Representation of ideals of relational structures

Christian Delhommé

E.R.M.I.T., Département de Mathématiques et d'Informatique, Université de La Réunion, 15, avenue René Cassin, BP 71551, 97715 Saint-Denis Messag. Cedex 9, France

Maurice Pouzet [1]

PCS, Université Claude-Bernard Lyon1, Domaine de Gerland -bât. Recherche [B], 50 avenue Tony-Garnier, F69365 Lyon cedex 07, France,

Gábor Sági [2]

Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Budapest H-1364, pf. 127, Hungary

Norbert Sauer [*],[3]

Department of Mathematics and Statistics, The University of Calgary, Calgary, T2N1N4, Alberta, Canada

Abstract

The *age* of a relational structure $\mathfrak{A}$ of signature μ is the set $age(\mathfrak{A})$ of its finite induced substructures, considered up to isomorphism. This is an ideal in the poset Ω_μ consisting of finite structures of signature μ and ordered by embeddability. If the structures are made of infinitely many relations and if, among those, infinitely many are at least binary then there are ideals which do not come from an age. We provide many examples. We particularly look at metric spaces and offer several problems. We also provide an example of an ideal I of isomorphism types of at most countable structures whose signature consists of a single ternary relation symbol. This ideal does not come from the set $age_\mathfrak{J}(\mathfrak{A})$ of isomorphism types of substructures of $\mathfrak{A}$ induced on the members of an ideal $\mathfrak{J}$ of sets. This answers a question due to R. Cusin and J.F. Pabion (1970).

Key words: Relational structures, metric spaces.

1 Introduction and basic notions

Let $\mathbb{N}$ be the set of non-negative integers, $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$ be the set of positive integers and $n \in \mathbb{N}^*$. A *n-ary relation* on a set A is a map $R : A^n \to \{0, 1\}$. A *signature* is a function $\mu : I \to \mathbb{N}^*$ from an *index set* I into $\mathbb{N}$. We write $\mu = (\mu_i; i \in I)$ as an indexed set. A *relational structure with signature* μ is a pair $\mathfrak{A} := (A; \mathbf{R}^{\mathfrak{A}})$ where $\mathbf{R}^{\mathfrak{A}} := (\mathsf{R}_i^{\mathfrak{A}})_{i \in I}$ is a set of relations on A, each relation $\mathsf{R}_i^{\mathfrak{A}}$ having arity μ_i. If $\mathfrak{A}$ is clear from the context then we will write $\mathbf{R}$ instead of $\mathbf{R}^{\mathfrak{A}}$ and R_i instead of $\mathsf{R}_i^{\mathfrak{A}}$. As much as possible we will denote relational structures by letters of the form $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, etc. and the corresponding base sets by A, B, C, etc. The *cardinality* of the relational structure $\mathfrak{A} := (A; \mathbf{R})$ is the cardinality of A, which, as usual, will be denoted by $|A|$.

The signature $\mu = (\mu_i; i \in I)$ is *unary, binary, ternary* and in general *n-ary* if the range of the function μ is $\{1\}$, $\{2\}$, $\{3\}$ or in general $\{n\}$. The signature $\mu = (\mu_i; i \in I)$ is at most *binary, ternary* and in general *n-ary* if the range of the function μ is a subset of $\{1, 2\}$, $\{1, 2, 3\}$ and in general $\{1, 2, 3, \ldots, n\}$. The signature μ is *finite* if the index set I is finite, it is *infinite* otherwise. If $S \subseteq \mathbb{N}^*$ then $\mu^{-1}[S]$ is the set of all indices $i \in I$ for which $\mu_i \in S$. If $|\mu^{-1}(\mathbb{N}^*)| = 1$ then μ is a *singleton signature*. For example, a relational structure with a binary singleton signature is a directed graph which may have loops.

A relational structure is *unary, binary, ternary, n-ary* and so on if its signature is unary, binary, ternary, n-ary, respectively.

Let $\mathfrak{A} := (A; \mathbf{R}^{\mathfrak{A}})$ and $\mathfrak{B} := (B; \mathbf{R}^{\mathfrak{B}})$ be two relational structures with common signature $\mu := (\mu_i; i \in I)$. A map $f : A \to B$ is an *isomorphism* of $\mathfrak{A}$ onto $\mathfrak{B}$ if f is bijective and for all $i \in I$ and $(x_0, x_1, \ldots, x_{\mu_i - 1}) \in A^n$:

$$R_i^{\mathfrak{A}}(x_0, x_1, \ldots, x_{\mu_i - 1}) \quad \text{if and only if} \quad R_i^{\mathfrak{B}}(f(x_0, f(x_1), \ldots, f(x_{\mu_i - 1})).$$

Let A' be a subset of B, the *substructure of $\mathfrak{B}$ induced on A'*, also called the *restriction of $\mathfrak{B}$ to A'* is the relational structure $\mathfrak{B}_{\restriction A'} := (A'; \mathbf{R}^{\mathfrak{A}'})$, where $\mathbf{R}_i^{\mathfrak{A}'}$ is the restriction of the map $\mathbf{R}_i^{\mathfrak{B}}$ to A'^{m_i}. A map $f : A \to B$ is an *embedding* of $\mathfrak{A}$ into $\mathfrak{B}$ if f is an isomorphism from $\mathfrak{A}$ onto $\mathfrak{B}_{\restriction f(A)}$. We write $\mathfrak{A} \leq \mathfrak{B}$ to indicate that there exists an embedding of $\mathfrak{A}$ into $\mathfrak{B}$.

* Corresponding author.

Email addresses: `delhomme@univ-reunion.fr` (Christian Delhommé), `pouzet@univ-lyon1.fr` (Maurice Pouzet), `sagi@renyi.hu` (Gábor Sági), `nsauer@math.ucalgary.ca` (Norbert Sauer).

Two relational structures are *isomorphic* or have the same *isomorphism type* if there is an isomorphism of one onto the other. We suppose that isomorphism types have been defined, we will denote $Is(\mathfrak{A})$ denotes the isomorphism type of $\mathfrak{A}$ and we will denote by Ω^*_μ the class of isomorphism types of relational structures with signature μ. We will denote by Ω_μ the subclass made of isomorphism types of finite relational structures. This class turns out to be a set (of size $\aleph_0$ if I is finite, and of size $2^{|I|}$ otherwise). The relation $\leq$ is a quasi-order on the class of relational structures with signature μ. It induces a quasi-ordering on the class Ω^*_μ and an ordering on Ω_μ, that we will also denote $\leq$.

Let $\mathfrak{A} := (A; \mathbf{R})$ be a relational structure with signature $\mu = (\mu_i; i \in I)$. The *skeleton*, skel($\mathfrak{A}$), of $\mathfrak{A}$ is the set $\{\mathfrak{A}_{\restriction F} : F \text{ is a finite subset of } A\}$. The *age*, age($\mathfrak{A}$), of $\mathfrak{A}$ is the set of isomorphism types of the elements of skel($\mathfrak{A}$). If $\mathfrak{B}$ is a relational structure then, by a slight abuse of notation, we allow ourselves to write $\mathfrak{B} \in$ age($\mathfrak{A}$) to indicate that the isomorphism type of $\mathfrak{B}$ is an element of the age of $\mathfrak{A}$. Note that age($\mathfrak{A}$) with the relation $\leq$ is also a poset.

A subset $\mathcal{A}$ of Ω_μ, is an *ideal* if :

(1) $\mathcal{A}$ is non-empty.
(2) $\mathcal{A}$ is an *initial segment*, that is $\mathfrak{B} \in \Omega_\mu$, $\mathfrak{C} \in \mathcal{A}$ and $\mathfrak{B} \leq \mathfrak{C}$ implies $\mathfrak{B} \in \mathcal{A}$.
(3) $\mathcal{A}$ is *up-directed*, that is $\mathfrak{B}, \mathfrak{B}' \in \mathcal{A}$ implies $\mathfrak{B}, \mathfrak{B}' \leq \mathfrak{C}$ for some $\mathfrak{C} \in \mathcal{A}$.

Clearly, the age of a relational structure is an ideal. As shown by Fraïssé, see [5], the converse holds for countable ideals and, hence, particularly in the case when μ is finite. If I is infinite, the converse also holds for every ideal consisting of the finite models of a set of universal first-order sentences (see Section 2). In his book, W. Hodges proposed to find an ideal for which the converse does not hold as an exercise for which he has no solution, see [6] Exercise 17 Chapter 7, p.332. Such an ideal, obtained with the first author, is described in [7]. It is made of binary relational structures coding metric spaces which isometrically embed into the real line equipped with the ordinary distance. Because of the existence of such an example, we may say that an ideal $\mathcal{A}$ of Ω_μ is *representable* if there is some relational structure $\mathfrak{A}$ such that age($\mathfrak{A}$) = $\mathcal{A}$ and it is κ-representable if there is some relational structure $\mathfrak{A}$ of cardinality κ such that age($\mathfrak{A}$) = $\mathcal{A}$.

One purpose of this paper is to point out the following

Problem 1 *Which ideals of Ω_μ are representable, which ideals are not?*

We present only partial results. We give first some examples of representable ideals, see Subsection 2.1. Examples lead us to consider the same problem for ideals made of finite metric spaces, ordered by isometrical embedding, see Subsection 2.4. The special case of ideals included into $age(\mathbb{R}^n)$, where $\mathbb{R}^n$ is

equipped with the euclidian distance, is left unresolved. In Subsection 2.5, we provide many more examples of ideals made of binary relational structures which are not representable. They are based on a notion of *ashes*.

In Theorem 1 below, we will characterize those signatures μ for which every ideal of Ω_μ is representable.

Theorem 1 *The following statements are equivalent:*

(i) *Every ideal of Ω_μ is representable.*
(ii) *The set $\mu^{-1}[\mathbb{N}^* \setminus \{1\}]$ is finite.*
(iii) *The set $\mathfrak{J}(\Omega_\mu)$ of ideals of Ω_μ, equipped with the product topology on $\mathfrak{P}(\Omega_\mu)$, is compact.*

Problem 2 *Let $\mathcal{A}$ be an ideal of Ω_μ. If the set $\mathfrak{J}(\mathcal{A})$ of ideals included into $\mathcal{A}$, equipped with the product topology on $\mathfrak{P}(\mathcal{A})$, is compact, does $\mathcal{A}$ have a representation?*

Note that if $\mathfrak{J}(\mathcal{A})$ is compact, this is the Stone space of the Boolean algebra generated by the subsets of $\mathcal{A}$ of the form $\uparrow \mathfrak{s} := \{\mathfrak{t} \in \mathcal{A} : \mathfrak{s} \leq \mathfrak{t}\}$ for $\mathfrak{s} \in \mathcal{A}$(cf [1]). We may represent members of this Boolean algebra by "sentences", replacing $\uparrow \mathfrak{s}$ by $\exists \mathfrak{s}$ and $\mathcal{A} \setminus \uparrow \mathfrak{s}$ by $\forall \neg \mathfrak{s}$ (this can be made more concrete by means of infinitary sentences). A positive solution of Problem 2 above amounts to a compactness theorem (for a counterpart, see Subsection 2.1).

The other purpose of this paper is to derive from this study a solution of a long standing question of Cusin and Pabion [3].

Indeed, on Ω_μ^* the same notion of ideals can be introduced. Since Ω_μ^* is a proper class, we extend the above stipulations by requiring that an ideal should be a set (and not a proper class). We say that an ideal $\mathcal{A}$ of Ω_μ^* is *bounded* if all its elements have cardinality less than some cardinal κ; it is *κ-bounded* if all elements have cardinality less than κ and for every $\lambda < \kappa$ there is an element of $\mathcal{A}$ of cardinality λ. It follows that every infinite ideal of Ω_μ is an $\aleph_0$-bounded ideal of Ω_μ^*.

In [3] Cusin and Pabion generalized the notions of age and ideal of isomorphism types as follows. For a relational structure $\mathfrak{A}$ and an ideal $\mathcal{J}$ of subsets of A they associated the set $\text{age}_{\mathcal{J}}(\mathfrak{A})$ consisting of isomorphism types of substructures induced by $\mathfrak{A}$ on elements of $\mathcal{J}$; more formally, $\text{age}_{\mathcal{J}}(\mathfrak{A}) := \{Is(\mathfrak{A}_{\restriction S}) \mid S \in \mathcal{J}\}$. If isomorphism types are quasi-ordered by embeddability, this set is an ideal of the quasi-ordered set Ω_μ^*.

Let us say that an ideal $\mathcal{C}$ of Ω_μ^* is *representable*, if there is a relational structure $\mathfrak{A}$ and an ideal $\mathcal{J}$ of subsets of A such that $\text{age}_{\mathcal{J}}(\mathfrak{A}) = \mathcal{C}$. Note that if $\mathcal{C}$ is an ideal of Ω_μ then it is representable in this more general sense if and only if

it is representable as defined previously. (To check this, let $\mathfrak{A}$ be a relational structure and let $\mathcal{J}$ be an ideal of finite subsets of A so that $\text{age}_{\mathcal{J}}(\mathfrak{A}) = \mathcal{C}$. Let B be the union of the elements of $\mathcal{J}$ and $\mathfrak{B} := \mathfrak{A}_{\restriction B}$. Then every finite subset F of B is in $\mathcal{J}$ because the singletons of B are elements of $\mathcal{J}$ and $\mathcal{J}$ is an updirected initial segment. Hence $\mathfrak{B}_{\restriction F} \in \mathcal{C}$ implying $\text{age}(\mathfrak{B}) \subseteq \mathcal{C}$. On the other hand every element of $\mathcal{C}$ is finite and hence an element of $\text{age}(\mathfrak{B})$.)

Cusin and Pabion asked the following question. Suppose that μ is a singleton signature. Is it then true, that every ideal of Ω^*_μ is representable? The answer is negative. In fact we will prove in Theorem 2, that if μ is a singleton ternary signature then there is an ideal of Ω^*_μ whose elements are countable relational structures and $\mathcal{A}$ is not representable.

Theorem 2 *Let μ be a singleton ternary signature. There is an $\aleph_1$-bounded ideal $\mathcal{A}$ in Ω^*_μ which is not representable.*

We do not know if there is an example with binary relations. In the case when μ is a singleton signature, we do not know if for every uncountable cardinal κ there exists a non-representable κ-bounded ideal of Ω^*_μ.

2 Representable and non-representable ideals

2.1 *Ideals defined by sets of universal sentences*

A sufficient condition for representability of an ideal $\mathcal{A}$ of Ω_μ with an arbitrary signature μ can be expressed in model-theoretic terms. As it is well-known, the class $mod(T)$ of models of a first-order theory T is an ideal of Ω^*_μ if and only if T is universal (that is can be axiomatized by universal sentences) and for every disjunction $\varphi \vee \psi$ of universal sentences, $\varphi \vee \psi \in T$ if and only if $\varphi \in T$ or $\psi \in T$ [2]. With the compactness theorem of first-order logic, it follows that an ideal $\mathcal{A}$ of Ω_μ, which consists of the finite models (up-to isomorphism) of a universal theory, is representable. Furthermore, if $\mathcal{A}$ is infinite, it is κ-representable for every $\kappa \geq |A|$.

The above condition on $\mathcal{A}$ can be easily translated in terms of *reducts* as follows.

Let $\mu : I \to \mathbb{N}^*$ be a signature. If I' is a subset of I, we denote by $\mu_{\restriction I'}$ the restriction of μ to I'. If $\mathfrak{A} := (A; (\mathsf{R}^{\mathfrak{A}}_i)_{i \in I})$ is a relational structure with signature μ, the I'-*reduct of* $\mathfrak{A}$ is the relational structure $\mathfrak{A}^{\restriction I'} := (A; (\mathsf{R}^{\mathfrak{A}}_i)_{i \in I'})$ of signature $\mu_{\restriction I'}$. If I' is finite, $\mathfrak{A}^{\restriction I'}$ is a *finite reduct* of $\mathfrak{A}$. If $\mathcal{C}$ is a class of relational structures of signature μ, we denote by $\mathcal{C}^{\restriction I'}$ the class of I'-reducts

of members of $\mathcal{C}$. We denote by $\widehat{\mathcal{C}}$ the class of relational structures $\mathfrak{A}$ such that $\mathfrak{A}^{\restriction I'} \in \mathcal{C}^{\restriction I'}$ for every finite $I' \subseteq I$. We say that $\mathcal{C}$ is *closed* if $\mathcal{C} = \widehat{\mathcal{C}}$. We use freely the same notations for classes made of isomorphism types of relational structures.

It is not hard to show that if $\mathcal{A}$ is an ideal of Ω_μ then $\widehat{\mathcal{A}}$ is an ideal too. And also that an ideal $\mathcal{A}$ is closed if and only if $\mathcal{A}$ is the set of finite models of a universal theory. Hence, we may recast the aforementioned fact as:

Theorem 3 *Every closed ideal $\mathcal{A}$ of Ω_μ is representable; if $\mathcal{A}$ is infinite, it is κ-representable for every $\kappa \geq |\mathcal{A}|$.*

A proof using compactness is a straightforward exercice. See [7] for a more detailed discussion.

2.2 The extension property

Let $\mathcal{A}$ be an ideal of Ω_μ. A relational structure $\mathfrak{A}$ with age included into $\mathcal{A}$ is *extendable w.r.t.* $\mathcal{A}$ if for every $\mathfrak{B} \in \mathcal{A}$ there is some $\mathfrak{C}$, with age included into $\mathcal{A}$, which extends both $\mathfrak{A}$ and $\mathfrak{B}$. An ideal $\mathcal{A}$ of Ω_μ has the *extension property* if every $\mathfrak{A} \in \Omega_\mu^*$ such that $|A| < \kappa := |\mathcal{A}|$ and $age(\mathfrak{A}) \subseteq \mathcal{A}$ is extendable w.r.t. $\mathcal{A}$.

Lemma 3 *If an ideal $\mathcal{A}$ of Ω_μ has the extension property then it is representable.*

Proof. Let $\kappa := |\mathcal{A}|$ and let $(\mathfrak{B}_\alpha)_{\alpha < \kappa}$ be an enumeration of the members of $\mathcal{A}$. We define a sequence $(\mathfrak{A}_\alpha)_{\alpha < \kappa}$ such that:

(1) $|A_\alpha| < \aleph_0$ if $\alpha < \omega$ and $|A_\alpha| \leq |\alpha|$, otherwise.
(2) $A_\alpha \subseteq A_{\alpha'}$ and $(\mathfrak{A}_{\alpha'})_{\restriction A_\alpha} = \mathfrak{A}_\alpha$ for every $\alpha < \alpha' < \kappa$.
(3) $age(\mathfrak{A}_\alpha) \subseteq \mathcal{A}$.
(4) $\mathfrak{B}_\alpha \leq \mathfrak{A}_{\alpha+1}$.

We start with $\mathfrak{A}_0$ equal to the relational structure on the empty set, and we use transfinite recursion. To get $\mathfrak{A}_{\alpha+1}$ we apply the extendibility property of $\mathcal{A}$ to $\mathfrak{A} := \mathfrak{A}_\alpha$ and $\mathfrak{B} := \mathfrak{B}_\alpha$. At limit stages we define $\mathfrak{A}_\alpha$ to be $\cup_{\gamma < \alpha} \mathfrak{A}_\gamma$. Clearly, $\mathfrak{A}_\kappa$ has age $\mathcal{A}$. □

Corollary 1 *Every countable ideal is representable.*

In view of Problem 2 we may ask:

Problem 4 *Let $\mathcal{A}$ be an ideal of Ω_μ. If $\mathfrak{J}(\mathcal{A})$ is compact, does $\mathcal{A}$ have the extension property?*

2.3 The amalgamation property

Let $\mathcal{C} \subseteq \Omega_\mu^*$. Let $f_1 : \mathfrak{A} \to \mathfrak{A}_1$ and $f_2 : \mathfrak{A} \to \mathfrak{A}_2$ be a pair of embeddings such that $\mathfrak{A}, \mathfrak{A}_1, \mathfrak{A}_2 \in \mathcal{C}$. We say that this pair *amalgamates* if there are two embeddings $g_1 : \mathfrak{A}_1 \to \mathfrak{B}$ and $g_2 : \mathfrak{A}_2 \to \mathfrak{B}$ such that $\mathfrak{B} \in \mathcal{C}$ and $g_1 \circ f_1 = g_2 \circ f_2$. We say that $\mathcal{C}$ has the *amalgamation property* if every pair of embeddings amalgamates. If this property holds for pairs of embeddings whose domain have size at most κ, we say that $\mathcal{C}$ has the κ-*amalgamation property*

Lemma 5 *Let $\mathcal{A}$ be an ideal of Ω_μ; if $\mathcal{A}$ has the amalgamation property, then the collection of countable $\mathfrak{A}$ whose age is included into $\mathcal{A}$ has the $\aleph_0$-amalgamation property. In particular, if $\mathcal{A}$ has size at most $\aleph_1$ then it has the extension property.*

Proof. One proves first that every pair of embedding $f_1 : \mathfrak{A} \to \mathfrak{A}_1$ and $f_2 : \mathfrak{A} \to \mathfrak{A}_2$ such that $\mathfrak{A}, \mathfrak{A}_1 \in \mathcal{A}$, $age(\mathfrak{A}_2) \subseteq \mathcal{A}$ and A_2 countable amalgamates. For that, one writes A_2 as an increasing sequence $(A_{2,n})_{n \in \omega}$ of finite sets containing the image of A and one successively amalgamates $\mathfrak{A}_1$ with the $\mathfrak{A}_{2,n}$. This allows to do the same when the condition on $\mathcal{A}_1$ is relaxed. $\quad\square$

Corollary 2 *If an ideal of Ω_μ has the amalgamation property and has size at most $\aleph_1$ then it is representable.*

Problems 6 *Let $\mathcal{A}$ be an ideal of Ω_μ. Suppose that $\mathcal{A}$ has the amalgamation property.*

(1) *Does $\mathcal{A}$ has a representation?*
(2) *Is there an homogeneous $\mathfrak{A}$ such that $age(\mathfrak{A}) = \mathcal{A}$?*

2.4 Metric spaces as relational structures and representability

Metric spaces can be encoded, in several ways, as binary relational structures in such a way that isometries correspond to embeddings. For example, let $\mu : \mathbb{Q}_+ \to \{2\}$. To each metric space $\mathbb{M} := (M, d)$, where d is a distance over the set M, we may associate the relational structure $rel(\mathbb{M}) := (M, (\delta_r)_{r \in \mathbb{Q}_+})$ of signature μ where $\delta_r(x, y) = 1$ if $d(x, y) \leq r$ and $\delta_r(x, y) = 0$ otherwise. With this definition, $d(x, y)$ is the infimum of the set of r's such that $\delta_r(x, y) = 1$, hence we may recover $\mathbb{M}$ from $rel(\mathbb{M})$. From this fact, it follows that:

(1) for two metric spaces $\mathbb{M} := (M, d)$, $\mathbb{M}' := (M', d')$, a map $f : M \to M'$ is an isometry from $\mathbb{M}$ into $\mathbb{M}'$ if and only this is an embedding from $rel(\mathbb{M})$ into $rel(\mathbb{M}')$.
(2) Moreover, if $\mathfrak{A}$ is a relational structure of signature μ such that every

induced substructure on at most 3 elements embeds into $\mathbb{M}'$, then there is a distance d on A such that $rel((A, d)) = \mathfrak{A}$.

If we compare metric spaces via isometric embeddings, the class $\mathcal{M}$, resp. $\mathcal{M}_{<\omega}$, of metric spaces, resp. finite metric spaces, is an ideal. Hence, $\mathcal{M}$ and $\mathcal{M}_{<\omega}$ yield an ideal of Ω_μ^* and of Ω_μ respectively. It make sense then to consider the representability of an ideal $\mathcal{C}$ of $\mathcal{M}_{<\omega}$. Because of item 1 above, its image $\mathcal{A}$ into Ω_μ^* is an ideal and because of item 2 the representability of $\mathcal{A}$ amounts to the representability of $\mathcal{C}$.

The ideal $\mathcal{M}_{<\omega}$ is representable, eg by the space $\ell_{\mathbb{R}}^\infty(\mathbb{N})$ of bounded sequences of reals, equipped with the "sup" distance. But, it turns out that there are plenty of non-representable ideals of $\mathcal{M}_{<\omega}$. We give some examples below.

Let $\mathbb{M} := (M, d)$ be a metric space. Let $a \in M$, we set $spec(\mathbb{M}, a) := \{d(a, x) : x \in \mathbb{M}\}$ and, for $r \in \mathbb{R}_+$, we set $B_{\mathbb{M}}(a, r) := \{x \in M : d(a, y) \leq r\}$. The *spectrum* of $\mathbb{M}$ is the set $spec(\mathbb{M}) := \bigcup\{spec(\mathbb{M}, a) : a \in M\}$. The *diameter of* $\mathbb{M}$ is $\delta(\mathbb{M}) := sup(spec(\mathbb{M}))$ and we set $d(\mathbb{M}) := inf(spec(\mathbb{M}) \setminus \{0\})$ (hence $\delta(\mathbb{M}) := +\infty$ if $\mathbb{M}$ is unbounded and $d(\mathbb{M}) := +\infty$ if $|M| \leq 1$). If $\mathcal{C}$ is a set of metric spaces, we set $d(\mathcal{C}) := inf\{d(\mathbb{M}) : \mathbb{M} \in \mathcal{C}\}$. Let $t \in \mathbb{R}_+$, we set $\omega_t(\mathbb{M}) := sup\{|X| : X \subseteq M$ and $d(\mathbb{M}_{\upharpoonright X}) \geq t\}$. We say that $\mathbb{M}$ is *t-totally bounded* if $\omega_t(\mathbb{M})$ is finite and that $\mathbb{M}$ is *totally bounded* if $\mathbb{M}$ is t-totally-bounded for every $t \in \mathbb{R}_+^*$. We say that $\mathbb{M}$ is *t-uniformly bounded* if $\omega_t(\mathbb{M}_{\upharpoonright X}) \leq \varphi_t(\delta(\mathbb{M}_{\upharpoonright X}))$ for some non-decreasing map $\varphi_t : \mathbb{R}_+ \to \mathbb{R}_+$ and every bounded subspace $\mathbb{M}_{\upharpoonright X}$ of $\mathbb{M}$.

Lemma 7 *Let $\mathbb{M}$ be a t-totally bounded metric space. Let $\mathcal{C} \subseteq age(\mathbb{M})$ be an ideal such that $d(\mathcal{C}) \geq t$. Then $\mathcal{C}$ is representable iff $\mathcal{C}$ is countable.*

Proof. Suppose that $\mathcal{C}$ is representable. Let $\mathbb{M}' := (M', d')$ be a representation.

Claim.
$$|X'| = \omega_t(\mathbb{M}'_{\upharpoonright X'}) \leq \varphi_t(\delta(\mathbb{M}'_{\upharpoonright X'}))$$
for every bounded subset X' of $\mathbb{M}'$.

Proof of the Claim. Since $age(\mathbb{M}') \subseteq \mathcal{C}$, we have $d(\mathbb{M}'_{\upharpoonright X'}) \geq t$. The equality follows. If the inequality above does not hold then X' contains a finite subset X'' with more than $s := \varphi_t(\delta(\mathbb{M}'_{\upharpoonright X'}))$ elements. But then for some finite subset X of $\mathbb{M}$ such that $\mathbb{M}_{\upharpoonright X}$ is isometric to $\mathbb{M}'_{\upharpoonright X''}$ we have $|X| = \omega_t(\mathbb{M}_{\upharpoonright X}) \leq \varphi_t(\delta(\mathbb{M}_{\upharpoonright X})) \leq s$. A contradiction.

From our claim, each ball in $\mathbb{M}'$ is finite, hence $\mathbb{M}'$ is countable. Thus $\mathcal{C} = age(\mathbb{M}')$ is countable. Conversely, if $\mathcal{C}$ is countable then it is representable from Corollary 1.

$\square$

Proposition 1 *Let $\mathbb{M}$ be an unbounded metric space whose group of isometries, $Aut(\mathbb{M})$, acts transitively on the elements of M. Suppose that for some $t \in \mathbb{R}_+^*$, $spec(\mathbb{M}) \cap [t, +\infty)$ is uncountable and every bounded subset of $\mathbb{M}$ is t-totally bounded, then $age_t(\mathbb{M}) := \{Is(\mathbb{M}_{\restriction X}) : d(\mathbb{M}_{\restriction X}) \geq t$ and X is finite$\}$ is a non-representable ideal of $age(\mathbb{M})$.*

Proof.

Claim 1. Let $t \in \mathbb{R}_+^*$, then $\mathbb{M}$ is t-uniformly bounded.

Proof of Claim 1. This follows from the fact that $Aut(\mathbb{M})$ is transitive and every bounded subspace is t-totally bounded. To see it, fix $a \in M$. Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ defined by setting $\varphi_t(r) := \omega_t(\mathbb{M}_{\restriction B_{\mathbb{M}}(a,r)})$. Let $\mathbb{M}_{\restriction X}$ be a bounded subspace of $\mathbb{M}$ and let $r := \delta(\mathbb{M}_{\restriction X})$. Since $Aut(\mathbb{M})$ is transitive, $\mathbb{M}_{\restriction X}$ is isometric to $\mathbb{M}_{\restriction X'}$ for some subset X' of $B_{\mathbb{M}}(a,r)$. Hence, $\omega_t(\mathbb{M}_{\restriction X}) = \omega_t(\mathbb{M}_{\restriction X'}) \leq \omega_t(\mathbb{M}_{\restriction X'}) = \varphi_t(r)$.

Claim 2. $age_t(\mathbb{M})$ is an uncountable ideal.

Proof of Claim 2. Since $\mathbb{M}$ is unbounded and $Aut(M)$ is transitive, $age_t(\mathbb{M})$ is an ideal. Let $a \in M$. Since $Aut(\mathbb{M})$ is transitive, $spec(\mathbb{M}, a) = spec(\mathbb{M})$, hence $spec(\mathbb{M}, a) \cap [t, +\infty[$ is uncountable. Thus $age_t(\mathbb{M})$ contains uncountably many 2-element metric spaces. $\square$

We consider more generally ideals made of metric spaces which omit a given set of distances. Precisely, let $\mathbb{M}$ be a metric space, $\kappa := |M|$ and let $A \subseteq \mathbb{R}_+^*$; we set $age_{-A}(\mathbb{M}) := \{Is(\mathbb{M}_{\restriction X}) : spec(\mathbb{M}_{\restriction X}) \cap A = \emptyset$ and X is finite$\}$. Given a type $\mathfrak{b} \in age(\mathbb{M})$, let $Orb(\mathfrak{b}, \mathbb{M}) := \{X \subseteq M : Is(\mathbb{M}_{\restriction X}) = \mathfrak{b}\}$; given $r \in R_+$ and $a \in M$, we set $S_{\mathbb{M}}(a,r) := \{x \in M : d(a,x) = r\}$.

Proposition 2 *Let $\mathbb{M}$ be a metric space and let $\kappa := spec(\mathbb{M})$. Suppose that κ is infinite and that there is a cardinal $\lambda < \kappa := spec(\mathbb{M})$ such that for every $\mathfrak{s} \in age(\mathbb{M})$, $Orb(\mathfrak{b}, \mathbb{M})$ contains a subset $\mathcal{X}_{\mathfrak{b}}$ of size at least κ such that $|\{F \in \mathcal{X}_{\mathfrak{b}} : F \cap S_{\mathbb{M}}(a,r) \neq \emptyset\}| \leq \lambda$ for every $a \in M, r \in \mathbb{R}_+$. Let $A \subseteq \mathbb{R}_+^*$ such that $|A| < \kappa$. Then $age_{-A}(\mathbb{M})$ is an ideal representable by some subspace of $\mathbb{M}$.*

Proof. We mimick the proof of Lemma 3. Let $(\mathfrak{b}_\alpha)_{\alpha < \kappa}$ be an enumeration of the members of $age_{-A}(\mathbb{M})$. We define a sequence $(X_\alpha)_{\alpha < \kappa}$ of subsets of M such that:

(1) $|X_\alpha| < \aleph_0$ if $\alpha < \omega$ and $|X_\alpha| \leq |\alpha|$, otherwise.
(2) $X_\alpha \subseteq X_{\alpha'}$
(3) $age(\mathbb{M}_{\restriction X_\alpha}) \subseteq age_{-A}(\mathbb{M})$.

(4) $\mathfrak{b}_\alpha \in \mathrm{age}(\mathbb{M}_{\upharpoonright X_{\alpha+1}})$.

We start with $X_0 := \emptyset$. To get $X_{\alpha+1}$ we select some subset $F \in Orb(\mathfrak{b}_\alpha, \mathbb{M})$ such that $\mathrm{age}(M_{\upharpoonright X_\alpha \cup F}) \in \mathrm{age}_{-A}(\mathbb{M})$. If this was imposssible, then for each $F \in \mathcal{X}_\mathfrak{b} \subseteq Orb(\mathfrak{b}_\alpha, \mathbb{M})$ we will find $(x_F, r_F) \in X_\alpha \times A$ such that $d(x_F, y_F) = r_F$ for some $y_F \in F$. Since $|\mathcal{X}_\mathfrak{b}| \geq \kappa > |X_\alpha \times A|$, there is a subset $\mathcal{X}'$ of size at least λ^+ and a pair (a, r) such that $(x_F, r_F) = (a, r)$ for all $F \in \mathcal{X}'$ but then $|\{F \in X_\mathfrak{b} : F \cap S_\mathbb{M}(a, r) \neq \emptyset\}| \geq \lambda^+$ contradicting our hypotheses on $Orb(\mathfrak{b}_\alpha, \mathbb{M})$. This allows to set $X_{\alpha+1} := X_\alpha \cup X$. At limit stages we define X_α to be $\bigcup\{X_\gamma : \gamma < \alpha\}$. □

Let $n \in \mathbb{N}^*$ and let $\mathbb{R}^n$ be the set of n-tuples of reals, equipped with the euclidian distance d_2. Then $(\mathbb{R}^n, d_2)$ satisfies the hypotheses of Proposition 1 above and this for every t. It also satisfies the hypotheses of Proposition 2 (fix a direction in $\mathbb{R}^n$ and in each $Orb(\mathfrak{b}, \mathbb{R}^n)$ select an orbit $\mathcal{X}_\mathfrak{b}$ according to this group and finally set $\lambda := \aleph_0$). The same facts hold if the euclidian distance is replaced by any distance associated with a vector space norm on $\mathbb{R}^n$. Then, we have the following:

Corollary 3 *Let $n \in \mathbb{N}^*$.*

- *For every positive real t, the set $\mathrm{age}_t((\mathbb{R}^n, d_2))$ of isometric types of finite subspaces $\mathbb{X}$ of $(\mathbb{R}^n, d_2)$ such that $d(\mathbb{X}) \geq t$ is a non representable ideal.*
- *For every subset $A \subset \mathbb{R}_+^*$ of size $\kappa < 2^{\aleph_0}$, the set $\mathrm{age}_{-A}(\mathbb{M})$ of isometric types of subspaces $\mathbb{X}$ of $(\mathbb{R}^n, d_2)$ whose distances does not belong to A is an ideal representable by a subset of $\mathbb{R}^n$.*

The example of a non-representable ideal given in [7] is $\mathrm{age}_1(\mathbb{R})$. By taking $A := \mathbb{Q}_+$, the second item of the corollary above asserts that there are subspaces $\mathbb{X}$ of the real line whose age is the set $\mathrm{age}_{-\mathbb{Q}_+}(\mathbb{R})$ made of all finite metric spaces with no rational non-zero distances. Such spaces are sections of the quotient $\mathbb{R}/\mathbb{Q}$ of the additive group $\mathbb{R}$ by the additive group $\mathbb{Q}$, but not every section provides such a space. The metric space $\mathbb{S}$ made of the unit circle with the arc length metric satisfies the hypotheses of Proposition 2. *We do not know if* $\mathrm{age}(\mathbb{S})$ *contains an non-representable ideal.*

In the case of $\mathbb{R}$ or even $\mathbb{R}^n$, we can say a little more. Let $(\mathbb{R}, d)$ where $d(x, y) := |x - y|$. Note first that a 3-element metric space isometrically embeds into $(\mathbb{R}, d)$ iff one distance is the sum of the two others; a 4-element metric space whose all 3-element subsets embed into $(\mathbb{R}, d)$ does not necessarily embed into $(\mathbb{R}, d)$ (think of four vertices forming a "rectangle" whose sides have length a and b and diagonal length $c := a + b$). However, all the ≤ 4-element subspaces of a metric space $\mathbb{M}$ embed isometrically into $(\mathbb{R}, d)$ if and only if $\mathbb{M}$ isometrically embeds into $(\mathbb{R}, d)$; moreover an embedding from $\mathbb{M}$ into $\mathbb{R}$ is determined by its values on any 2-element subset of $\mathbb{M}$. This extends: *all $\leq n + 3$-elements subspaces of a metric space $\mathbb{M}$ embed into $(\mathbb{R}^n, d_2)$ iff $\mathbb{M}$ embeds into $(\mathbb{R}^n, d_2)$.*

From this, we immediately have:

- if $\mathcal{C} \subseteq age((\mathbb{R}^n, d_2))$ is a representable ideal, all its representations embed into $(\mathbb{R}^n, d_2)$. Hence have cardinality at most the continuum.

This is a substantial difference with to the representability of closed ideals.

Let us mention that in the case of the real line, the two problems in Problems 4 have a positive answer.

Lemma 8 *Let $\mathbb{R}$ be the real line equipped with the ordinary distance. Let $\mathcal{C} \subseteq age(\mathbb{R})$ be an ideal. Then $\mathcal{C}$ has the 2-amalgamation property if and only if there is a homogeneous metric space $\mathbb{D}$ whose age is $\mathcal{C}$. Moreover, if $\mathcal{C}$ contains at least a 3-element metric space, then $\mathbb{D} = (G, d_{\restriction G})$ where G is an additive subgroup of $\mathbb{R}$ and $d_{\restriction G}$ is induced by the distance on R.*

Proof. We just give a hint. Let $\mathcal{C} \subseteq age(\mathbb{R})$ be an ideal. Let $V := \cup \{spec(\mathbb{M}) : \mathbb{M} \in \mathcal{C}\}$.

Case 1. $|V| \leq 2$. In this case $V = \{0, v\}$ and $\mathbb{D} := (V, d_{\restriction V})$ has the required property.

Case 2. $|V| \geq 2$. Set $G := V \cup -V$.

Claim 1. If $\mathcal{C}$ has the 2-amalgamation property then G is a subgroup of $\mathbb{R}$. The proof of this claim breaks into three parts; we leave the verification to the reader.

Subclaim 1. For every finite subset F of V there is some $\mathbb{M} \in \mathcal{C}$ and $a \in M$ such that $F \subseteq spec(\mathbb{M}, a)$.

Subclaim 2. V is unbounded.

Subclaim 3. $y - x \in V$ and $x + y \in V$ for every $x, y \in V$ with $x \leq y$.

Claim 2. Let G be an additive subgroup of $\mathbb{R}$ and $\mathbb{D} := (G, d_{\restriction G})$. Let f be an isometry from a subset A of G onto a subset A. Then f extends to an isometry.

Indeed, we may suppose $A \neq \emptyset$. Let $x \in A$ and $x' := f(x) \in A'$. Let $g^+(y) := x' + y - x$ for all $y \in G$ and $g^-(y) := x' - y + x$ for all $y \in G$. These two maps are isometries from $\mathbb{D}$ into iself and one of these extends f. This proves Claim 2. $\qquad\square$

Note that the 1-amalgamation property provides a representative $\mathbb{M}$ with $Aut(\mathbb{M})$ transitive. As an example let $M := a \cdot \mathbb{Z} \cup (b + a \cdot \mathbb{Z})$ with $0 < b < \frac{a}{2}$ and $\mathbb{M} := (M, d_{\restriction M})$.

Problems 9 *(1) Describe the amagamable ideals and the homogeneous sub-spaces of $(\mathbb{R}^n, d_2)$;*
(2) Characterize the representable ideals of $(\mathbb{R}^n, d_2)$.

2.5   A construction of non-representable ideals

Let S be a set. A set $\mathcal{S}$ of finite subsets of S is an *ash on S* if:

(1) $\{s\} \in \mathcal{S}$ for every $s \in S$.
(2) For every finite subset $\mathcal{F}$ of $\mathcal{S}$ there exists an element $s \in S \setminus \bigcup \mathcal{F}$ so that $\{s\} \cup F \in \mathcal{S}$ for every set $F \in \mathcal{F}$.
(3) For every subset S' of S with $|S'| = |S|$ there is a finite subset F of S' with $F \notin \mathcal{S}$.

Note that there is no ash on a finite set.

An ash can be obtained as follows. Let κ be an infinite cardinal and $\mathcal{T}$ a family which consists of κ sets of cardinality κ^+ for which there is no finite subset $F \subseteq \bigcup \mathcal{T}$ with $X \cap F \neq \emptyset$ for all $X \in \mathcal{T}$. (For example the elements of $\mathcal{T}$ are disjoint.) Let $0 \neq n \in \omega$ and let $\mathcal{S}$ be the set of finite subsets $F \subseteq S := \bigcup \mathcal{T}$ with $|F \cap X| \leq n$ for all $X \in \mathcal{T}$. Then $\mathcal{S}$ is an ash on S.

Let $\mu := S \to \{2\}$. Let $\mathcal{S}$ be an ash on S and let $\mathcal{A}_{\mathcal{S}}$ be the collection of all finite relational structures $\mathfrak{A}$ in Ω_μ for which for all $x, y, z \in A$ and $s, t \in S$:

(1) $\neg R_s(x, x)$.
(2) If $R_s(x, y)$ then $R_s(y, x)$.
(3) If $R_s(x, y)$ and $R_t(x, y)$ then $s = t$ and if $R_s(x, y)$ and $R_s(x, z)$ then $y = z$.
(4) If $x \neq y$ then there exists an element $r \in S$ so that $R_r(x, y)$.
(5) Every non-empty subset of the set $\{r \in S : \exists u \in A : R_r(x, u)\}$ is an element of $\mathcal{S}$.

Note that the elements of $\mathcal{A}_{\mathcal{S}}$ are graphs with several types of edges, a type of edge for every element of S.

Lemma 10 *Let $\mathcal{S}$ be an ash on the set S. Then $\mathcal{A}_{\mathcal{S}}$ is an ideal of Ω_μ.*

Proof. It follows directly from the definition that $\mathcal{A}_{\mathcal{S}}$ is closed under induced substructures. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two elements of $\mathcal{A}_{\mathcal{S}}$ with $A \cap B = \emptyset$. Item 2 of the definition of ash allows us to determine successively an edge type for every pair (x, y) with $x \in A$ and $y \in B$, satisfying items 1 to 5 of the definition of $\mathcal{A}_{\mathcal{S}}$. It follows that $\mathcal{A}_{\mathcal{S}}$ is updirected. $\qquad\square$

Lemma 11 *Let $\mathcal{S}$ be an ash on the set S. Then $\mathcal{A}_{\mathcal{S}}$ is not representable.*

Proof. Assume for a contradiction that there is a relational structure $\mathfrak{A}$ whose age is equal to $\mathcal{A}_{\mathcal{S}}$.

Let $x \in A$. It follows from items 3 and 4 of the definition of $\mathcal{A}_{\mathcal{S}}$ that there exists an injection $f : A \setminus \{x\} \rightarrow S$ so that $R_{f(y)}(x, y)$ for every element $y \in A \setminus \{x\}$.

Because every two element structure of $\mathcal{A}_{\mathcal{S}}$ is isomorphic to an induced substructure of $\mathfrak{A}$ it follows that $|A| \geq |S|$. Hence $|f[A \setminus \{x\}]| = |S|$ which in turn implies using item 3 of the definition of an ash that there is a finite subset $F \subseteq f[A \setminus \{x\}]$ with $F \notin \mathcal{S}$. But this leads to a contradiction because the substructure of $\mathfrak{A}$ induced by the set $\{x\} \cup \{y : \exists s \in F : R_s(x, y)\}$ is not an element of $\mathcal{A}_{\mathcal{S}}$ according to item 5 of the definition of $\mathcal{A}_{\mathcal{S}}$. $\qquad\square$

With the theorem of R. Fraïssé asserting that every ideal with countable signature is representable, this yields:

Corollary 4 *Let $\mathcal{S}$ be an ash on the set S. Then $|S| > \aleph_0$.*

The graph G is an *ash-graph* if it has the following two properties:

- For every finite subset $F \subseteq G$ there exists a vertex $v \in G \setminus F$ which is adjacent to every vertex in F.
- The graph G does not contain a complete subgraph K with $|K| = |G|$.

Note that the set of finite subsets F of G which contain an element adjacent to all the other elements of F is an ash on G.

An ash-graph G can be obtained as follows. Let $\mathbb{M} := (M; d)$ be a metric space with the properties:

- For every finite subset F of M there exists an element $x \in M$ with $d(x, y) \geq 1$ for all $y \in F$.
- Every subset W of M with $|W| = |M|$ contains two elements x, y with $x \neq y$ and $d(x, y) < 1$.

Such a metric space is an *ash-space*. The set of real numbers is an example of an ash-space.

Let $\mathbb{M} := (M; d)$ be an ash-space. Then the graph G with vertex set M in wich two different vertices are adjacent if and only if their distance is larger than or equal to 1 is an ash-graph. On the other hand if G is an ash-graph, then the metric space $(G; d)$ with $d(x, y) = 1$ if x is adjacent to y and $d(x, y) = \frac{1}{2}$ if x is not adjacent to y is an ash-space.

Let P := $(P; \leq)$ be an up-directed poset which does not contain a maximal element and no chain of size $|P|$. Such a poset is an *ash-poset*. Let P := $(P; \leq)$

be an ash-poset. Let $\mathcal{P}$ be the set of finite subsets F of P which contain an element x with $x \geq y$ for all $y \in F$. It follows that $\mathcal{P}$ is an ash on P.

For example, let $\kappa > \aleph_0$ be a cardinal and let P be the poset on the set of finite subsets of κ with $\subseteq$ as the order relation. Then P is an ash-poset.

Let $\mathcal{S}$ be an ash on the set S. We used a particular construction to obtain a non-representable age. There are many other ways. For an example, we can generalise in an obvious way from binary to n-ary relations and we can define generalized edges of some type $s \in S$ as n-tuples for which a relation of the form R_s holds. Of course the relations do not have to be symmetric.

3 Proof of Theorem 1

We will use the following fact. Let P be a poset and let $\mathcal{J}(P)$ be the set of ideals of P. We think of $\mathcal{J}(P)$ as being equipped with the topology induced by the product topology on the power set of P. Then *$\mathcal{J}(P)$ is compact if and only if P is a finite union of principal final segments and $\uparrow x \cap \uparrow y$ is a finite union of principal final segments, for all $x, y \in P$* (for a proof, see [1]).

Applying this to a non empty initial segment $\mathcal{C}$ of Ω_μ and observing that Ω_μ has a least element and does not have an infinite descending chain, we get that $\mathcal{J}(\mathcal{C})$ is compact if and only if for every $\mathfrak{A}, \mathfrak{B} \in \mathcal{C}$ there are at most finitely many non-isomorphic $\mathfrak{C} \in \mathcal{C}$ such that:

(1) $\mathfrak{A}, \mathfrak{B} \leq \mathfrak{C}$.
(2) If $x \in \mathfrak{C}$ then $\mathfrak{A} \not\leq \mathfrak{C}_{\restriction C \setminus \{x\}}$ or $\mathfrak{B} \not\leq \mathfrak{C}_{\restriction C \setminus \{x\}}$.

Lemma 12 *$\mathcal{J}(\Omega_\mu)$ is compact if and only if $\mu^{-1}[\mathbb{N}^* \setminus \{1\}]$ is finite.*

Proof. Let us check that if $\mathcal{C} := \Omega_\mu$ and $\mu^{-1}[\mathbb{N} \setminus \{1\}]$ is finite then only finitely many non-isomorphic $\mathfrak{C}$ satisfy Conditions 1 and 2 above. Let $\mathfrak{A}, \mathfrak{B} \in \mathcal{C}$. Let $m := |A|$ and $n := |B|$. Suppose $\mathfrak{C}$ satisfies Conditions 1 and 2 and let $r_{\mathfrak{C}} := |C|$. We may suppose that $C = \{1, \dots, r_{\mathfrak{C}}\}$. Let $A_{\mathfrak{C}}, B_{\mathfrak{C}} \subseteq C$ such that $\mathfrak{C}_{\restriction A_{\mathfrak{C}}} \cong \mathfrak{A}$ and $\mathfrak{C}_{\restriction B_{\mathfrak{C}}} \cong \mathfrak{B}$. Clearly $A_{\mathfrak{C}} \cup B_{\mathfrak{C}} = C$, hence $r_{\mathfrak{C}} \leq m + n$. If there are infinitely many non-isomorphic $\mathfrak{C}$ satisfying Conditions 1 and 2, there are infinitely many for which $r_{\mathfrak{C}}, A_{\mathfrak{C}}, B_{\mathfrak{C}}, \mathfrak{C}_{\restriction A_{\mathfrak{C}}}$ and $\mathfrak{C}_{\restriction B_{\mathfrak{C}}}$ are independent of $\mathfrak{C}$. Let r, A, B such a triple. Since $A \cup B = \{1, \dots, r\}$, all unary relations on $\{1, \dots, r\}$ are entirely determined. Let $I' := \mu^{-1}[\mathbb{N}^* \setminus \{1\}]$. Since I' is finite, the number of relational structures of signature $\mu_{\restriction I'}$ defined on $\{1, \dots, r\}$ is finite but then one cannot define infinitely many relational structures of signature μ on this set. A contradiction.

Let $\underline{2}$ be the constant map from $\mathbb{N}$ to $\{2\}$.

Claim 1. If $\mu^{-1}[\mathbb{N}^* \setminus \{1\}]$ is infinite then $\mathfrak{J}(\Omega_{\underline{2}})$ can be mapped continuously into $\mathfrak{J}(\Omega_\mu)$ by a one-to-one map.

Proof of Claim 1. Let $\varphi : \omega \to \mu^{-1}[\mathbb{N}^* \setminus \{1\}]$ be a one-to-one map and let $rang(\varphi)$ be its range. For every $\mathfrak{A} := (A, (R_i)_{i<\omega}) \in \Omega_{\underline{2}}^*$, let $F(\mathfrak{A}) := (A, (S_i)_{i \in I})$ where $S_i : A^{n_i} \to \{0\}$ if $i \notin rang(\varphi)$ and $S_i((x_1, \ldots, x_{n_i})) := R_j((x_1, x_2))$ if $i := \varphi(j)$. Clearly,

(1) $F(\mathfrak{A}_{\restriction B}) = F(\mathfrak{A})_{\restriction B}$ for every $\mathfrak{A} \in \Omega_{\underline{2}}^*$ and $B \subseteq A$;
(2) If $\mathfrak{A}, \mathfrak{A}' \in \Omega_{\underline{2}}^*$, a map f is an isomorphism from $F(\mathfrak{A})$ onto $F(\mathfrak{A}')$if and only if f is an isomorphism from $\mathfrak{A}$ onto $\mathfrak{A}'$.

Consequently, F defines an embedding from $\Omega_{\underline{2}}$ onto an initial segment of Ω_μ. This map induces a continuous embedding from $\mathfrak{J}(\Omega_{\underline{2}})$ into $\mathfrak{J}(\Omega_\mu)$.

Claim 2. $\mathfrak{J}(\Omega_{\underline{2}})$ is not compact.

Proof of Claim 2. Let $\mathfrak{A}, \mathfrak{B}$, where $A := \{0\}$, $\mathsf{R}_i^{\mathfrak{A}} : A^2 \to \{1\}$, $B := \{1\}$, $\mathsf{R}_i^{\mathfrak{B}} : B^2 \to \{0\}$ for $i < \mathbb{N}$. Let $\mathfrak{C}_n$ where $C_n := \{0, 1\}$, $\mathsf{R}_n^{\mathfrak{C}}(x, y) = 1$ iff $(x, y) \in \{(0,0), (0,1)\}$ and $\mathsf{R}_i^{\mathfrak{C}}(x, y) = 1$ iff $(x, y) = (0, 0)$ in case $i \neq n$. The $\mathfrak{C}_n$'s satisfy Conditions 1 and 2 above. Hence, $\mathfrak{J}(\Omega_{\underline{2}})$ cannot be compact.

It follows from Claims 1 and 2, that $\mathfrak{J}(\Omega_\mu)$ cannot be compact if $\mu^{-1}[\mathbb{N}^* \setminus \{1\}]$ is infinite. This completes the proof. □

Lemma 13 $\mu^{-1}[\mathbb{N}^* \setminus \{1\}]$ *is finite if and only if every ideal* $\mathcal{A} \in \mathfrak{J}(\Omega_\mu)$ *is representable.*

Proof. Suppose that $\mu^{-1}[\mathbb{N}^* \setminus \{1\}]$ is finite. Let $\mathcal{A} \in \mathfrak{J}(\Omega_\mu)$. For each $\mathfrak{s} \in \mathcal{A}$, let $\uparrow \mathfrak{s} := \{\mathfrak{t} \in \mathcal{A} : \mathfrak{s} \leq \mathfrak{t}\}$. The set $\mathcal{F} := \{X \subseteq \mathcal{A} :\uparrow \mathfrak{s} \subseteq X$ for some $\mathfrak{s} \in \mathcal{A}\}$ is a filter. Let $\mathcal{U}$ be an ultrafilter on $\mathcal{A}$ containing it. Let $\mathcal{A}^*$ be the set of non-empty members of $\mathcal{A}^*$. For each $\mathfrak{s} \in \mathcal{A}^*$, let $\mathfrak{S}_\mathfrak{s}$ such that $Is(\mathfrak{S}_\mathfrak{s}) = \mathfrak{s}$ and $S_\mathfrak{s} := \{1, \ldots, |\mathfrak{s}|\}$. Let $\mathfrak{A} := \Pi_{\mathfrak{s} \in \mathcal{A}^*} \mathfrak{S}_\mathfrak{s}/\mathcal{U}$ be the ultraproduct of the $\mathfrak{S}_\mathfrak{s}$' s . Let $B := \{(x_\mathfrak{s})_{\mathfrak{s} \in \mathcal{A}^*} :$ there is a $\mathfrak{t} \in \mathcal{A}$ such that $\{\mathfrak{s} : \mathfrak{S}_{\mathfrak{s} \restriction \{x_\mathfrak{s}\}} = \mathfrak{t}\} \in \mathcal{U}\}$.

Claim. $\mathrm{age}(\mathfrak{A}_{\restriction B}) = \mathcal{A}$.

First, $\mathcal{A} \subseteq \mathrm{age}(\mathfrak{A}_{\restriction B})$. Indeed, let $\mathfrak{s} \in \mathcal{A}$. For every $\mathfrak{s}' \geq \mathfrak{s}$ select an embedding $\varphi_{\mathfrak{s}'}$ of $\mathfrak{S}_\mathfrak{s}$ into $\mathfrak{S}_{\mathfrak{s}'}$. Let $X := \{(x_{\mathfrak{s}'}^i)_{\mathfrak{s}' \in \mathcal{A}} : 1 \leq i \leq |\mathfrak{s}|$ and $x_{\mathfrak{s}'}^i := \varphi_{\mathfrak{s}'}(i)$ for each $\mathfrak{s}' \geq \mathfrak{s}\}$. Then $X \subseteq B$ and $\mathrm{age}(\mathfrak{A}_{\restriction X}) = \mathfrak{s}$.

Next, $\mathrm{age}(\mathfrak{A}_{\restriction B}) \subseteq \mathcal{A}$. Indeed, let $Y := \{(y_\mathfrak{s}^i)_{\mathfrak{s} \in \mathcal{A}} : 1 \leq i \leq n\}$ be a n-element subset of B. We claim that here is some $\mathfrak{s} \in \mathcal{A}$ such that the projection $p_\mathfrak{s}$ from B onto $S_\mathfrak{s}$ induces an isomorphism from $\mathfrak{A}_{\restriction Y}$ onto $\mathfrak{S}_{\mathfrak{s} \restriction \{y_\mathfrak{s}^i : 1 \leq i \leq n\}}$. Due to the choice of the ultrafilter, it is obvious that there is some $\mathfrak{s}$ such that $p_\mathfrak{s}$ preseves the unary relations. Now, let $I' := \mu^{-1}[\mathbb{N}^* \setminus \{1\}]$; since there are only

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finitely many relational structures of signature $\mu_{\upharpoonright I'}$ on an n-element set, we may find $\mathfrak{s}$ such that the other relations can be preserved. From this $\mathfrak{A}_{\upharpoonright Y} \in \mathcal{A}$.

Suppose that $\mu^{-1}[\mathbb{N}^* \backslash \{1\}]$ is infinite. According to Claim 1 of Lemma 12, $\mathfrak{J}(\Omega_{\underline{2}})$ can be mapped continuously into $\mathfrak{J}(\Omega_\mu)$ by a one-to-one map. According to Corollary 3, $\Omega_{\underline{2}}$ contains non-representable ideals. If $\mathcal{A}$ is a non-representable ideal of $\Omega_{\underline{2}}$ then, as it is easy to check, its image is a non-representable ideal of Ω_μ.

With this, the proof of Lemma 13 is complete. □

4 Proof of Theorem 2

Let $\underline{2} : \mathbb{N} \to \{2\}$ and let $\Omega^*_{(3)}$ be the class of relational structures containing a single ternary relation. To each relational structure $\mathfrak{A} := (A; \mathbf{R}) \in \Omega^*_{\underline{2}}$ such that $A \cap \mathbb{N} = \emptyset$, we associate $F(\mathfrak{A}) := (A \cup \mathbb{N}, T) \in \Omega^*_{(3)}$ such that $T := \tilde{X} \cup Y$, where $X := \{(x, y, z) \in \mathbb{N}^3 \backslash \{(1, 0, 1)\} : x + y = z\} \cup \{(1, 0, 0)\}$ and $Y := \{(x, y, z) \in A^2 \times \mathbb{N} : (x, y) \in \mathsf{R}^{\mathfrak{A}}_z\}$.

Claim 1. Let $\mathfrak{A} := (A; \mathbf{R})$, $\mathfrak{A}' := (A'; \mathbf{R}')$ be two binary relational structures as above, then:

(1) A map $f : A \to A'$ is an isomorphism from $\mathfrak{A}$ into $\mathfrak{A}'$ if and only if the map $F(f) := f \cup 1_{\mathbb{N}}$ extending f by the identity on $\mathbb{N}$ is an isomorphism from $F(\mathfrak{A})$ into $F(\mathfrak{A}')$.
(2) Every isomorphism $g : F(\mathfrak{A}) \to F(\mathfrak{A}')$ is of the form $F(f)$ for some isomorphism $f : \mathfrak{A} \to \mathfrak{A}'$.

Proof of Claim 1. Part (1) is straightforward to check.

Part (2). Let $g : F(\mathfrak{A}) \to F(\mathfrak{A}')$ be an isomorphism. First, g is the identity on $\mathbb{N}$. Indeed, as is is easy to see, each element of $\mathbb{N}$ is definable by an existential formula. To be precise, 0 is the unique element x of $A' \cup \mathbb{N}$ such that $(x, x, x) \in T'$, hence $g(0) = 0$. Also, $(0, y, z) \in T'$ implies that $y = z$ and $y \in \mathbb{N}$, from which follows that $g(\mathbb{N}) \subseteq \mathbb{N}$. Furthermore, $g(1) = 1$ since 1 is the only element $x \neq 0$ of A' such that $(x, 0, 0) \in T'$. Since $n + 1$ is the unique element x of A' such that $(n, 1, x) \in T'$, we have $g(n+1) = n+1$ for all n. Second, since g maps $\mathbb{N}$ onto $\mathbb{N}$, it maps A into A'. Clearly, for every $z \in \mathbb{N}$, we have $(x, y) \in \mathsf{R}^{\mathfrak{A}}_z$ if and only if $(g(x), g(y)) \in \mathsf{R}^{\mathfrak{A}'}_z$ that is $f := g_{\upharpoonright A}$ is an isomorphism from $\mathfrak{A}$ into $\mathfrak{A}'$, proving that $g = F(f)$. □

Let $\mathcal{C}$ be a set of relational structures, we denote by $\downarrow \mathcal{C}$ the set of isomorphism types of relational structures which embed into some member of $\mathcal{C}$.

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Claim 2. Let $\mathcal{C}$ be a subset of $\Omega_{\underline{2}}^{*}$ made of relational structures as above. Then $\downarrow\mathcal{C}$ is a representable ideal if and only if $\downarrow F[\mathcal{C}]$ is a representable ideal of $\Omega_{(3)}^{*}$.

Proof of Claim 2. Suppose that $\downarrow\mathcal{C}$ is representable. Let $\mathfrak{A} := (A; \mathbf{R})$ be a relational structure and $\mathfrak{J}$ be an ideal of subsets of A such that $\mathcal{A}_{\mathfrak{J}}(\mathfrak{A}) = \downarrow\mathcal{C}$. With no loss of generality, we may suppose $A \cap \mathbb{N} = \emptyset$. Let $\mathfrak{B} := F(\mathfrak{A})$ and $\mathfrak{K} := \{X \subseteq A \cup \mathbb{N} : X \cap A \in \mathfrak{J}\}$. From Part (1) of Claim 1 we obtain that $\mathcal{A}_{\mathfrak{K}}(\mathfrak{B}) = \downarrow F[\mathcal{C}]$ proving that $\downarrow F[\mathcal{C}]$ is representable.

Conversely, suppose that $\downarrow F[\mathcal{C}]$ is representable. Let $\mathfrak{B} := (B; T)$ be a relational structure consisting of a single ternary relation T and let $\mathfrak{K}$ be an ideal of subsets of B such that $\mathcal{A}_{\mathfrak{K}}(\mathfrak{B}) = \downarrow F[\mathcal{C}]$. For each $\mathfrak{A} \in \mathcal{C}$, there is some $X_{\mathfrak{A}} \in \mathfrak{K}$ and an isomorphism $g_{\mathfrak{A}}$ from $F(\mathfrak{A})$ onto $\mathfrak{B}_{\restriction X_{\mathfrak{A}}}$. We claim that $g_{\mathfrak{A}\restriction\mathbb{N}}$ is independent of $\mathfrak{A}$. To see that, take $\mathfrak{A}, \mathfrak{A}' \in \mathcal{C}$. Since $\mathfrak{K}$ is an ideal, it contains the union Y of the ranges of $g_{\mathfrak{A}}$ and $g_{\mathfrak{A}'}$ hence there is some $Y' \in \mathfrak{K}$ such that $\mathfrak{B}_{\restriction Y'}$ is isomorphic to a member of $F[\mathcal{C}]$ and there is some embedding h from $\mathfrak{B}_{\restriction Y}$ into $\mathfrak{B}_{\restriction Y'}$. According to Part (2) of Claim (1), $h \circ g_{\mathfrak{A}} = h \circ g_{\mathfrak{A}'}$ proving our claim. Identifying $\mathbb{N}' := g_{\mathfrak{A}}[\mathbb{N}]$ to $\mathbb{N}$ allows us to define a relational structure $\mathfrak{A}$ such that $\mathfrak{B} = F(\mathfrak{A})$. For $\mathfrak{J} := \{X \setminus \mathbb{N} : X \in \mathfrak{K}\}$ we have $\mathcal{A}_{\mathfrak{J}}(\mathfrak{A}) = \downarrow\mathcal{C}$ proving that $\downarrow\mathcal{C}$ is representable.

Taking for $\mathcal{C}$ a non-representable ideal of finite binary relational structures (as given by Corollary 3), we get an ideal of countable ternary relations which is not-representable. With this the proof of Theorem 2 is complete. $\qquad\square$

5   Conclusion

We just scratched the surface of Problem 1. We posed the question of the representability of ideals of metric spaces. Besides Problems 2, 6, and 9, we offer a very basic one:

Problem 14 *Does the representability of an ideal of Ω_μ depend only upon its order structure? In the most general form, the problem is this. Let $\mathcal{A}, \mathcal{A}'$ be two ideals of $\Omega_\mu, \Omega_{\mu'}$ which are order isomorphic; is $\mathcal{A}$ representable if and only if $\mathcal{A}'$ is representable?*

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