Additional information

Classroom notes for PMAT 603.51, Fall 2006

1 A theorem of Fraïssé

Let $\mathcal{A} = (A; \mathbb{L})$ and $\mathcal{B} = (B; \mathbb{L})$ and $\mathcal{C} = (C; \mathbb{L})$ and $\mathcal{D} = (D; \mathbb{L})$ be relational structures and α an embedding of $\mathcal{C}$ to $\mathcal{A}$ and g an embedding of $\mathcal{C}$ to $\mathcal{B}$ and f an embedding of $\mathcal{A}$ to $\mathcal{D}$ and β an embedding of $\mathcal{B}$ to $\mathcal{D}$.

The triple $(\mathcal{D}, f, \beta)$ is an *amalgamation* of the quintuple $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \alpha, g)$ if the diagram comutes, that is $\beta \circ g = f \circ \alpha$. The quintuple $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \alpha, g)$ is an *amalgamation instance*.

The set $\mathbf{A}$ of finite relational structures has *amalgamation*, or is a set of relational structures *with amalgamation*, if every amalgamation instance of its elements has amalgamation.

Let $\mathbf{A}$ be a countable age. The quadruple $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \alpha, (g_i; i \in n))$ is a *generalized amalgamation instance* in $\mathbf{A}$ if the structures $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ are elements of $\mathbf{A}$ and α is an embedding of $\mathcal{C}$ into $\mathcal{A}$ and $n \in \omega$ and g_i is an emebedding of $\mathcal{C}$ into $\mathcal{B}$ for every $i \in n$.

The triple $(\mathcal{D}, (f_i; i \in n), \beta)$ is an amalgamation of the generalized amalgamation instance $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \alpha, (g_i; i \in n))$ if β is an embedding of $\mathcal{B}$ into $\mathcal{D}$ and f_i is an embedding of $\mathcal{A}$ into $\mathcal{D}$ for every $i \in n$ with $f_i \circ \alpha = \beta \circ g_i$.

Lemma 1.1. *Let $\mathbf{A}$ be a countable age and $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \alpha, (g_i; i \in n))$ a generalized amalgamation instance.*

Then there exists a structure $\mathcal{D} \in \mathbf{A}$ and an amalgamation $(\mathcal{D}, (f_i; i \in n), \beta)$ of the amalgamaation instance $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \alpha, (g_i; i \in n))$.

Proof. By induction on n. If $n = 0$ let $\mathcal{D} = \mathcal{B}$ and β the idendity map on $\mathcal{B}$. Let $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \alpha, (g_i; i \in n+1))$ be a generalized amalgamation instance in $\mathbf{A}$.

Then $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \alpha, (g_i; i \in n))$ is a generalized amalgamation instance in $\mathbf{A}$ and hence there exists a structure $\mathcal{B}' \in \mathbf{A}$ and an embedding β' of $\mathcal{B}$ into $\mathcal{B}'$

and for every $i \in n$ an embedding f'_i of $\mathcal{A}$ into $\mathcal{B}'$ so that $f'_i \circ \alpha = \beta' \circ g_i$ for every $i \in n$.

It follows that the quintuple $(\mathcal{A}, \mathcal{B}', \mathcal{C}, \alpha, \beta' \circ g_n)$ is an amalgamation instance and hence it has amalgamation $(\mathcal{D}, f_n, \beta'')$ with $\mathcal{D} \in \mathbf{A}$ and β'' an embedding of $\mathcal{B}'$ into $\mathcal{D}$ and f_n an embedding of $\mathcal{A}$ into $\mathcal{D}$ so that $f_n \circ \alpha = (\beta'' \circ \beta') \circ g_n$.

Let $\beta := \beta'' \circ \beta'$ and $f_i := \beta'' \circ f'_i$ for every $i \in n$. Then:

$$f_i \circ \alpha = \beta'' \circ f'_i \circ \alpha = \beta'' \circ \beta' \circ g_i = \beta \circ g_i$$

for every $i \in n$. Hence $f_i \circ \alpha = \beta \circ g_i$ for every $i \in n+1$. □

Theorem 1.1 (Fraïssé)**.** *Let* $\mathbf{A}$ *be a countable age with amalgamation. Then there exists a homogeneous structure* $\mathcal{H}$ *with* $\mathrm{age}(\mathcal{H}) = \mathbf{A}$.

Proof. Let

$$\mathfrak{T} := \{(\mathcal{A}, \mathcal{C}, \alpha) : \mathcal{A}, \mathcal{C} \in \mathbf{A} \text{ and } \alpha \text{ an embedding of } \mathcal{C} \text{ into } \mathcal{A}_n\}.$$

Let $\big((\mathcal{A}_n, \mathcal{C}_n, \alpha_n); n \in \omega\big)$ be an ω-sequence of triples in $\mathfrak{T}$ in which every element of $\mathfrak{T}$ appears infinitely often.

We construct recursively structures $\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2, \dots$. Let $\mathcal{B}_0 := \emptyset$. Let $\{g_i^{(n)}; i \in m\}$ be the set of all embeddings of $\mathcal{C}_n$ into $\mathcal{B}_n$. Then the quintuple $(\mathcal{A}_n, \mathcal{B}_n, \mathcal{C}_n, \alpha_n, (g_i^{(n)}; i \in m))$ is a generalized amalgamation instance. It has an amalgamation, say $(\mathcal{D}, (f_i^{(n)}; i \in m), \beta_n)$. Let $\mathcal{B}_{n+1} := \mathcal{D}$. Note that β_n is an embedding of $\mathcal{B}_n$ into $\mathcal{B}_{n+1}$.

Let $\mathcal{H}$ be the limit of the sequence $\mathcal{B}_n$ under the embeddings β_n. For $m < n$ let $\gamma_{m,n} := \beta_{n-1} \circ \beta_{n-2} \circ \cdots \circ \beta_m$ and γ_m the embedding of $\mathcal{B}_m$ into $\mathcal{H}$ so that $\gamma_m = \gamma_n \circ \gamma_{m,n}$.

Each of the structures $\mathcal{B}_n$ is an element of $\mathbf{A}$ and hence $\mathrm{age}(\mathcal{H}) \subseteq \mathbf{A}$. Let $\mathcal{A} \in \mathbf{A}$. Then $(\mathcal{A}, \emptyset, \emptyset) \in \mathfrak{T}$ and hence there is an $n \in \omega$ so that $(\mathcal{A}_n, \mathcal{C}_n, \alpha_n) = (\mathcal{A}, \emptyset, \emptyset)$.

The empty function is an embedding of the empty set into $\mathcal{B}_n$ and hence there is an embedding of $\mathcal{A}$ into $\mathcal{B}_{n+1}$. Hence $\mathcal{A} \in \mathrm{age}(\mathcal{H})$ and hence $\mathrm{age}(\mathcal{H}) = \mathbf{A}$.

In order to prove that $\mathcal{H}$ is homogeneous we will show that it has the mapping extension property.

Let $\mathcal{A}, \mathcal{C} \in \mathbf{A}$ and α an embedding of $\mathcal{C}$ into $\mathcal{A}$ and g an embedding of $\mathcal{C}$ into $\mathcal{H}$. It follows from the construction of $\mathcal{H}$ that there is an $m \in \omega$ and an embedding h of $\mathcal{C}$ into $\mathcal{B}_m$ so that $g = \gamma_m \circ h$.

The triple $(\mathcal{A}, \mathcal{C}, \alpha)$ is an element of $\mathfrak{T}$ and hence there is an $n \in \omega$ with $(\mathcal{A}, \mathcal{C}, \alpha) = (\mathcal{A}_n, \mathcal{C}_n, \alpha_n)$ and $n \geq m$. Then $g = \gamma_n \circ \gamma_{m,n} \circ h$ and $g' := \gamma_{m,n} \circ h$ is an embedding of $\mathcal{C}$ into $\mathcal{B}_n$. Hence there exists an embedding f' of $\mathcal{A}$ into $\mathcal{B}_{n+1}$ with $f' \circ \alpha = \beta_n \circ g'$. The function $f := \gamma_{n+1} \circ f'$ is an embedding of $\mathcal{A}$ into $\mathcal{H}$ with:

$$f \circ \alpha = \gamma_{n+1} \circ \beta_n \circ g' = \gamma_{n+1} \circ \beta \circ \gamma_{m,n} \circ h = \gamma_{n+1} \circ \gamma_{m,m+1} \circ h = g.$$

□

2 Topological dynamics and partitions

Let $\mathbb{L}^*$ be a relational signature containing the distinguished order symbol $<$ and let $\mathbf{A}^*$ be a Fraïssé order class on $\mathbb{L}^*$. We denote by $\mathbb{L}$ the relational language $\mathbb{L}^* \setminus \{<\}$ and by $\mathbf{A}$ the set of all relational $\mathbb{L}$-structures which can be ordered under $<$ to be an element of $\mathbf{A}^*$. Let $(\mathcal{H}, <) := \mathrm{Flim}(\mathbf{A}^*)$ and H the base set of $\mathcal{H}$.

Theorem 2.1 (Nguyen). *The following are equivalent:*

1. $\mathbf{A}$ *is a Fraïssé class and* $\mathcal{H} = \mathrm{Flim}(\mathbf{A})$.

2. $\mathbf{A}^*$ *is reasonable.*

Theorem 2.2. *Let G be a topological group. Then there is a minimal G-flow $M(G)$ such that for any minimal G-flow X there is a surjective homomorphism $f : M(G) \to X$. The minimal G-flow $M(G)$ is unique up to isomorphism.*

Definition 2.1. *The minimal G-flow $M(G)$ is called the* universal minimal flow of G.

The following three theorems are from the paper of Kechris-Pestov-Todorcevic on this webpage.

Theorem 2.3. *The following are equivalent:*

1. $\mathrm{Aut}(\mathcal{H}, <)$ *is extremely amenable.*

2. $\mathbf{A}^*$ *is a Ramsey class.*

Definition 2.2. *The set* $\mathbf{A}^*$ *has the* ordering property *if for all* $\mathcal{C} \in \mathbf{A}$ *there is a* $\mathcal{B} \in \mathbf{A}$ *so that for all orderings* $(\mathcal{C}, <), (\mathcal{B} <) \in \mathbf{A}^*$ *there exists an embedding of* $(\mathcal{C}, <)$ *into* $(\mathcal{B}, <)$*.*

Definition 2.3. *The linear order* $\prec$ *on* H *is* $\mathbf{A}^*$-admissible *if the age of the structure* $(\mathcal{H}; \prec)$ *is a subset of* $\mathbf{A}^*$*. Let* $X_{\mathbf{A}^*}$ *denote the set of all* $\mathbf{A}^*$ *admissible orderings of* H*.*

Theorem 2.4. *Let* $\mathbf{A}^*$ *be reasonable. Then the following are equivalent:*

1. $X_{\mathbf{A}^*}$ *is a minimal* $\mathrm{Aut}(\mathbf{A})$*-flow.*

2. $\mathbf{A}^*$ *satisfies the ordering property.*

Theorem 2.5. *If* $\mathbf{A}^*$ *has the Ramsey and the ordering properties then the universal minimal flow of* $\mathrm{Aut}(\mathcal{H})$ *is* $X_{\mathbf{A}^*}$*.*