)$ . Pick  $x' \in B, y' \in C$ . Let  $f$  be an isometry from  $\mathbb{M}$  onto itself such that  $f(x') = y'$ . Then  $f(B) = C$ . For two elements  $x', y'$  of  $B$ , set  $x' \equiv y'$  if  $d(x', y') < r$ . This relation is an equivalence relation whose number of classes is the degree of  $x := B$  in the poset  $P := \text{Nerv}(\mathbb{M})$ . The desired conclusion follows.

(iv)  $\Rightarrow$  (ii) Let  $f$  be an isometry from a finite subset  $A$  of  $M$  onto a subset  $B$  of  $M$ . Let  $x \in M \setminus A$ . We prove that  $f$  extends to an isometry defined on  $A \cup \{x\}$ . If  $A$  is empty, we may send  $x$  onto any element  $b$  of  $M$ . If  $A$  is non-empty, set  $r := \min(\{d(x, y) : y \in A\})$ . In order to extend  $f$  we only need to send  $x$  onto some  $b \in M$  such that  $f(B(x, r)) = B(b, r) \cap f(A)$ . There is some  $u \in P$  such that  $x \wedge x' = u$  for all  $x' \in B(x, r) \cap A$  and moreover  $v(u) = r$ . Select  $y \in f(B(x, r))$ . Since  $v[\downarrow x] = v[\downarrow y]$  there is some  $u' \in \downarrow y$  such that  $v(u') = r$ . Since  $d_P(u) = d_P(u')$ , there is  $b \in M$  such that  $y' \wedge b = u'$  for all  $y' \in f(A)$ . Such an element will do.

(ii)  $\Rightarrow$  (i). Let  $\lambda := \text{Spec}(\mathbb{M}) \setminus \{0\}$  ordered with the dual of the order induced by the natural order on  $\mathbb{R}$ , let  $w : \lambda \cup \{\infty\} \rightarrow \mathbb{R}_+$  with  $w(x) := x$  for  $x \in \lambda$  and  $w(\infty) := 0$  and let  $\bar{a} : \lambda \rightarrow \omega + 1$  such that  $\bar{a} \circ w = d_P$  (such a map exists because of (iv) Condition 1).

**Claim**  $\mathbb{M}$  is isometric to  $(\omega^{[\bar{a}]}, d_w)$ . According to the implications (i)  $\Rightarrow$  (iv)  $\Rightarrow$  (ii) proved above,  $(\omega^{[\bar{a}]}, d_w)$  is homogeneous. Since  $\mathbb{M}$  is homogeneous, it suffices to prove that  $(\omega^{[\bar{a}]}, d_w)$  and  $\mathbb{M}$  have the same age to get the desired conclusion. From the implication (iii)  $\Rightarrow$  (iv), the reduced valued trees associated to  $(\omega^{[\bar{a}]}, d_w)$  and  $\mathbb{M}$  are isomorphic by an isomorphism which preserves the degree. From Lemma 2.5,  $(\omega^{[\bar{a}]}, d_w)$  and  $\mathbb{M}$  have the same age.  $\square$

**Proposition 2.8.** *The space  $(\omega^{[\lambda]}, d_w)$  is the countable homogeneous ultrametric space  $\mathbb{U}lt_V$  associated with  $V$ .*

*Proof.* We only need to prove that every finite ultrametric space  $\mathbb{M} := (M, d)$  with spectrum included into  $V$  embeds isometrically into  $(\omega^{[\lambda]}, d_w)$ .

We argue by induction on the number  $n$  of elements of  $M$ . If  $n \leq 1$ , the result is obvious. Suppose  $n \geq 2$ . Let  $x \in M$ . We may suppose that there is an isometric embedding  $f$  of  $\mathbb{M}_{-x} := \mathbb{M}_{\upharpoonright M \setminus \{x\}}$  into  $(\omega^{[\lambda]}, d_w)$ . We prove that  $f$  extends to  $\mathbb{M}$ . Set  $r := \min(\{d(x, y) : y \in M \setminus \{x\}\})$  and  $\mu \in \lambda$  such that  $w(\mu) = r$ . In order to extend  $f$  we only need to find some element  $b \in \omega^{[\lambda]}$  such that  $f(B(x, r)) = B(b, r) \cap f(M \setminus \{x\})$ . For every  $\bar{b}', \bar{b}'' \in f(B(x, r))$  we have  $b'_{\mu'} = b''_{\mu'}$  for all  $\mu' < \mu$ . Select  $b \in \omega^{[\lambda]}$  such that  $b_{\mu'} = b'_{\mu'}$  for all  $\mu' < \mu$  and  $b_\mu \in \omega \setminus \{b'_\mu : \bar{b}' \in f(B(x, r))\}$ .  $\square$

## 2.1 Indivisible ultrametric spaces

**Definition 2.9.** Let  $\mathbb{M} := (M; d)$  be a metric space,  $a \in M$  and  $0 \leq r < s$ . Then

$$\mathcal{R}_a(r, s) := \{x \in M \mid r \leq d(a, x) < s\}.$$

**Lemma 2.10.** *Let  $\mathbb{M} = (M, d)$  be an indivisible ultrametric space. Then the spectrum of every element of  $M$  is dually well founded.*

*Proof.* Let  $a \in M$ . Suppose for a contradiction that  $r_0 = 0 < r_1 < r_2 < r_3 < \dots$  is an infinite sequence of reals in the spectrum of  $a$ . Let  $s$  be its supremum. Cover  $M$  by a family  $\mathcal{B} := \{\mathcal{R}_{a_\alpha}(0, s) : \alpha < \kappa\}$  of open balls of radius  $s$  such that  $a_\alpha \notin M_\alpha := \cup\{\mathcal{R}_{a_\beta}(0, s) : \beta < \alpha\}$  (with the convention that if  $s = \infty$  then  $\mathcal{B}$  consists of  $M$ ). Since  $d$  is an ultrametric distance, these balls are pairwise disjoint and therefore, the rings  $\mathcal{R}_{a_\alpha}(r_i, r_{i+1})$  make-up a partition of  $M$ . Let:

$$\mathcal{E} := \bigcup_{\alpha < \kappa, i \in \omega} \mathcal{R}_{a_\alpha}(r_{2i}, r_{2i+1}) \text{ and } \mathcal{O} := \bigcup_{\alpha < \kappa, i \in \omega} \mathcal{R}_{a_\alpha}(r_{2i+1}, r_{2i+2})$$

and let  $f$  be an isometry of  $M$  into  $M$ . Let  $\alpha < \kappa$  and  $i \in \omega$  so that  $f(a) \in \mathcal{R}_{a_\alpha}(r_i, r_{i+1})$ . Let  $b \in M$  with  $d(a, b) = r_{i+1}$ .

Then  $d(f(a), f(b)) = r_{i+1}$  and because  $d(f(a), a_\alpha) < r_{i+1}$  it follows that that  $d(a_\alpha, f(b)) = r_{i+1} < s$ . Thus  $f(b) \in \mathcal{R}_{a_\alpha}(r_{i+1}, r_{i+2})$ .  $\square$

**Corollary 2.11.** *If an ultrametric space is indivisible then the collection of balls, once ordered by inclusion, is dually well-founded and the diameter is attained.*

*Proof.* Let  $(B_n)_{n < \omega}$  be an increasing sequence of balls of an ultrametric space  $\mathbb{M} := (M, d)$ . Pick  $a \in \cap\{B_n : n \in \mathbb{N}\}$ . Since  $\mathbb{M}$  is ultrametric,  $a$  is the center of each  $B_n$  thus their radii belong to the spectrum of  $a$ . If  $\mathbb{M}$  is indivisible, then from Lemma 2.10 above  $\text{Spec}(\mathbb{M}, a)$  is dually well-founded, thus the sequence is eventually constant. Let  $s$  be the maximum of  $\text{Spec}(\mathbb{M}, a)$ . Let  $x, y \in M$ . We have  $d(x, y) \leq \max(\{d(x, a), d(y, a)\}) \leq s$ , hence  $s$  is the maximum of the spectrum of  $\mathbb{M}$ , that is the diameter of  $\mathbb{M}$ .  $\square$

**Theorem 2.12.** *Let  $\mathbb{M}$  be a denumerable ultrametric space. The following properties are equivalent:*

- (i)  $\mathbb{M}$  is isometric to some  $\text{Ult}_V$ , where  $V$  is dually well-ordered;
- (ii)  $\mathbb{M}$  is point-homogeneous,  $P := (\text{Nerv}(\mathbb{M}), \supseteq)$  is well founded and the degree of every non maximal element is infinite;
- (iii)  $\mathbb{M}$  is homogeneous and indivisible;

*Proof.* (i)  $\Rightarrow$  (ii) By definition,  $\text{Ult}_V$  is homogeneous, hence point-homogeneous. In fact, according to Proposition 2.8,  $\mathbb{M}$  is isometric to some  $(\omega^{[\lambda]}, d_w)$  where  $\lambda$  is a well-ordered chain. Thus, from Lemma 2.6,  $P := (\text{Nerv}(\mathbb{M}), \supseteq)$  is well-founded and the degree of every non maximal element is infinite.

(iii)  $\Rightarrow$  (i) Suppose that (iii) holds. Theorem 2.7 asserts that  $\mathbb{M}$  is isometric to some  $(\omega^{[\bar{a}]}, d_w)$ . Since  $\mathbb{M}$  is indivisible, it follows from Lemma 2.10 that  $V := \text{Spec}(\mathbb{M})$  is well-founded, hence we may suppose that  $\lambda$  is an ordinal. To conclude it suffices to prove that  $a_\mu = \omega$  for every  $\mu < \lambda$ . Let  $\mu < \lambda$ ; set  $r := w(\mu)$ . First, observe that  $M = \cup \mathcal{B}$  where  $\mathcal{B}$  is a collection of pairwise disjoint balls, all of diameter  $r$ . Next, each member  $B$  of  $\mathcal{B}$  is the union of  $a_\mu$  balls  $B_i$  each of smaller diameter than  $r$ . Indeed, since  $\mathbb{M}$  is point-homogeneous, all balls having the same radius are isometric spaces, thus it suffices to prove this property for the ball  $B := \mathcal{B}_0(r)$ , where 0 is the ordinal sequence which only takes value 0. This is easy: set  $\bar{x}_i := (b_\nu)_{\nu < \lambda}$  where  $i < a_\mu$ ,  $b_\nu = 0$  if  $\nu \neq \mu$  and  $b_\mu := i$  otherwise, set  $r^+ := w(\mu^+)$  where  $\mu^+ := \mu + 1$  if  $\mu + 1 < \lambda$  and  $\mu^+ := \infty$  otherwise, then  $B := \cup\{B(\bar{x}_i, r^+) : i < a_\mu\}$ . With these two observations we have  $M = \cup\{M_i : i < a_\mu\}$  where  $M_i := \cup\{B_i : B \in \mathcal{B}\}$ . Clearly, there is no isometry from  $\mathbb{M}$  into an  $\mathbb{M}_i$  hence if  $a_\mu < \omega$ ,  $\mathbb{M}$  cannot be indivisible.

(ii)  $\Rightarrow$  (iii) According to Theorem 2.7,  $\mathbb{M}$  is homogeneous. Let us show that it is indivisible. Let  $f : M \rightarrow 2$  be a partition of  $M$  into two parts. Set  $\mathcal{F}_0$  be the set of balls  $B \in \text{Nerv}(\mathbb{M})$  such that there is some isometry  $\varphi_B$  from  $B$  into  $B \cap f^{-1}(0)$  and let  $M_0 := \cup \mathcal{F}_0$ .

**Claim 1** There is an isometry from  $M_0$  to  $M_0 \cap f^{-1}(0)$ .

Indeed, let  $\mathcal{F}'_0$  be the subset of  $\mathcal{F}_0$  made of its maximal members (w.r.t. inclusion). Let  $\varphi := \cup\{\varphi_B : B \in \mathcal{F}'_0\}$ . Since balls are either disjoint or comparable,  $\varphi$  is a map and, since  $P := (Nerv(\mathbb{M}), \supseteq)$  is well-founded,  $M_0 = \cup\mathcal{F}'_0$ , hence the domain of  $\varphi$  is  $M_0$ .

For  $B$  in  $Nerv(\mathbb{M})$ , set  $Pred(B) := \max(\{B' : B' \subset B, B' \in Nerv(\mathbb{M})\})$ .

**Claim 2** If  $B \notin \mathcal{F}_0$  then  $Pred(B) \cap \mathcal{F}_0$  is finite.

Indeed, suppose not. Then, since the space is point-homogeneous, all members of  $Pred(B)$  have the same radius and there is an isometry  $\psi$  from  $B$  into  $B$  which transforms each member of  $Pred(B)$  to a member of  $Pred(B) \cap \mathcal{F}_0$ . Let  $\varphi := \cup\{\varphi_{B'} : B' \in Pred(B) \cap \mathcal{F}_0\}$ . Then  $\varphi$  is an isometry from  $\cup(Pred(B) \cap \mathcal{F}_0)$  into  $B \cap f^{-1}(0)$ . Consequently,  $\varphi \circ \psi$  is an isometry from  $B$  into  $B \cap f^{-1}(0)$ , thus  $B \in \mathcal{F}_0$ , a contradiction.

Suppose that  $M \notin \mathcal{F}_0$ . We construct an isometry  $h$  from  $M$  into  $f^{-1}(1) \setminus M_0$  as follows. We start with an enumeration  $(x_n)_{n < \omega}$  of the elements of  $M$ . According to Claim 1,  $M \setminus M_0 \neq \emptyset$ . We may also suppose that it contains an element of  $f^{-1}(1)$  (otherwise the union of the identity map on  $M \setminus M_0$  and an isometry as constructed in Claim 1, is an isometry from  $M$  into  $f^{-1}(0)$ ). Let  $y_0$  such an element. We set  $h(x_0) := y_0$ .

Suppose  $h$  be defined for all  $m, m < n$ . Let  $p := \min(\{d(x_m, x_n) : m < n\})$ . Let  $I := \{i, i < n : d(x_i, x_n) := p\}$ . Let  $B := \mathcal{B}_{h(i)}(p)$  for  $i \in I$ . This set does not depend upon the choice of  $i$ . Since  $h(i) \in f^{-1}(1) \setminus M_0$ ,  $B \notin \mathcal{F}_0$ . For each  $i \in I$  let  $B'_i$  such that  $h(i) \in B'_i \in Pred(B)$ . According to Claim 2, there is some  $B'' \in Pred(B) \setminus \mathcal{F}_0$  which is distinct from all the  $B'_i$ s. As in our first step,  $B'' \setminus M_0$  is nonempty and in fact contains an element, say  $y_n$  of  $f^{-1}(1)$ . We set  $h(x_n) := y_n$ .

□

### 3 Divisibility of metric spaces

The sequence  $a_0, a_1, \dots, a_{n-1}, a_n$  of elements in a metric space  $\mathbb{M} := (M; d)$  is an  $\epsilon$ -chain joining  $a_0$  and  $a_n$  if  $d(a_i, a_{i+1}) \leq \epsilon$  for all  $i \in n$ . The space  $\mathbb{M}$  is *Cantor connected* if any two of its elements can be joined by an  $\epsilon$ -chain for any  $\epsilon > 0$ . The *Cantor connected component of an element  $a \in M$*  is the largest Cantor connected subset of  $M$  containing  $a$ . The space  $\mathbb{M}$  is *totally Cantor disconnected* if the Cantor connected component of every  $a$  reduce to  $a$ . See [8] for more details and references.

For  $a \in M$  let  $\lambda_\epsilon(a)$  be the supremum of all reals  $l \leq 1$  for which there exists an  $\epsilon$ -chain  $a_0, a_1, \dots, a_{n-1}, a_n$  with  $d(a_0, a_n) \geq l$  containing  $a$ . (The

condition  $l \leq 1$  saves us from having to consider the special case  $\infty$ .) Let

$$\lambda(a) := \sup\{l \in \mathbb{R} \mid \forall \epsilon > 0 (\lambda_\epsilon(a) \geq l)\}.$$

A space  $(M; d)$  is *restricted* if  $\lambda(a) = 0$  for all  $a \in M$ . It follows that every restricted space is totally Cantor disconnected. There are totally Cantor disconnected spaces which are not restricted. Here is an example with a finite diameter :

**Example 3.1.** Let  $(M; d)$  be the metric space so that:

1.  $M = \{(0, 0)\} \cup \{(m, n) \in \mathbb{N} \times \mathbb{N} \mid m < n\}$
2.  $d((0, 0), (m, n)) = \frac{m+1}{n}$
3.  $d((m_1, n), (m_2, n)) = \frac{|m_1 - m_2|}{n}$
4.  $d((m_1, n_1), (m_2, n_2)) = \frac{m_1+1}{n_1} + \frac{m_2+1}{n_2}$  when  $n_1 \neq n_2$ .

This example falls into the following category:

**Definition 3.2.** A *spider* is a metric space  $(M; d)$  so that

1.  $M = \{(0, 0)\} \cup \{(m, n) \in \mathbb{N} \times \mathbb{N} \mid m < n\}$
2.  $d((0, 0), (n-1, n)) \geq r$  for some non-negative  $r$  and all  $n \in \mathbb{N}^*$
3.  $d((0, 0), (0, n)) \leq r_n$  and  $d((m, n), (m+1, n)) \leq r_n$  for all  $n \in \mathbb{N}^*$  where  $\lim_{n \rightarrow \infty} r_n = 0$ .

We have easily

**Lemma 3.3.** *A metric space is restricted if it does not isometrically embed a spider.*

**Lemma 3.4.** *Let  $c \in M$  and  $0 \leq r_0 < r_1 < r_2 < r_3$  and  $a \in \mathcal{R}_c(r_0, r_1)$  and  $b \in \mathcal{R}_c(r_2, r_3)$  then:*

1.  $d(a, b) > r_2 - r_1$ .
2.  $d(x, y) < 2r_2$  for all  $x, y \in \mathcal{R}_c(r_1, r_2)$ .
3. *If  $0 < \epsilon < \min\{r_1 - r_0, r_3 - r_2\}$  and  $x_0, x_1, x_2, \dots, x_{n-1}$  is an  $\epsilon$ -sequence with  $x_i \notin \mathcal{R}_c(r_0, r_1) \cup \mathcal{R}_c(r_2, r_3)$  for all  $i \in n$  but with  $x_i \in \mathcal{R}_c(r_1, r_2)$  for at least one  $i \in n$ , then  $x_i \in \mathcal{R}_c(r_1, r_2)$  for all  $i \in n$ .*



4. Let  $f$  be an isometry of  $M$  with  $f[M] \cap (\mathcal{R}_c(r_0, r_1) \cup \mathcal{R}_c(r_2, r_3)) = \emptyset$  and let  $z \in M$  with  $\lambda(z) > 2r_2$ . Then  $f(z) \notin \mathcal{R}_c(r_1, r_2)$ .

*Proof.* Items 1 and 2 follow from the triangle inequality. Item 3 follows from item 1 and item 4 follows from items 2 and 3.  $\square$

**Definition 3.5.** Let  $c \in M$  and  $0 < l$ . Then

$$\mathcal{E}_c(l) := \bigcup_{n \geq 2, n \text{ even}} \mathcal{R}_c\left(\frac{l(n-1)}{n}, \frac{ln}{n+1}\right)$$

and

$$\mathcal{O}_c(l) := \bigcup_{n \text{ odd}} \mathcal{R}_c\left(\frac{l(n-1)}{n}, \frac{ln}{n+1}\right).$$

**Theorem 3.6.** Let  $\mathbb{M} = (M; d)$  be a countable metric space. If there exists an element  $a \in M$  with  $\lambda(a) > 0$  then  $\mathbb{M}$  is divisible.

*Proof.* Since  $M$  is countable, it can be covered by a family of pairwise disjoint open balls with radius less than  $\frac{\lambda(a)}{2}$ . In fact, there exists a subset  $C$  of  $M$  and for every  $c \in C$  a positive real  $l_c$  so that:

1.  $l_c \neq d(x, y)$  for every  $c \in C$  and  $x, y \in M$ .
2.  $2l_c < \lambda(a)$  for every  $c \in C$ .
3. For every element  $x \in M$  there is one and only one element  $c \in C$  with  $x \in \mathcal{R}_c(0, l_c)$ .

(After enumerating  $M$  into an  $\omega$ -sequence  $m_0, m_1, m_2, m_3, \dots$  such a set  $C$  and function  $l$  can be constructed step by step exhausting all of the elements of  $M$ .)

Let

$$\mathcal{E} := \bigcup_{c \in C} \mathcal{E}_c(l_c) \text{ and } \mathcal{O} := \bigcup_{c \in C} \mathcal{O}_c(l_c).$$

Then  $\mathcal{E} \cup \mathcal{O} = M$  and  $\mathcal{E} \cap \mathcal{O} = \emptyset$ .

Assume for a contradiction that there is an isometry  $f$  which maps  $M$  into  $\mathcal{E}$ . Then there is a  $c \in C$  so that  $f(a) \in \mathcal{E}_c(l_c)$ . But this is not possible according to Lemma 3.4 item 4. Similarly it is not possible that  $f$  maps  $M$  into  $\mathcal{O}$ .  $\square$

**Corollary 3.7.** *A countable metric space which is indivisible is restricted and hence totally Cantor disconnected.*

The second part of the conclusion of the corollary above extends to uncountable metric spaces.

**Theorem 3.8.** *Let  $\mathbb{M}$  be a metric space and  $r$  be a positive real, then there is a partition into two parts  $A_0$  and  $A_1$  which contains no Cantor connected subspace of diameter larger than  $r$ .*

**Lemma 3.9.** *Let  $\mathbb{M}$  be a metric space and  $r$  be a positive real number. Then there is a sequence  $(E_\mu)_{\mu < \lambda}$  such that:*

1.  $E_0 = \emptyset$  and each  $E_\mu$  is open in  $M$
2. the sequence is strictly increasing and continuous, that is  $E_\mu$  is the union of  $E_\nu$  for  $\nu < \mu$  if  $\mu$  is a limit ordinal,
3. the union covers  $M$
4.  $F_\mu := E_{\mu+1} \setminus E_\mu$  has diameter at most  $r$  and decomposes into two sets  $A_{\mu,0}$  and  $A_{\mu,1}$  such that each Cantor connected subspace  $Y$  of  $A_{\mu,i}$  is contained into some subset  $\mathcal{B}_Y$  of  $F_\mu$  such that  $d(y, A_{\mu,i} \cap \mathcal{B}_Y) \geq \epsilon_Y$  for some  $\epsilon_Y > 0$  and every  $y \in M \setminus (\mathcal{B}_Y \cup E_\mu)$ .

*Proof.* Suppose the sequence defined for all  $\nu$ ,  $\nu < \mu$ . If  $\mu$  is a limit ordinal, set  $E_\mu := \bigcup\{E_\nu : \nu < \mu\}$ . If  $\mu$  is a successor, say  $\mu := \nu + 1$ , pick  $x \in E' := M \setminus E_\nu$ , set  $\mathcal{R}'_x(0, r/2) := \{y \in E' : d(x, y) < r/2\}$  and set  $E_\mu := E_\nu \cup \mathcal{R}'_x(0, r/2)$ . Decompose  $\mathcal{R}'_x(0, r/2)$  into countably many crowns  $\mathcal{R}'_x(\frac{r(n-1)/2}{n}, \frac{rn/2}{n+1})$  as in the proof of Theorem 3.6, the union of the even ones gives  $A_{\nu,0}$ , the rest gives  $A_{\nu,1}$ .

Finally any Cantor connected subspace  $Y$  of  $A_{\mu,i}$  must be included into  $\mathcal{R}'_x(\frac{r(n-1)/2}{n}, \frac{rn/2}{n+1})$  for some  $n$ , and therefore the required  $\mathcal{B}_Y$  may be taken to be  $\mathcal{R}'_x(0, s_Y)$  with  $\frac{rn/2}{n+1} < s_Y < \frac{r(n+1)/2}{n+2}$  with  $\epsilon_Y := s_Y - \frac{rn/2}{n+1}$ . □

**Proof of Theorem 3.8** Let  $A_i := \bigcup\{A_{\mu,i} : \mu < \lambda\}$ . Then  $A_i$  contains no Cantor connected subspace  $X$  of diameter larger than  $r$ .

Indeed, suppose the contrary. Let  $\mu$  be minimum such that  $E_\mu$  meets  $X$ . Clearly  $\mu$  is a successor, say  $\mu = \nu + 1$ . Let  $x \in X_\nu := X \cap F_\nu$ . Let  $Y$  be the Cantor connected component of  $x$  in  $A_{\nu,i}$  and let  $\mathcal{B}_Y$  given by the

above lemma. **Claim**  $X \subseteq \mathcal{B}_Y$ . Indeed suppose not, let  $y \in X \setminus \mathcal{B}_Y$ , let  $\epsilon$ ,  $0 < \epsilon < \epsilon_Y$  and  $x_0 := x, \dots, x_k, \dots, x_n = y$  be an  $\epsilon$  path contained in  $X$ . Let  $\ell$  be least index such that  $x_\ell \notin \mathcal{B}_Y$ . From  $x_{\ell-1} \in A_{\mu,i} \cap \mathcal{B}_Y$ , we get  $d(x_\ell, A_{\mu,i} \cap \mathcal{B}_Y) < \epsilon_Y$ . A contradiction.

Since  $X \subseteq B_Y \subseteq F_\mu$ , the diameter of  $X$  is at most  $r$ . The proof is complete.

**Definition 3.10.** Let  $\mathbb{M} := (M; d)$  be totally Cantor disconnected. Then

$$d^*(x, y) := \inf\{\epsilon > 0 \mid \text{there exists an } \epsilon\text{-sequence containing } x \text{ and } y\}.$$

**Lemma 3.11.** *Let  $\mathbb{M} := (M; d)$  be totally Cantor disconnected. Then  $\mathbb{M}^* := (M; d^*)$  is an ultrametric space.*

*Proof.* Let  $x, y, z \in M$  with  $d^*(x, y) \geq d^*(x, z) \geq d^*(y, z)$ . Then for every  $\epsilon > d^*(x, z)$  there are  $\epsilon$ -sequences joining  $x$  to  $z$  and  $z$  to  $y$ , then one joining  $x$  to  $y$ , hence  $d^*(x, y) \leq \epsilon$ . Thus  $d^*(x, y) \leq d^*(x, z)$ .  $\square$

(See [8], Theorem 1 and Lemma 8.)

**Theorem 3.12.** *Let  $\mathbb{M} := (M; d)$  be a countable homogeneous indivisible metric space then  $\mathbb{M}^*$  is an homogeneous indivisible ultrametric space.*

*Proof.* Since  $\mathbb{M}$  is indivisible it is totally Cantor disconnected, hence  $d^*$  is well defined. Since  $\mathbb{M}$  is homogeneous then  $d(x, y) = d(x', y')$  implies  $d^*(x, y) = d^*(x', y')$  for all  $x, y, x', y' \in M$ . From this property, every local isometry of  $\mathbb{M}$  is a local isometry of  $\mathbb{M}^*$ . Hence, since  $M$  is indivisible,  $M^*$  is indivisible. Since every automorphism of  $M$  is an automorphism of  $M^*$ ,  $M^*$  is point-homogeneous. According to Theorem 2.7,  $M^*$  is homogeneous.  $\square$

**Theorem 3.13.** *Let  $\mathbb{M}$  be a homogeneous metric space and  $V := \text{Spec}(\mathbb{M})$ . If  $\mathbb{M}$  is totally Cantor disconnected and every three element metric space  $\mathbb{T}$  with  $\text{Spec}(\mathbb{T}) \subseteq V$  embeds into  $\mathbb{M}$  then the set  $V \setminus \{0\}$  is either contained into an interval of the form  $[a \rightarrow +\infty)$  for some  $a \in \mathbb{R}_+ \setminus \{0\}$  or into an union of intervals of the form  $\cup\{[a_{2(n+1)}, a_{2n+1}] : n < \omega\} \cup [a_0 \rightarrow +\infty)$  where  $\{a_n : n < \omega\}$  is a sequence such that  $a_{2n+1} \leq \frac{a_{2n}}{2}$ .*

*Proof. Claim* For every  $w \in V^* := \text{Spec}(\mathbb{M}^*), ]\frac{w}{2}, w[ \cap V = \emptyset$ .

Suppose the contrary. Pick  $r \in ]\frac{w}{2}, w[ \cap V = \emptyset$ . Since  $w \in V^*$ , we may find  $x, y$  such that  $d^*(x, y) = w$ . Let  $n < \omega$  and  $\epsilon := 2r$ , then there is an  $\epsilon$ -sequence  $x_0, \dots, x_n$  containing  $x, y$ . For  $i < n$ , let  $\mathbb{T}_i := (\{x_i, x_{i+1}, z_i\}, d_i)$  where  $d_i(x_i, x_{i+1}) := d(x_i, x_{i+1}), d_i(x_i, z_i) = d_i(x_{i+1}, z_i) := r$ . Each  $\mathbb{T}_i$  is a metric space whith spectrum included into  $V$ , hence can be isometrically embedded into  $\mathbb{M}$ . Since  $M$  is homegenous, we may suppose that  $z_i \in M$  and that the embedding is the inclusion. By adding the  $z_i$ 's to the  $x_i$ 's we get a  $r$ -sequence containing  $x$  and  $y$ . Since  $r < w$  this gives a contradiction.

Since every element of  $V^*$  is the infimum of elements of  $V$  it also follows that  $]\frac{w}{2}, w[ \cap V^* = \emptyset$ .

Let  $\alpha := \text{Inf}(V \setminus \{0\})$ . If  $\alpha \neq 0$  set  $a := \alpha$ ; in this case  $V \setminus \{0\} \subseteq [a \rightarrow +\infty)$ . If  $\alpha = 0$  then, since every element de  $V \setminus \{0\}$  majorizes some element of  $V^* \setminus \{0\}$  it follows that  $\text{Inf}(V^* \setminus \{0\}) = 0$  too. Let  $\{a_{2n} : n < \omega\}$  be a strictly decreasing sequence of elements of  $V^*$  which converges to 0. Set  $a_{2n+1} := a_{2n}$ . From the Claim  $]\frac{a_{2n}}{2}, a_{2n}[ \cap V = \emptyset$ , hence  $a_{2n+2} \leq a_{2n+1}$ . The rest follows. □

**Theorem 3.14.** *Every unbounded metric space is divisible.*

*Proof.* Let  $\mathbb{M} := (M; d)$  be an unbounded metric space. Construct a sequence of reals  $r_0, r_1, r_2, \dots$  and a sequence  $a_0, a_1, a_2, \dots$  of elements of  $M$  so that for every integer  $i \in \mathbb{N}$

1.  $d(a_0, a_{i+1}) > 2r_i$ .
2.  $d(a_0, a_{i+1}) + r_i < r_{i+1}$ .

Let  $r_0 := 0$  and  $a_0 \in M$  be arbitrary. Suppose that  $(r_i : i \leq n)$  and  $(a_i : i \leq n)$  have already been constructed. From the fact that  $\mathbb{M}$  is unbounded, we can find  $a_{n+1} \in M$  such that  $d(a_0, a_{n+1}) > 2r_n$ . Next, choose  $r_{n+1} > d(a_0, a_{n+1}) + r_n$ . Note that the set  $\{r_i : i \in \mathbb{N}\}$  such constructed is unbounded.

Let, given any  $c \in M$ ,

$$\mathcal{E} := \bigcup_{i \in \mathbb{N}} \mathcal{R}_c(r_{2i}, r_{2i+1}) \text{ and } \mathcal{O} := \bigcup_{i \in \mathbb{N}} \mathcal{R}_c(r_{2i+1}, r_{2i+2}).$$

We prove that there is no isometric embedding of  $M$  into  $\mathcal{E}$  or into  $\mathcal{O}$ .

Let  $f$  be an isometric embedding of  $M$  into  $M$ . Let  $i$  be minimal so that  $d(c, f(a_0)) < r_i$ ; notice that  $i > 0$  and  $f(a_0) \in \mathcal{R}_c(r_{i-1}, r_i)$ . We have:

$$\begin{aligned} d(c, f(a_{i+1})) &\geq d(f(a_0), f(a_{i+1})) - d(c, f(a_0)) = \\ &= d(a_0, a_{i+1}) - d(c, f(a_0)) > 2r_i - r_i = r_i. \end{aligned}$$

Also:

$$d(c, f(a_{i+1})) \leq d(c, f(a_0)) + d(f(a_0), f(a_{i+1})) \leq r_i + d(a_0, a_{i+1}) < r_{i+1}.$$

It follows that  $f(a_{i+1}) \in \mathcal{R}_c(r_i, r_{i+1})$ . Therefore  $f[M]$  intersects both  $\mathcal{E}$  and  $\mathcal{O}$ . □

## 4 Divisibility of the bounded Urysohn space

In [6], Hjorth shows that the Urysohn space  $\mathbb{U}_{\mathbb{Q}_+}$  is divisible, and asks whether the corresponding bounded space has the same property. We show that it does, and in fact this generalizes to bounded Urysohn spaces for which the spectrum  $V$  satisfies a density condition. In the sequel  $V$  will denote a countable subset of  $\mathbb{R}_+$  containing 0 and satisfying the four-values condition.

**Proposition 4.1.** *If for some  $r > 0$ ,  $V \cap [0, r]$  is dense in  $[0, r]$  then the diameter of each Cantor connected component of  $\mathbb{U}_V$  is at least  $r$ .*

*Proof.* Let  $a \in U_V$  and  $\ell \in V \cap (0, r]$ . Let  $b \in U$  such that  $d(a, b) = \ell$ . For any  $n \in \omega$  choose successively  $a_0, \dots, a_n$  such that:

$$a_0 := \frac{\ell}{n}, a_k := \frac{\ell}{n}k + \epsilon_k \in V \cap \left[\frac{\ell}{n}(k-1), \frac{a_{k-1}}{2}\right] \text{ for } 1 \leq k < n \text{ and } a_n := \ell.$$

Let  $x_0 = a$ ,  $x_n = b$ ,  $x_1, x_2, \dots, x_{n-1}$  be elements not in  $U$  and  $X := \{x_0, x_1, x_2, \dots, x_n\}$ . Let  $d' : X \times X \rightarrow V$  defined by  $d'(x_i, x_{i+k}) = a_k$  for  $1 \leq k \leq n$  and  $d'(x_i, x_i) := 0$ . With our choice,  $\epsilon_{i+j} \leq \epsilon_i + \epsilon_j$  for all  $i, j$  such that  $i + j \leq n$ , thus  $\mathbb{X} := (X, d') \in \mathcal{M}_V$ .

Hence we can use the mapping extension property of the Urysohn space and obtain an embedding  $f$  of the space  $\mathbb{X}$  into  $U$  which is the identity map on  $x_0$  and  $x_n$ .

Since this can be done for any  $n$  we conclude that the Cantor connected component of  $a$  contains  $b$ , hence its diameter is at least  $\ell$ . Since this holds for every  $\ell \in V \cap (0, r]$ , this diameter is at least  $r$ . □

**Remark 4.2.** If  $V$  is residuated, Proposition 4.1 follows from Proposition 1.8. Indeed, in this case  $\mathbb{V} := (V, d_V)$  is a metric space, thus for every  $a \in \mathbb{U}_V$  there is isometric embedding of  $\mathbb{V}$  into  $\mathbb{U}_V$  which maps 0 onto  $a$  (Point 5 of Section 1.1 together with the homogeneity of  $\mathbb{U}_V$ ). The Cantor connected component of 0 contains  $V \cap [0, r]$ , thus its image has diameter at least  $r$ . Notice that when  $V$  is an initial segment of  $\mathbb{Q}_+$ ,  $d_V$  is just its usual distance, in which case Proposition 1.8 is not required.

**Corollary 4.3.** *The countable homogeneous metric space  $\mathbb{U}_{\mathbb{Q}_+, \leq 1} := (U; d)$  having all rational numbers less than or equal to one as distances is Cantor connected.*

**Theorem 4.4.** *If for some  $r > 0$ ,  $V \cap [0, r]$  is dense in  $[0, r]$  then  $\mathbb{U}_V$  is divisible.*

*Proof.* Follows from Proposition 4.1 and from Corollary 3.7. □

In particular, we have:

**Theorem 4.5.** *The Urysohn space  $\mathbb{U}_{\mathbb{Q}_+, \leq 1}$  is divisible.*

In the remainder of this section, we investigate certain conditions which guarantee that the bounded Urysohn space  $\mathbb{U}_{\mathbb{Q}_+, \leq 1}$  isometrically embeds into “large” parts of itself. These give some measure of the indivisibility of the space.

We first wish to extend the notion of an orbit and its socket. Indeed notice that if  $S$  is an orbit of the Urysohn space  $\mathbb{U}_V = (U; d)$  with socket  $B = \{b_i \mid i \in n\}$  and  $s$  and  $t$  elements in  $S$ , then  $d(b_i, s) = d(b_i, t)$  for all  $i \in n$  because there exists an isometry which fixes  $B$  element-wise and maps  $s$  to  $t$ . Hence we are led to the following definition.

**Definition 4.6.** A *distance-socket* (or simply *d-socket*)  $\mathfrak{B}$  of  $\mathbb{U}_V$  is a sequence of the form

$$\langle (b_0, d_0), (b_1, d_1), \dots, (b_{n-1}, d_{n-1}) \rangle$$

so that for all  $i, j \in n$ :

1.  $b_i \in U$  and  $d_i \in V$ .
2.  $d_i + d_j \geq d(b_i, b_j)$ .
3.  $d_i + d(b_i, b_j) \geq d_j$ .

The set  $\text{vert}(\mathfrak{B})$  of *vertices* of  $\mathfrak{B}$  is the set  $\{b_0, b_1, \dots, b_{n-1}\}$ , and the set of *distances* of  $\mathfrak{B}$  is the set  $\{d_0, d_1, \dots, d_{n-1}\}$ .

An orbit therefore naturally gives rise to a corresponding socket and d-socket. But it also follows that given a d-socket

$$\mathfrak{B} = \langle (b_0, d_0), (b_1, d_1), \dots, (b_{n-1}, d_{n-1}) \rangle,$$

the set of all  $s \in U$  so that  $d(s, b_i) = d_i$  for all  $s \in S$  and  $i \in n$  is an orbit of  $\mathbb{U}_V$  with socket  $B = \{b_i \mid i \in n\}$  (Conditions 1. 2. and 3. of the definition ensure that the set is not empty).

We first show that under certain conditions an orbit itself contains an isometric copy of the bounded Urysohn space.

**Lemma 4.7.** *Let  $S$  be an orbit of the Urysohn space  $\mathbb{U}_V = (U; d)$  with corresponding d-socket  $\mathfrak{B} = \langle (b_0, d_0), (b_1, d_1), \dots, (b_{n-1}, d_{n-1}) \rangle$ .*

*If  $\ell := \min\{d_i \mid i \in n\}$ , then the metric subspace of  $\mathbb{U}_V$  induced by  $S$  is an isometric copy of the Urysohn space  $\mathbb{U}_{V, \leq 2\ell}$ .*

*Proof.* Let  $i \in n$  be such that  $\ell = d_i$ . Then  $d(s, b_i) = \ell$  for every element  $s \in S$  and hence it follows from the triangle inequality that  $d(s, t) \leq 2\ell$  for any two elements  $s$  and  $t$  of  $S$ .

Let  $F := (F, d')$  be an element in the age of  $\mathbb{U}_{V, \leq 2\ell}$  so that  $F \cap U \subseteq S$  and the metric subspace of  $\mathbb{U}_{V, \leq 2\ell}$  induced by  $F \cap S$  is equal to the metric subspace of  $F$  induced by  $F \cap S$ . According to the mapping extension property, it suffices to show that there exists an embedding of  $F$  into  $S$ .

Let  $G := (\{b_i \mid i \in n\} \cup F, \bar{d})$  be the metric space for which  $\bar{d}$  agrees with  $d$  on  $F \cap U$  and  $\bar{d}$  agrees with  $d'$  on  $F \setminus U$ , and  $\bar{d}(x, b_i) = d_i$  for all  $x \in F \setminus S$  and all  $i \in n$ . The function  $\bar{d}$  satisfies the triangle inequality and hence  $G$  is an element of the age of  $\mathbb{U}_V$ .

It follows from the mapping extension property of  $\mathbb{U}_V$  that there exists an embedding  $f$  of  $G$  into  $U$  which fixes the elements of  $G \cap U$ . It follows from the condition that  $\bar{d}(x, b_i) = d_i$  for all  $x \in F \setminus S$  and all  $i \in n$ , that the elements of  $F$  are mapped by  $f$  into  $S$ .  $\square$

If  $V'$  is an initial segment of  $V$  then  $\mathbb{U}_{V'}$  embeds into  $\mathbb{U}_V$ . Hence, if we compare orbits of an Urysohn space by isometric embedding, it follows from Lemma 4.7 above that they form a chain. This is important as you may recall (see [2] and [14]) that a necessary condition for indivisibility is that the ages of the orbits of an homogeneous structure  $H$  form a chain.

**Corollary 4.8.** *The orbits of an Urysohn metric space form a chain.*

## 4.1 Semi-scattered spaces

We suppose that 0 is an accumulation point of  $V$ . We show that there are certain small subsets of the Urysohn space  $\mathbb{U}_V$  that can be avoided by any isometrical embedding.

Let  $W$  be a subset of  $V$  such that 0 is an accumulation point of  $W$ . Thus, in particular,  $\mathcal{M}_W$  contains members of arbitrary small diameter.

**Definition 4.9.** An element  $a$  of  $\mathbb{M} := (M, d) \in \mathcal{M}_V$  is a *W-sub-isolated point* if for every  $\epsilon > 0$ , there is some non-trivial member  $\mathbb{X} \in \mathcal{M}_W$  of diameter at most  $\epsilon$  such that  $a$  does not belong to the union of the isometric copies of  $\mathbb{X}$  in  $M$ . Let  $M'$  be the set of elements of  $M$  which are not  $W$ -sub-isolated and  $\mathbb{M}'$  be the restriction of  $\mathbb{M}$  to  $M'$ .

Clearly every isolated point is  $W$ -sub-isolated. Thus the following decomposition generalizes the Cantor-Bendixson decomposition of scattered spaces.

**Definition 4.10.** Given a metric space  $\mathbb{M}$ , define for each ordinal  $\alpha$  a metric space  $\mathbb{M}^{(\alpha)}$  by

1.  $\mathbb{M}^{(0)} := \mathbb{M}$ .
2.  $\mathbb{M}^{(\alpha)} := (\mathbb{M}^{(\beta)})'$  if  $\alpha := \beta + 1$ .
3.  $\mathbb{M}^{(\alpha)} := \bigcap_{\beta < \alpha} \mathbb{M}^{(\beta)}$  if  $\alpha$  is a limit ordinal.

Clearly,  $\beta \leq \alpha \Rightarrow \mathbb{M}^{(\alpha)} \subseteq \mathbb{M}^{(\beta)}$ , hence this ordinal sequence is eventually constant. If it is eventually empty, we say that  $\mathbb{M}$  is *W-sub-scattered*.

In the sequel  $W := V$  (and  $V$  is a countable subset of  $\mathbb{R}_+$  containing 0, for which 0 is an accumulation point, and satisfying the four-values condition).

**Theorem 4.11.** *For every V-sub-scattered subspace of  $\mathbb{U}_V$  the complementary subspace is isometric to  $\mathbb{U}_V$ .*

*Proof.* First observe that the notion of  $V$ -sub-isolation is hereditary, *i.e.* a  $V$ -sub-isolated point of a metric space is  $V$ -sub-isolated in any subspace it lies in. It easily follows that every subspace of a  $V$ -sub-scattered metric space is also  $V$ -sub-scattered. Since  $V$  satisfies the four-values condition, then, for every positive real  $\ell$ , the metric space  $\mathbb{U}_{V, \leq \ell}$  has diameter at most  $\ell$ , and in particular it has no  $V$ -sub-isolated point, since it is homogeneous and it embeds the non singleton  $\mathbb{U}_{V, \leq \ell'}$  for any  $\ell' \leq \ell$ .



Now given a subspace  $\mathbb{M}$  of  $\mathbb{U}_V$ , if  $\mathbb{M}$  is not isometric to  $\mathbb{U}_V$ , then it follows from Lemma 4.7 and Point 9 of Section 1.1 that the complementary subspace embeds  $\mathbb{U}_{V, \leq \ell}$  for some positive  $\ell$ , and therefore, since  $\mathbb{U}_{V, \leq \ell}$  is not  $V$ -sub scattered, that complementary subspace is not  $V$ -sub scattered either.  $\square$

**Example 4.12.** For example, given a subspace  $\mathbb{M}$  of such a  $\mathbb{U}_V$ , the complementary subspace is isometric to  $\mathbb{U}_V$  whenever

- $\mathbb{M}$  is topologically scattered.
- $\mathbb{M}$  is  $V$ -semi discrete, i.e. for every point  $a$  of  $\mathbb{M}$ ,  $0$  is an accumulation point of  $V \setminus \text{Spec}(a, \mathbb{M})$ .

## 4.2 The case of the Urysohn space $\mathbb{U}_{\mathbb{Q}_+, \leq 1}$ .

Our final goal is to show that an isometric embedding can avoid a set containing elements close to a sequence of relatively far elements.

**Lemma 4.13.** *Let  $\mathfrak{B} := \langle (b_0, d_0), (b_1, d_1), \dots, (b_{n-1}, d_{n-1}) \rangle$  be a  $d$ -socket with orbit  $S$  in  $\mathbb{U}_{\mathbb{Q}_+, \leq 1}$ . Let  $a \in U$ ,  $r \in \mathbb{Q}_+ \cap [0, 1]$ , and  $x \in S$  such that  $d(a, x) \leq r$ .*

*Then if  $d(a, b_i) \geq r$  for all  $i \in n$ , there exists an element  $y \in S$  with  $d(a, y) = r$ .*

*Proof.* It suffices to show that  $\mathfrak{B}' := \langle (b_0, d_0), (b_1, d_1), \dots, (b_{n-1}, d_{n-1})(a, r) \rangle$  is again a  $d$ -socket.

To fulfill the requirements of Definition 4.6, it remains to verify that for every  $i \in n$  the following inequalities hold:

1.  $r + d_i \geq d(a, b_i)$ .
2.  $d_i + d(a, b_i) \geq r$ .
3.  $r + d(a, b_i) \geq d_i$ .

But by the triangle inequality with  $x$  we have  $d(a, b_i) \leq d(a, x) + d_i \leq r + d_i$ . Trivially we have  $r \leq d(a, b_i) \leq d_i + d(a, b_i)$ , and finally  $d_i \leq d(a, x) + d(a, b_i) \leq r + d(a, b_i)$ .  $\square$

**Theorem 4.14.** Let  $\mathbb{U}_{\mathbb{Q}_+, \leq 1} = (U; d)$  be the bounded Urysohn space. Fix  $R := \{r_i \mid i \in \omega\}$  a set of rationals in the interval  $(0, 1]$ ,  $A := \{a_i \mid i \in \omega\}$  a subset of  $U$  so that  $d(a_i, a_j) \geq r_i + r_j$  for all  $i, j \in \omega$  with  $i \neq j$ , and let

$$X := \bigcup_{i \in \omega} \{x : d(a_i, x) < r_i\}.$$

Then the metric subspace of  $\mathbb{U}_{\mathbb{Q}_+, \leq 1}$  induced by  $U \setminus X$  is an isometric copy of  $\mathbb{U}_{\mathbb{Q}_+, \leq 1}$ .

*Proof.* Notice that if  $y$  is any element at distance from some  $a_i$  greater than or equal to  $r_i$  for some  $i \in \omega$ , then  $y \in U \setminus X$ .

Let  $S$  be an orbit of  $\mathbb{U}_{\mathbb{Q}_+, \leq 1}$  with socket  $F \subseteq U \setminus X$ . Given Point 9 of Section 1.1, let us check that  $S$  meets  $U \setminus X$ . Let  $s \in S$ . If  $s \notin X$  then there is nothing to prove. Otherwise there exists an  $i \in \omega$  so that  $d(a_i, s) < r_i$ . But then it follows from Lemma 4.13 that there is an element  $y \in S$  with  $d(a_i, y) = r_i$ , and hence  $y \in U \setminus X$ . This completes the proof.  $\square$

## 5 Conclusion

As a conclusion, we propose the following problems (some have been already mentioned in the text) and remarks.

- Problems 5.1.**
1. Let  $V := \{0, 1, 2, 3, 4\}$ . Is the Urysohn metric space  $U_V$  indivisible?
  2. A metric space which embeds into a countable indivisible space has a bounded diameter and does not embed a spider (see 3.2). Does the converse hold?
  3. Give necessary and sufficient conditions on a countable subset  $V$  of  $\mathbb{R}_+$  such that  $U_V$  exists and is indivisible.
  4. For which countable subsets  $V$  of  $\mathbb{R}_+$  containing  $\{0\}$ , the collection of countable metric spaces with values in  $V$  has a universal member?
  5. Is the Urysohn metric space  $U_{\mathbb{Q}_+}$  weakly indivisible in the sense that for every partition of its domain into two parts, if one part does not embed a finite metric space with rational values, the other does?
  6. Describe the countable indivisible ultrametric spaces.

7. Supposing that the continuum hypothesis holds, extend the construction of indivisible homogeneous ultrametric spaces given in the countable case to cardinality  $\aleph_1$ .

**Remarks 5.2.** In [1], we solve a problem similar to Problem 2. We prove that an ultrametric space  $\mathbb{M}$  embeds into a countable indivisible ultrametric space if and only if  $\mathbb{M}$  is countable and contains no infinite strictly increasing sequence of balls. We also construct large classes of countable indivisible ultrametric spaces.

## References

- [1] C.Delhomme, C.Laflamme, M.Pouzet, N.Sauer, Indivisible ultrametric spaces, lecture at the Workshop on the Urysohn space, May 21-24, 2006 Beer-Sheva, Israel, preprint 2006.
- [2] M. El-Zahar, N.W. Sauer, Indivisible homogeneous directed graphs and a Game for Vertex Partitions, *Discrete Mathematics*, **291** (2005) no. 1-3, 99-113.
- [3] R. Fraïssé, Theory of Relations, Revised Edition, in: *Studies in Logic and the Foundations of Mathematics*, **145**, North Holland 2000.
- [4] A. W. Hales and R. I. Jewett, Regularity and positional games, *Trans. Amer. Math. Soc.*, **106** (1963), 222-229.
- [5] J.Jezek and J.Nesetril, Ramsey varieties, *Europ. J. Combinatorics* **4**(1983), 143-147.
- [6] G. Hjorth, An oscillation theorem for groups of isometries, *manuscript*
- [7] K. Kuratowski : Topologie, volume 1, 4 ième Édition. Varsovie, 1958.
- [8] A. J. Lemin, On ultrametrization of general metric spaces, *Proceedings of the American Mathematical Society* **131** (2002) no. 3, 979-989.
- [9] A. Lemin, The category of ultrametric spaces is isomorphic to the category of complete, atomic, tree-like, and real graduated lattices  $LAT^*$ , *Algebra universalis* **50** (2003) 35-49.
- [10] L.Nguyen VanThé, Big Ramsey degrees and indivisibility in classes of ultrametric spaces, preprint, July 5, 2005, 13pp.

- [11] M. Pouzet, Relation impartible, Dissertationnes, **103** (1981), 1-48.
- [12] M. Pouzet and I.G.Rosenberg, Ramsey properties for classes of relational structures, *Europ. J. Combinatorics* **6**(1985), 361-368.
- [13] M. Pouzet and B. Roux, Ubiquity in category for metric spaces and transition systems, *Europ. J. Combinatorics* **17**(1996), 291-307.
- [14] N.W. Sauer, Canonical vertex partitions, *Combinatorics Probability and Computing* **12** (2003) no. 5-6, 671-704.
- [15] P. Urysohn, Sur un espace métrique universel, *Bull. Sci. Math. (2)* **51** (1927), 43-64.