Fraïssé Limits, Ramsey Theory, and Topological Dynamics of Automorphism Groups

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0 Introduction

(A) We study in this paper some connections between the Fraïssé theory of amalgamation classes and ultrahomogeneous structures, Ramsey theory, and topological dynamics of automorphism groups of countable structures.

A prime concern of topological dynamics is the study of continuous actions of (Hausdorff) topological groups G on (Hausdorff) compact spaces X. These are usually referred to as (compact) *G-flows*. Of particular interest is the study of *minimal G-flows*, those for which every orbit is dense. Every G-flow contains a minimal subflow. A general result of topological dynamics asserts that every topological group G has a *universal minimal flow* $M(G)$, a minimal G-flow which can be homomorphically mapped onto any other minimal G-flow. Moreover, this is uniquely determined, by this property, up to isomorphism. (As usual a *homomorphism* $\pi : X \to Y$ between G-flows is a continuous G-map and an *isomorphism* is a bijective homomorphism.) For separable, metrizable groups G, which are the ones that we are interested in here, the universal minimal flow of G is an inverse limit of manageable, i.e., metrizable G-flows, but itself may be very complicated, for example non-metrizable. In fact, for the "simplest" infinite G, i.e., the countable discrete ones, $M(G)$ is a very complicated compact G-invariant subset of the space βG of ultrafilters on G and is always non-metrizable.

Rather remarkably, it turned out that there are topological groups G for which $M(G)$ is actually trivial, i.e., a singleton. This is equivalent to saying that G has a very strong fixed point property, namely every G-flow has a fixed point (i.e., a point x such that $g \cdot x = x$, $\forall g \in G$). (For separable, metrizable groups this is also equivalent to the fixed point property restricted to metrizable G-flows.) Such groups are said to have the *fixed point on compacta property* or be *extremely amenable.* The latter name comes from one of the standard characterizations of second countable locally compact amenable groups. A second countable locally compact group G is *amenable* iff every metrizable G-flow has an invariant (Borel probability) measure. However, no locally compact group can be extremely amenable, because, by a theorem of Veech [77], every such group admits a *free* G-flow (i.e., a flow for which $g \cdot x = x \Rightarrow g = 1_G$). This probably explains the rather late emergence of extreme amenability. The first examples of extremely amenable groups were constructed by Herer-Christensen [75]. They found Polish abelian so-called *pathological groups*, i.e., topological groups with no non-trivial unitary representations. Then they showed (see Theorem 4 in their paper) that every amenable pathological group is extremely amenable. Remarkably though it turned out that a lot of important (non-locally compact) Polish groups are indeed extremely amenable. Gromov-Milman [83] showed that the unitary group of infinite dimensional separable Hilbert space is extremely amenable, Furstenberg-Weiss and Glasner [98] showed that the group of measurable maps from $I = [0, 1]$ to the unit circle $\mathbb{T}$ is extremely amenable, Pestov [98a] (see also Pestov [98]) showed that the groups $H_+(I)$, $H_+(\mathbb{R})$ of orientation preserving homeomorphisms of $I, \mathbb{R}$, resp., are extremely amenable, and Pestov [98a] showed that the group $\mathrm{Aut}(\langle \mathbb{Q}, < \rangle)$ of automorphisms of the rationals is extremely amenable. More recently, Pestov [02] proved that the universal Polish group $\mathrm{Iso}(\boldsymbol{U})$, of all isometries of the Urysohn space $\boldsymbol{U}$, is extremely amenable, and Giordano-Pestov [02] showed that the groups $\mathrm{Aut}(I, \lambda)$ (resp., $\mathrm{Aut}^*(I, \lambda)$) of measure preserving automorphisms of I with Lebesgue measure λ (resp., measure-class preserving automorphisms of I, λ) is extremely amenable.

In most known examples of extremely amenable groups, beginning with the result by Gromov and Milman [83] on the unitary group, extreme amenability is established by using the phenomenon of concentration of measure on high-dimensional structures, see, e.g., Milman and Schechtman [86] or Ledoux [01]. However, we will not touch upon this subject here, referring the reader instead to the introductory article by Pestov [02b] and references therein.

Beyond the extremely amenable groups there were very few cases of metrizable universal minimal flows that had been computed. The first such example is in Pestov [98a], where the author shows that the universal minimal flow of $H_+(\mathbb{T})$, the group of orientation preserving homeomorphisms of the circle, has as a universal minimal flow its natural (evaluation) action on $\mathbb{T}$. Then Glasner-Weiss [02] showed that the universal minimal flow of S_∞, the infinite symmetric group of all permutations of $\mathbb{N}$, is its canonical action on the space of all linear orderings on $\mathbb{N}$. Finally, we have recently received a preprint of Glasner-Weiss [03], that shows that the universal minimal flow of $H(2^{\mathbb{N}})$, the group of homeomorphisms of the Cantor space, is its canonical action on the space of maximal chains of compact subsets of $2^{\mathbb{N}}$, a space introduced in Uspenskij [00].

(B) Motivated by Pestov's result that $\mathrm{Aut}(\langle \mathbb{Q}, < \rangle)$ is extremely amenable and the Glasner-Weiss computation of the universal minimal flow of S_∞, we develop in this paper a general framework, in which such results can be viewed as special instances. In particular, this gives many new examples of automorphism groups that are extremely amenable and calculations of universal minimal flows. There are two main ingredients that come into play here. The first is the Fraïssé theory of amalgamation classes and ultrahomogeneous structures, and the second is the structural Ramsey theory that arises in the works of Graham, Leeb, Rothchild, Nešetřil and Rödl. As repeatedly stressed by one of the present authors (see, e.g., Pestov [02b]), extreme amenability is related to Ramsey-type phenomena. For instance, Pestov's proof that $\mathrm{Aut}(\langle \mathbb{Q}, < \rangle)$ is extremely amenable depends on the classical finite Ramsey theorem and in fact it is equivalent to it. Generalizing this, we will see that, once things are put in the proper context, extreme amenability of automorphism groups and calculation of universal minimal flows turn out to have equivalent formulations in terms of concepts that have arisen in structural Ramsey theory.

(C) Let us first review some basic facts of the Fraïssé theory. A (countable) *signature* consists of a set of symbols $L = \{R_i\}_{i \in I} \cup \{f_j\}_{j \in J}$ (I, J countable), to each of which there is an associated *arity* $n(i) \in \{1, 2, \dots\}$ ($i \in I$) and $m(j) \in \mathbb{N}$ ($j \in J$). We call R_i the *relation symbols* and f_j the *function symbols* of L. A *structure* for L is of the form $\boldsymbol{A} = \langle A, \{R_i^{\boldsymbol{A}}\}_{i \in I},\ \{f_j^{\boldsymbol{A}}\}_{j \in J} \rangle$, where $A \neq \emptyset$, $R_i^{\boldsymbol{A}} \subseteq A^{n(i)}$, $f_j^{\boldsymbol{A}} : A^{m(j)} \to A$. The set A is called the *universe* of the structure. An *embedding* between structures $\boldsymbol{A}, \boldsymbol{B}$ for L is an injection $\pi : A \to B$ such that $R_i^{\boldsymbol{A}}(a_1, \dots, a_{n(i)}) \Leftrightarrow R_i^{\boldsymbol{B}}(\pi(a_1), \dots, \pi(a_{n(i)}))$ and $\pi(f_j^{\boldsymbol{A}}(a_1, \dots, a_{m(j)})) = f_j^{\boldsymbol{B}}(\pi(a_1), \dots, \pi(a_{m(j)}))$. If π is the identity, we say that $\mathbf{A}$ is a *substructure* of $\mathbf{B}$. An *isomorphism* is an onto embedding. We write $\boldsymbol{A} \leq \boldsymbol{B}$ if $\boldsymbol{A}$ can be embedded in $\boldsymbol{B}$ and $\boldsymbol{A} \cong \boldsymbol{B}$ if $\boldsymbol{A}$ is isomorphic to

$\boldsymbol{B}$.

A class $\mathcal{K}$ of finite structures for L is *hereditary* if $\boldsymbol{A} \leq \boldsymbol{B} \in \mathcal{K}$ implies $\boldsymbol{A} \in \mathcal{K}$. It satisfies the *joint embedding property* if for any $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{K}$ there is $\boldsymbol{C} \in \mathcal{K}$ with $\boldsymbol{A} \leq \boldsymbol{C},\ \boldsymbol{B} \leq \boldsymbol{C}$. Finally, it satisfies the *amalgamation property* if for any embeddings $f : \boldsymbol{A} \to \boldsymbol{B},\ g : \boldsymbol{A} \to \boldsymbol{C}$, with $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathcal{K}$, there is $\boldsymbol{D} \in \mathcal{K}$ and embeddings $r : \boldsymbol{B} \to \boldsymbol{D}$ and $s : \boldsymbol{C} \to \boldsymbol{D}$, such that $r \circ f = s \circ g$. We call $\mathcal{K}$ a *Fraïssé class* if it is hereditary, satisfies joint embedding and amalgamation, contains only countably many structures, up to isomorphism, and contains structures of arbitrarily large (finite) cardinality.

If now $\boldsymbol{A}$ is a countable structure, which is locally finite (i.e., finitely generated substructures are finite), its *age*, $\text{Age}(\boldsymbol{A})$, is the class of all finite structures which can be embedded in $\boldsymbol{A}$. We call $\boldsymbol{A}$ *ultrahomogeneous* if every isomorphism between finite substructures of $\boldsymbol{A}$ can be extended to an automorphism of $\boldsymbol{A}$. We call a locally finite, countably infinite, ultrahomogeneous structure a *Fraïssé structure*.

There is a canonical 1-1 correspondence between Fraïssé classes and structures, discovered by Fraïssé. If $\boldsymbol{A}$ is a Fraïssé structure, then $\text{Age}(\boldsymbol{A})$ is a Fraïssé class. Conversely, if $\mathcal{K}$ is a Fraïssé class, then there is a unique Fraïssé structure, the *Fraïssé limit of* $\mathcal{K}$, denoted by $\text{Flim}(\mathcal{K})$, whose age is exactly $\mathcal{K}$. Here are a couple of examples: the Fraïssé limit of the class of finite linear orderings is $\langle \mathbb{Q}, < \rangle$, and the Fraïssé limit of the class of finite graphs is the *random graph*.

(D) We now come to structural Ramsey theory. Let $\mathcal{K}$ be a hereditary class of finite structures in a signature L. For $\boldsymbol{A} \in \mathcal{K},\ \boldsymbol{B} \in \mathcal{K}$ with $\boldsymbol{A} \leq \boldsymbol{B}$, we denote by $\binom{\boldsymbol{B}}{\boldsymbol{A}}$ the set of all substructures of $\boldsymbol{B}$ isomorphic to $\boldsymbol{A}$. If $\boldsymbol{A} \leq \boldsymbol{B} \leq \boldsymbol{C}$ are in $\mathcal{K}$ and $n = 2, 3, \ldots$, we write

$$\boldsymbol{C} \to (\boldsymbol{B})^{\boldsymbol{A}}_n,$$

if for every coloring $c : \binom{\boldsymbol{C}}{\boldsymbol{A}} \to \{1, \ldots, n\}$, there is $\boldsymbol{B}' \in \binom{\boldsymbol{C}}{\boldsymbol{B}}$ which is homogeneous, i.e., $\binom{\boldsymbol{B}'}{\boldsymbol{A}}$ is monochromatic. We say that $\mathcal{K}$ satisfies the *Ramsey property* if for every $\boldsymbol{A} \leq \boldsymbol{B}$ in $\mathcal{K}$ and $n \geq 2$, there is $\boldsymbol{C} \in \mathcal{K}$ with $\boldsymbol{B} \leq \boldsymbol{C}$ such that $\boldsymbol{C} \to (\boldsymbol{B})^{\boldsymbol{A}}_n$. For example, the classical finite Ramsey theorem is equivalent to the statement that the class of finite linear orderings has the Ramsey property. Also Nešetřil and Rödl showed that the class of finite ordered graphs has the Ramsey property, and Graham-Leeb-Rothchild showed that the class of finite-dimensional vector spaces over a finite field has the Ramsey property.

(E) Consider now automorphism groups $\mathrm{Aut}(\boldsymbol{A})$ of countably infinite structures $\boldsymbol{A}$, for which we may as well assume that $A = \mathbb{N}$. Thus, with the pointwise convergence topology, $\mathrm{Aut}(\boldsymbol{A})$ is a closed subgroup of S_∞, the infinite symmetric group. Conversely, given a closed subgroup $G \leq S_\infty$, G is the automorphism group of some structure on $A = \mathbb{N}$ (in some signature).

Given a closed subgroup $G \leq S_\infty$ and $F \subseteq \mathbb{N}$ a finite nonempty set, we call the G-orbit $G \cdot F = \{g \cdot F : g \in G\}$, where $g \cdot F = \{g(i) : i \in F\}$, the *$G$-type* of F. A *G-type* σ is the G-type of some finite set. If ρ, σ are G-types, we write

$$\rho \leq \sigma \Leftrightarrow \exists F' \in \rho \exists F \in \sigma (F' \subseteq F).$$

If $\rho \leq \sigma,\ F \in \sigma$, put

$$\binom{F}{\rho} = \{F' \subseteq F : F' \in \rho\}.$$

Finally, if $\rho \leq \sigma \leq \tau$ are G-types, we put

$$\tau \to (\sigma)^\rho_n,$$

for $n = 2, 3, \dots$, if for every $F \in \tau$ and coloring $c : \binom{F}{\rho} \to \{1, \dots, n\}$, there is $F_0 \in \binom{F}{\sigma}$ which is homogeneous, i.e., $\binom{F_0}{\rho}$ is monochromatic. If for every $n = 2, 3, \dots$, and G-types $\rho \leq \sigma$, there is a G-type τ with $\sigma \leq \tau$ and $\tau \to (\sigma)^\rho_n$, we say that G has the *Ramsey property*. We also say that $G \leq S_\infty$ *preserves* an *ordering* if there is a linear ordering on $\mathbb{N}, \prec$, such that for all $g \in G$,

$$m \prec n \Leftrightarrow g(m) \prec g(n).$$

We now have:

Theorem 1. *Let $G \leq S_\infty$ be a closed subgroup. Then the following are equivalent:*

(i) G is extremely amenable.

(ii) (a) G preserves a linear ordering and (b) G has the Ramsey property.

Assume now L is a signature containing a distinguished binary relation symbol $<$. An *order structure* $\boldsymbol{A}$ for L is a structure $\mathbf{A}$ for which $<^{\boldsymbol{A}}$ is a linear ordering. An *order class* $\mathcal{K}$ for L is one for which all $\boldsymbol{A} \in \mathcal{K}$ are order structures.

Using Theorem 1 we now obtain:

Theorem 2. *The extremely amenable closed subgroups of* S_∞ *are exactly the groups of the form* $\mathrm{Aut}(\boldsymbol{F})$*, where* $\boldsymbol{F}$ *is the Fraïssé limit of a Fraïssé order class with the Ramsey property.*

Another way to formulate this result is the following:

Theorem 2′. *Let* $\mathcal{K}$ *be a Fraïssé order class and* $\boldsymbol{F}$ *its Fraïssé limit. Then the following are equivalent.*

(i) $\mathrm{Aut}(\boldsymbol{F})$ *is extremely amenable.*

(ii) $\mathcal{K}$ *has the Ramsey property.*

(F) We can now use this, and known results of structural Ramsey theory, to find many new examples of extremely amenable automorphism groups. Notice that, by the preceding result, the extreme amenability of these groups is in fact equivalent to the corresponding Ramsey theorem.

Consider the class of finite ordered graphs. Its Fraïssé limit is the random graph with an appropriate linear ordering. We call it the *random ordered graph*. Let K_n be the complete graph with n elements, $n = 3, 4, \ldots$. Consider the class of K_n-free finite ordered graphs, whose Fraïssé limit we call the *random K_n-free ordered graph*. Next consider the class of finite ordered graphs which are equivalence relations (i.e., can be written as a disjoint union of K_n's). Its Fraïssé limit is the rationals with the usual order and an equivalence relation with infinitely many classes, which are all dense in $\mathbb{Q}$. Finally consider the class of finite linear orderings. Its Fraïssé limit is $\langle \mathbb{Q}, < \rangle$.

All of the above classes satisfy the Ramsey property. This is due to Nešetřil-Rödl [77],[83] (see also Nešetřil [89] and [95]) for the graph cases, and it is of course the classical Finite Ramsey Theorem for the last case. So all the corresponding automorphism groups of their Fraïssé limits are extremely amenable.

This can be generalized to hypergraphs. Let $L_0 = \{R_i\}_{i \in I}$ be a finite relational signature. A *hypergraph of type* L_0 is a structure $\boldsymbol{A}_0 = \langle A_0, \{R_i^{\boldsymbol{A}_0}\}_{i \in I} \rangle$ in which $(a_1, \ldots a_{n(i)}) \in R_i^{\boldsymbol{A}} \Rightarrow a_1, \ldots, a_{n(i)}$ are distinct, and $R_i^{\boldsymbol{A}_0}$ is closed under permutations. Thus, essentially, $R_i^{\boldsymbol{A}_0} \subseteq [A_0]^{n(i)} =$ the set of subsets of

A_0 of cardinality $n(i)$. Consider the class of all finite ordered hypergraphs of type L_0, whose Fraïssé limit we call the *random ordered hypergraph of type* L_0. More generally, for every class $\mathcal{A}$ of finite irreducible hypergraphs of type L_0 (where $\boldsymbol{A}_0$ is *irreducible* if it has at least two elements and for every $x \neq y$ in A_0 there is $i \in I$ with $\{x, y\} \subseteq R_i^{\boldsymbol{A}_0}$), let $\mathcal{O}Forb(\mathcal{A})$ be the class of all finite ordered hypergraphs of type L_0 which omit $\mathcal{A}$ (i.e., no element of $\mathcal{A}$ can be embedded in them). We call the Fraïssé limit of $\mathcal{O}Forb(\mathcal{A})$ the *random* $\mathcal{A}$*-free ordered hypergraph of type* L_0. Again Nešetřil-Rödl [77], [83] showed that these classes have the Ramsey property, so the corresponding automorphism groups are extremely amenable.

There are similar results for metric spaces. Consider the class of finite ordered metric spaces with rational distances. Its Fraïssé limit is the so-called rational Urysohn space with an appropriate ordering. We call it the *ordered rational Urysohn space*. In response to an inquiry by the authors, Nešetřil [03] verified that the class of finite ordered metric spaces has the Ramsey property. Thus the automorphism group of the ordered rational Urysohn space is extremely amenable. We also show how this result can be used to give a new proof of the result of Pestov [02] that the isometry group of the Urysohn space is extremely amenable.

We next consider some other kinds of examples. We first look at the class of all finite *convexly ordered equivalence relations*, where convexly ordered means that each equivalence class is convex (whenever two elements are in it every element between them is also in it). Their Fraïssé limit is the rationals with the usual ordering and an equivalence relation whose classes are convex, order isomorphic to the rationals, and moreover the set of classes itself is ordered like the rationals. We show that the automorphism group of this structure is extremely amenable. This implies that the corresponding class has the Ramsey property, a fact that can also be proved directly.

Next we consider finite-dimensional vector spaces over a fixed finite field F. A *natural ordering* on such a vector space is one induced antilexicographically by an ordering of a basis. These were considered in Thomas [86], who showed that the class of naturally ordered finite-dimensional spaces over F is a Fraïssé class. We call its limit the $\aleph_0$*-dimensional vector space over* F *with the canonical ordering*. The Ramsey property for the class of naturally ordered finite-dimensional vector spaces over F is easily seen to be equivalent to the Ramsey property for the class of finite-dimensional vector spaces over F, which was established in Graham-Leeb-Rothchild [72]. So the corresponding automorphism group of the Fraïssé limit is extremely amenable.

Finally, we consider the class of naturally ordered finite Boolean algebras, where a *natural ordering* on a finite Boolean algebra is one antilexicographically induced by an ordering of its atoms. By analogy with Thomas' result, we show that this is also a Fraïssé class, and we call its limit the *countable atomless Boolean algebra with the canonical ordering*. The Ramsey property for the class of naturally ordered finite Boolean algebras is again easily seen to be equivalent to the Ramsey property for the class of finite Boolean algebras and this is trivially equivalent to the Dual Ramsey Theorem of Graham-Rothchild [71]. Thus the corresponding automorphism group is extremely amenable.

We summarize:

Theorem 3. *The automorphism groups of the following structures are extremely amenable:*

(i) The random ordered graph.

(ii) The random K_n-free ordered graph, $n = 3, 4, \ldots$

(iii) The rationals with the usual ordering and an equivalence relation with infinitely many classes, all of which are dense.

(iv) (Pestov [98a]) The rationals with the usual ordering.

(v) The random ordered hypergraph of type L_0 and more generally the random $\mathcal{A}$-free ordered hypergraph of type L_0, for any class $\mathcal{A}$ of irreducible finite hypergraphs of type L_0.

(vi) The ordered rational Urysohn space.

(vii) The rationals with the usual ordering and an equivalence relation whose classes are convex, ordered like the rationals, and moreover the set of classes itself is ordered like the rationals.

(viii) The $\aleph_0$-dimensional vector space over a finite field with the canonical ordering.

(ix) The countable atomless Boolean algebra with the canonical ordering.

(G) Finally we use the results in **(E)**, and some additional considerations, to compute universal minimal flows. In **(E)** we have seen a host of examples of Fraïssé order classes $\mathcal{K}$ in a signature $L \supseteq \{<\}$. Let $L_0 = L \setminus \{<\}$, the signature without the distinguished symbol for the ordering. For any structure $\boldsymbol{A}$ for L, we denote by $\boldsymbol{A}_0 = \boldsymbol{A}|L_0$ its *reduct* to L_0, i.e., $\boldsymbol{A}_0$ is the structure $\boldsymbol{A}$ with $<^{\boldsymbol{A}}$ dropped. Denote also by $\mathcal{K}_0 = \mathcal{K}|L$ the class of all reducts $\boldsymbol{A}_0 = \boldsymbol{A}|L_0$ for $\boldsymbol{A} \in \mathcal{K}$. When $\mathcal{K}$ satisfies a mild (and easily verified in every case we are interested in) condition, in which case we call $\mathcal{K}$

reasonable (see 5.1 below for the precise definition), then $\mathcal{K}_0$ is a Fraïssé class, whose limit is the reduct of the Fraïssé limit of $\mathcal{K}$. Put $\boldsymbol{F}_0 = \mathrm{Flim}(\mathcal{K}_0)$, $\boldsymbol{F} = \mathrm{Flim}(\mathcal{K})$, so that $\boldsymbol{F}_0 = \boldsymbol{F}|L_0$, i.e., $\boldsymbol{F} = \langle \boldsymbol{F}_0, <^{\boldsymbol{F}} \rangle$. In particular, $F_0 = F$. Put $<^{\boldsymbol{F}} = <_0$. It is natural now to look at the action of $\mathrm{Aut}(\boldsymbol{F}_0)$ on the space of all linear orderings on F_0. Denote then by $X_{\mathcal{K}}$ the orbit closure $\overline{G \cdot <_0}$ of $<_0$ in this action. It is easy to see that $X_{\mathcal{K}}$ is the space of all linear orderings $\prec$ on F_0 which have the property that for any finite substructure $\boldsymbol{B}_0$ of $\boldsymbol{F}_0$, $\boldsymbol{B} = \langle \boldsymbol{B}_0, \prec |B_0 \rangle \in \mathcal{K}$. We call these $\mathcal{K}$-*admissible* orderings. This is clearly a compact $\mathrm{Aut}(\boldsymbol{F}_0)$-invariant subset of $2^{F_0 \times F_0}$ in the natural action of $\mathrm{Aut}(\boldsymbol{F}_0)$ on $2^{F_0 \times F_0}$, so $X_{\mathcal{K}}$ is an $\mathrm{Aut}(\boldsymbol{F}_0)$-flow. If $\mathcal{K}$ has the Ramsey property, it turns out that it is the universal minimal flow of $\mathrm{Aut}(F_0)$ precisely when $\mathcal{K}$ additionally satisfies a natural property called the ordering property, which also plays an important role in structural Ramsey theory (see Nešetřil-Rödl [78] and Nešetřil [95]). We say that $\mathcal{K}$ satisfies the *ordering property* if for every $\boldsymbol{A}_0 \in \mathcal{K}_0$, there is $\boldsymbol{B}_0 \in \mathcal{K}_0$ such that for every linear ordering $\prec$ on A_0 and every linear ordering $\prec'$ on B_0, if $\boldsymbol{A} = \langle \boldsymbol{A}_0, \prec \rangle \in \mathcal{K}$ and $\boldsymbol{B} = \langle \boldsymbol{B}_0, \prec' \rangle \in \mathcal{K}$, then $\boldsymbol{A} \leq \boldsymbol{B}$. Then we have:

Theorem 4. *Let $L \supseteq \{<\}$ be a signature, $L_0 = L \setminus \{<\}$, $\mathcal{K}$ a reasonable Fraïssé order class in L, $\mathcal{K}_0 = \mathcal{K}|L_0$, $\boldsymbol{F} = \mathrm{Flim}(\mathcal{K})$, $\boldsymbol{F}_0 = \mathrm{Flim}(\mathcal{K}_0) = \boldsymbol{F}|L_0$. Put $G_0 = \mathrm{Aut}(\boldsymbol{F}_0)$. Let $X_{\mathcal{K}}$ be the G_0-flow of $\mathcal{K}$-admissible orderings on F_0 $(= F)$. Then the following are equivalent:*

(i) $X_{\mathcal{K}}$ is a minimal G_0-flow,

(ii) $\mathcal{K}$ satisfies the ordering property,

and when $\mathcal{K}$ satisfies the Ramsey property, these are also equivalent to:

(iii) $X_{\mathcal{K}}$ is the universal minimal G_0-flow.

Now all the classes $\mathcal{K}$, considered in **(E)** above, except for the class of finite ordered equivalence relations, satisfy the ordering property. This is due to Nešetřil-Rödl [78] for the case of graphs and hypergraphs, Nešetřil [03] for metric spaces, and is easily verified in all the other cases. Therefore, we have the following computations of universal minimal flows:

Theorem 5. *(i) Consider the following groups:*

(a) The automorphism group of the random graph.

(b) The automorphism group of the random $\mathcal{K}_n$-free graph, $n = 2, 3, \dots$

(c) S_∞ (the automorphism group of the structure $\langle \mathbb{N} \rangle$).

(d) The automorphism group of the random hypergraph of type L_0.

(e) The automorphism group of the random $\mathcal{A}$-free hypergraph of type L_0, where $\mathcal{A}$ is a class of irreducible finite hypergraphs of type L_0.

(f) The automorphism group of the rational Urysohn space.

Then their actions on the space of linear orderings on the universe of each structure is the universal minimal flow.

(ii) The universal minimal flow of the automorphism group of the equivalence relation on a countable set with infinitely many classes, each of which is infinite, is its action on the space of all linear orderings on that set for which each equivalence class is convex.

(iii) The universal minimal flow of the automorphism group $GL(\boldsymbol{V}_F)$ *of the* $\aleph_0$*-dimensional vector space* $\boldsymbol{V}_F$ *over a finite field* F*, is its action on the space of all orderings on* V_F*, whose restrictions to finite-dimensional subspaces are natural.*

(iv) The universal minimal flow of the automorphism group of the countable atomless Boolean algebra $\boldsymbol{B}_\infty$*, is its action on the space of all linear orderings on* B_∞*, whose restrictions to finite subalgebras are natural.*

In particular, in all these cases, the universal minimal flow is metrizable.

Of course (i), (c) is the result of Glasner-Weiss [02]. Very recently, Glasner-Weiss [03] computed the universal minimal flow of $H(2^{\mathbb{N}})$, the homeomorphism group of the Cantor space $2^{\mathbb{N}}$, as the space of maximal chains of compact subsets of $2^{\mathbb{N}}$, which is metrizable. Since the group in (iv) above is, by Stone duality, isomorphic to $H(2^{\mathbb{N}})$, we have another proof that the universal minimal flow is metrizable and a different description of this flow. Of course these two flows are isomorphic and in fact an explicit isomorphism can be found.

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1 Topological dynamics

(A) We will survey here some basic concepts and results of topological dynamics, which we will need in this paper. More detailed treatments can be found in Ellis [69], Auslander [88], de Vries [93], Pestov [98, 99], Glasner [00], Uspenskij [02].

Recall that an action $(g, x) \in G \times X \mapsto g \cdot x \in X$ of a topological group G on a topological space X is *continuous* if it is continuous as a map from $G \times X$ into X. We will consider continuous actions of (Hausdorff) topological groups G on (non-$\emptyset$) compact, Hausdorff spaces X. Actually we are primarily interested in metrizable topological groups G and in fact only separable metrizable ones. So although we will state in this survey several standard results for general topological groups, we will often give a sketch of an argument for the metrizable case only. It should be kept in mind that if G is metrizable (equivalently is Hausdorff and has a countable nbhd basis at the identity), then G admits a right-invariant compatible metric d_r, which of course can always be taken to be bounded by 1, by replacing it, if necessary, by $\frac{d_r}{1+d_r}$. See, e.g., Berberian [74, p. 28].

Let G be a topological group and X a compact, Hausdorff space. If we equip $H(X)$, the group of homeomorphisms of X, with the compact-open topology, i.e., the topology with subbasis $\{f \in H(X) : f(K) \subseteq V\}$, with $K \subseteq X$ compact, $V \subseteq X$ open, then $H(X)$ is a topological group, and a continuous action of G on X is simply a continuous homomorphism of G into $H(X)$. We will also refer to a continuous action of G on X as a *G-flow* on X. If the action is understood, we will often simply use X to refer to the flow.

Given a G-flow on X and a point $x \in X$, the *orbit* of x is the set

$$G \cdot x = \{g \cdot x : g \in G\}$$

and the *orbit closure* of x, the set

$$\overline{G \cdot x}.$$

This is a G-invariant, compact subset of X. In general, a (non-$\emptyset$) compact, G-invariant subset $Y \subseteq X$ defines a *subflow* by restricting the G-action to Y. A G-flow on X is *minimal* if it contains no proper subflows, i.e., there is no (non-$\emptyset$) compact G-invariant set other than X. Thus X is minimal iff every orbit is dense. A simple application of Zorn's Lemma shows that every G-flow X contains a minimal subflow $Y \subseteq X$.

Among minimal flows of a given group G, there is a largest (universal) one, called the *universal minimal flow*. To define this, we first need the concept of homomorphism of G-flows. Let X, Y be two G-flows. A *homomorphism* of the G-flow X to the G-flow Y is a continuous map $\pi : X \to Y$, which is also a G-*map*, i.e.,

$$\pi(g \cdot x) = g \cdot \pi(x), \quad x \in X, g \in G.$$

Notice that if Y is minimal, then any homomorphism of X into Y is surjective. An *isomorphism* of X to Y is a bijective homomorphism $\pi : X \to Y$ (notice then that π^{-1} is also a homomorphism). We now have the following basic fact in topological dynamics. (For a proof see Auslander [88, Ch. 8], or Uspenskij [02, §3].)

Theorem 1.1 *Given a topological group G, there is a minimal G-flow $M(G)$ with the following property: For any minimal G-flow X there is a homomorphism $\pi : M(G) \to X$. Moreover, $M(G)$ is uniquely determined up to isomorphism by this property.*

The space $M(G)$ is called the *universal minimal flow* of G. In order to get an intuition about this space, we will discuss a standard way of looking at it. For that we first need to discuss the concept of a *pointed G-flow* or *ambit*.

Let G be a topological group. A G-*ambit* is a G-flow X with a distinguished point $x_0 \in X$, whose orbit is dense in X. We often abbreviate this by (X, x_0). A *homomorphism of G-ambits* $(X, x_0), (Y, y_0)$ is a homomorphism $\pi : X \to Y$ of the G-flows such that $\pi(x_0) = y_0$. If such a homomorphism exists, it is clearly unique. Similarly we define the concept of isomorphism of G-ambits.

It is another basic fact of topological dynamics that there is again a largest (universal) G-ambit.

Theorem 1.2 *Given a topological group G, there is a G-ambit $(S(G), s_0)$ with the following property: For any G-ambit (X, x_0) there is a homomorphism of $(S(G), s_0)$ to (X, x_0). Moreover, $(S(G), s_0)$ is uniquely determined up to isomorphism by this property.*

The space $(S(G), s_0)$, often simply written as $S(G)$, is called the *greatest G-ambit*.

The uniqueness part of the preceding result is obvious from the definitions. To establish existence, we will describe a particularly useful way of constructing the greatest ambit.

Consider the space

$$RUC^b(G)$$

of bounded right-uniformly continuous functions $x : G \to \mathbb{C}$. Recall that $x : G \to \mathbb{C}$ is *right-uniformly continuous* if for each $\epsilon > 0$ there is a nbhd V of the identity 1_G of G so that

$$gh^{-1} \in V \Rightarrow |x(g) - x(h)| < \epsilon.$$

If G is metrizable, with right-invariant compatible metric d_r, then right-uniformly continuous means uniformly continuous with respect to d_r:

$$\forall \epsilon \exists \delta (d_r(g,h) < \delta \Rightarrow |x(g) - x(h)| < \epsilon).$$

Under pointwise addition, multiplication and conjugation, and with the sup norm $\|x\|_\infty = \sup\{|x(g)| : g \in G\}$, $RUC^b(G)$ is an abelian C^*-algebra which is unital (with multiplicative identity the constant 1 function). Denote by $S(G)$ the maximal ideal space of $RUC^b(G)$, i.e., the space of all continuous homomorphisms $\varphi : RUC^b(G) \to \mathbb{C}$. Equipped with the topology generated by the maps $\hat{x} : S(G) \to \mathbb{C}, \hat{x}(\varphi) = \varphi(x)$, for $x \in RUC^b(G)$ (i.e., the smallest topology in which all $\hat{x}, x \in RUC^b(G)$, are continuous), this is a compact, Hausdorff space. Moreover, by the Gelfand-Naimark theorem, $RUC^b(G)$ can be canonically identified with $C(S(G))$, the C^*-algebra of all continuous complex-valued functions on $S(G)$, identifying $x \in RUC^b(G)$ with $\hat{x}$ (see, e.g., Rudin [73, 11.18]).

Now G acts continously by C^*-automorphisms on $RUC^b(G)$ by left-shift

$$g \cdot x(h) = x(g^{-1}h)$$

and thus acts canonically on $S(G)$ via

$$g \cdot \varphi(x) = \varphi(g^{-1} \cdot x).$$

It is also easy to check that this action is continuous, so $S(G)$ is a G-flow. We will now identify a canonical element of $S(G)$, that will turn it into an ambit.

For each $g \in G$, let $\varphi_g \in S(G)$ be defined by $\varphi_g(x) = x(g), x \in RUC^b(G)$. Then one can see that $g \mapsto \varphi_g$ is a homeomorphism of G with a dense subset of $S(G)$. For example, when G is metrizable with bounded compatible right-invariant metric d_r, and $g_0 \neq h_0$, then for $x(g) = d_r(g, h_0), \varphi_{g_0}(x) \neq \varphi_{h_0}(x)$ so $\varphi_{g_0} \neq \varphi_{h_0}$, i.e., this map is 1-1. The verification that it is homeomorphism is straightforward. Finally $\{\varphi_g : g \in G\}$ is dense in $S(G)$, since, otherwise, there is $f \in C(S(G))$, so that $f = \hat{x}$ for some $x \in RUC^b(G)$, with $f \neq 0$ but

$f(\varphi_g) = \hat{x}(\varphi_g) = x(g) = 0, \forall g \in G$, which implies that $x = 0$, thus $f = 0$, a contradiction.

So from now on we will identify g with φ_g and think of G as a dense subset of $S(G)$. Moreover G is an invariant subset of $S(G)$ and the restriction of the action to G is simply left-translation: $(g, h) \mapsto gh$. We now have the following standard fact.

Theorem 1.3 *The G-ambit $(S(G), 1_G)$ is the greatest G-ambit.*

Proof. Since the orbit of 1_G in $S(G)$ is G, which is dense, clearly $(S(G), 1_G)$ is a G-ambit. Consider now an arbitrary G-ambit (X, x_0). Suppose $f \in C(X)$. Define then $f^* : G \to \mathbb{C}$ by

$$f^*(g) = f(g \cdot x_0).$$

We verify that $f^* \in RUC^b(G)$. Since the action of G on X is continuous, an easy compactness argument shows that given $\epsilon > 0$, there is a nbhd V of the identity of G such that $g \in V$ implies $|f(g \cdot x) - f(x)| < \epsilon, \forall x \in X$. So if $gh^{-1} \in V$, then $|f^*(g) - f^*(h)| = |f(g \cdot x_0) - f(h \cdot x_0)| = |f(gh^{-1} \cdot (h \cdot x_0)) - f(h \cdot x_0)| < \epsilon$, and thus $f^* \in RUC^b(G)$ (f^* is clearly bounded).

Identifying, as usual, $RUC^b(G)$ with $C(S(G))$, the map $f \mapsto f^*$ is a unital C^*-algebra monomorphism of $C(X)$ into $C(S(G))$. Now it is a well known fact that a unital C^*-algebra monomorphism $\pi : C(K) \to C(L)$, where K, L are (non-$\emptyset$) compact spaces, is of the form $\pi(f) = f \circ \Pi$ for a *uniquely* determined continuous surjection $\Pi : L \to K$ (see, e.g., Constantinescu [01, 2.4.3.6]). From this it also follows that if K, L are actually G-flows, and we let G act on $C(K), C(L)$ by shift, $g \cdot f(x) = f(g^{-1} \cdot x)$, then if π is a G-map, Π is also a G-map. Applying this to $f \mapsto f^*$, we see that there is a homomorphism of G-flows $\Phi : S(G) \to X$ with $f^* = f \circ \Pi$. It only remains to check that $\Pi(1_G) = x_0$. But for any $f \in C(X)$, $f^*(1_G) = f(x_0) = f(\Pi(1_G))$, so we must have $\Pi(1_G) = x_0$, and the proof is complete. $\dashv$

¿From this it is now immediate to obtain the following description of the universal minimal flow of G.

Corollary 1.4 *Let $M(G)$ be a minimal subflow of $S(G)$ (i.e., $M(G)$ is a minimal G-invariant compact subset of $S(G)$). Then $M(G)$ is the universal minimal flow (up to isomorphism).*

Proof. Let X by any minimal G-flow. Fix $x_0 \in X$. Then (X, x_0) is a G-ambit, so let $\pi : (S(G), 1_G) \to (X, x_0)$ be a homomorphism. Then clearly the restriction of π to $M(G)$ is also a homomorphism, and we are done. $\dashv$

In particular, it follows from the uniqueness part of Theorem 1.1, that all minimal subflows of $S(G)$ are isomorphic. (This uniqueness part, which is not proved here, is based on techniques from semigroup theory.) As we will soon see, the space $M(G)$ can be extremely complicated, e.g., non-metrizable, even when the group G is very "small", e.g., a countable discrete G. However, we will verify that when G is separable, metrizable, $M(G)$ is at least an inverse limit of metrizable G-flows.

Fix a topological group G. An *inverse system* of G-flows consists of a directed set $\langle I, \preceq \rangle$, a family $\{X_i\}_{i \in I}$ of G-flows and a family of homomorphisms $\pi_{ij} : X_j \to X_i$, for each $i \preceq j$, such that $\pi_{ii} =$ the identity of X_i, and $i \preceq j \preceq k \Rightarrow \pi_{ik} = \pi_{ij} \circ \pi_{jk}$. The *inverse limit* $\lim_{\leftarrow} X_i$ is the G-flow defined as follows: Consider the product topological space $\prod_{i \in I} X_i$ and let

$$\lim_{\leftarrow} X_i = \{\{x_i\}_i \in \prod_i X_i : \forall i \preceq j (\pi_{ij}(x_j) = x_i)\}.$$

By a simple application of compactness, $\lim_{\leftarrow} X_i \neq \emptyset$ and is clearly a compact subset of $\prod_{i \in I} X_i$. The group G acts on $\lim_{\leftarrow} X_i$ coordinatewise: $g \cdot \{x_i\} = \{g \cdot x_i\}$ and this is clearly a continuous action. Define

$$\pi_i : \lim_{\leftarrow} X_i \to X_i$$

by $\pi_i(\{x_i\}_{i \in I}) = x_i$. Then π_i is a homomorphism and if $i \preceq j$, then $\pi_i = \pi_{ij} \circ \pi_j$. Finally, if X is a G-flow and there are homomorphisms $\varphi_i : X \to X_i$ with $i \preceq j \Rightarrow \varphi_i = \varphi_{ij} \circ \varphi_j$, then there is a unique homomorphism $\varphi : X \to \lim_{\leftarrow} X_i$ such that $\varphi_i = \pi_i \circ \varphi$.

Similarly we define inverse systems of G-ambits. We now have the following folklore fact.

Theorem 1.5 *Let G be a separable, metrizable group. Then the greatest ambit $(S(G), 1_G)$ is the inverse limit of a system of metrizable G-ambits. Similarly, the universal minimal flow is the inverse limit of a system of metrizable minimal G-flows.*

Proof. We will first derive the second assertion from the first. Suppose $\{(X_i, x_i^0)\}$ is an inverse system of metrizable G-ambits, so that $(X, x_0) = \lim_{\leftarrow}(X_i, x_i^0)$ is the greatest G-ambit. Let $\pi_i : X \to X_i$ be the corresponding homomorphism. In particular, $X = \lim_{\leftarrow} X_i$ as a G-flow. Now fix a minimal subflow $M \subseteq X$, so that, as we have seen earlier, M is the universal minimal flow. Put $\pi_i(M) = M_i \subseteq X_i$. Then M_i is a subflow of X_i and it is clearly minimal and metrizable. So it is enough to check that $M = \lim_{\leftarrow} M_i$, which is an easy compactness argument.

For the first assertion, fix a countable dense set $D \subseteq G$. Let $(I, \preceq)$ be the following directed set: I consists of all separable, closed, unital G-invariant C^*-subalgebras of $RUC^b(G)$. Since a closed, unital C^*-subalgebra of $RUC^b(G)$ is G-invariant iff it is D-invariant, clearly $\bigcup I = RUC^b(G)$. Let for $A, B \in I$

$$A \preceq B \Leftrightarrow A \subseteq B.$$

For $A \in I$ denote by X_A the maximal ideal space of A, which is compact metrizable, since A is separable. Since G acts continuously by C^*-automorphisms on A, it acts continuously on X_A. Moreover, as before, we can identify each $g \in G$ with an element of X_A, so that G can be thought as a dense invariant subset of X_A and the G action on it is by left-translation. Thus again $(X_A, 1_G)$ is a metrizable G-ambit. We view as usual A as identified with $C(X_A)$ via $x \mapsto \hat{x}_A$. When $A \preceq B$, the identity is an injective unital C^*-homomorphism from A to B, so there is a unique surjection $\pi_{AB} : X_B \to X_A$ such that for $x \in A$, $\hat{x}_B = \hat{x}_A \circ \pi_{AB}$, therefore for $\varphi \in X_B, \hat{x}_A(\pi_{AB}(\varphi)) = \hat{x}_B(\varphi)$ or $\pi_{AB}(\varphi)(x) = \varphi(x)$, i.e., $\pi_{AB}(\varphi) = \varphi|A$. Similarly, there is a surjection $\pi_A : S(G) \to X_A$ given by $\pi_A(\varphi) = \varphi|A$ for each $A \in I$, so that $\pi_A = \pi_{AB} \circ \pi_B$ for $A \preceq B$. Moreover, $\pi_A(1_G) = \pi_{AB}(1_G) = 1_G$. Thus $\varphi \mapsto (\pi_A(\varphi))_{A \in I}$ is a homomorphism from $(S(G), 1_G)$ to $\lim_{\leftarrow}(X_A, 1_G)$. Also given $(\varphi_A) \in \lim_{\leftarrow} X_A$, we have, for $A \preceq B$, that $\varphi_A = \varphi_B|A$, so that there is a unique $\varphi \in S(G)$ with $\pi_A(\varphi) = \varphi_A$. Thus $\varphi \mapsto (\pi_A(\varphi))_{A \in I}$ is an isomorphism of the G-ambit $(S(G), 1_G)$ and $\lim_{\leftarrow}(X_A, 1_G)$. ⊣

(B) We will now discuss the case of infinite countable discrete groups G and see that in this case $M(G)$ is an extremely large space, in particular it is not metrizable.

For an infinite countable discrete G, it is clear that $RUC^b(G)$ is identical with $\ell^\infty(G)$, the C^*-algebra of bounded complex functions on G with the supremum norm. It is then easy to see that $S(G)$ is identical with βG, the space of ultrafilters on G with the topology whose basis consists of the sets $\hat{A} = \{U \in \beta G : A \in U\}$, for $\emptyset \neq A \subseteq G$. This is a non-metrizable, compact, Hausdorff space. The action of G on $S(G)$ is given by

$$A \in g \cdot U \Leftrightarrow g^{-1}A \in U$$

for $A \subseteq G$.

The copy of G in $S(G)$ consists simply of the principal ultrafilters. So the distinguished point is the principal ultrafilter on 1_G. Finally, $M(G)$ is any minimal subflow of βG.

First we point out that G acts *freely* on $S(G)$, i.e., $g \cdot x \neq x,\ \forall g \neq 1_G,\ x \in S(G)$ (this is due to Ellis [60]). For that it is of course enough to find a free G-flow X (since $S(G)$ can be homomorphically mapped to X). Again it is enough to find for each $g \in G,\ g \neq 1_G$, a G-flow X_g, such that $g \cdot x \neq x,\ \forall x \in X_g$. Because then $X = \prod_{g \in G \setminus \{1_G\}} X_g$ with the action $h \cdot (x_g) = (h \cdot x_g)$ works. Consider the shift action of G on $3^G,\ g \cdot p(h) = p(g^{-1}h)$. Suppose we can find $p_g \in 3^G$ such that $(*) : \forall h \in G(g \cdot (h \cdot p_g)(1_G) \neq (h \cdot p_g)(1_G))$. Then we can take X_g to be the orbit closure of p_g. Now $(*)$ is equivalent to: $\forall h \in G(p_g(h^{-1}g^{-1}) \neq p_g(h^{-1}))$ or $\forall h(p_g(h) \neq p_g(hg))$. Consider a coset $h\langle g\rangle$ of the cyclic subgroup $\langle g\rangle$. Define p_g on $h\langle g\rangle$, so that $p_g(hg^k) \neq p_g(hg^{k+1})$, for any $k \in \mathbb{Z}$. (We used 3^G instead of 2^G to take care of the case when g has finite order.) This clearly works.

The space $M(G)$ is quite big, see e.g., the references in de Vries [93, p. 391, **11**]. Let us verify for instance that it is not metrizable. The space $M(G)$ is a closed subset of βG. If it was metrizable and infinite, it would have a non-trivial convergent sequence, which is impossible in the extremally disconnected space βG (see Engelking [77], Ex. 6.2.G(a) on p. 456). So if $M(G)$ is metrizable, it has to be finite, contradicting the fact that G acts freely on $M(G)$.

(C) More generally, Veech [77] has shown that when G is locally compact, then G acts freely on $S(G)$ and thus on $M(G)$. For a simpler version of the proof see Pym [99]. See also Adams-Stuck [93] for the second countable case. In an Appendix to this paper, we also give a new proof of Veech's Theorem. Note that Veech's Theorem implies that if G is second countable, then G admits a free metrizable G-flow. To see this, notice that it is enough to find for each $1_G \neq g \in G$ a metrizable G-flow X_g with $g \cdot x \neq x$, if $x \in X_g$. Indeed, if we can do that, then, by compactness, there is an open nbhd V_g of g with $1_G \notin V_g$ and $h \in V_g \Rightarrow (h \cdot x \neq x,\ \forall x \in X_g)$. Find now $g_0, g_1, \dots \in G \setminus \{1_G\}$ so that $\{V_{g_n}\}_{n \in \mathbb{N}}$ is an open cover of $G \setminus \{1_G\}$. Then $X = \prod_n X_{g_n}$ with the coordinatewise action is a free metrizable G-flow. (This argument comes from Adams-Stuck [93].) So fix $g \in G,\ g \in 1_G$, in order to find X_g. Write $S(G) = \lim_{\leftarrow} X_i,\ X_i$ a metrizable G-flow. If none of the X_i can be $X_g,\ \{x_i \in X_i : g \cdot x_i = x_i\} = Y_i$ is non-$\emptyset$, and if $\pi_{ij} : X_j \to X_i$, for $i \preceq j$, are the corresponding homomorphisms, then $\pi_{ij}(Y_j) \subseteq Y_i$, so the inverse limit of $(Y_i, \pi_{ij}|Y_j)$ is non-$\emptyset$ and thus there is $\{y_i\} \in S(G)$ with $g \cdot \{y_i\} = \{y_i\}$, a contradiction, as G acts freely on $S(G)$.

We also show in an Appendix that when G is non-compact, locally compact, $M(G)$ is non-metrizable. Stronger results in special cases were obtained in Turek [95], Lau-Milner-Pym [99]. Of course when G is compact, $M(G)$ is G itself with the left-translation action.

(D) Rather remarkably, there are groups G for which $M(G)$ trivializes, i.e., consists of a single point. Such groups are called extremely amenable. Thus a topological group is *extremely amenable* if any G-flow X has a fixed point, i.e., there is $x \in X$ with $g \cdot x = x$, $\forall g \in G$. (For this reason, sometimes extremely amenable groups are described as groups having the *fixed point on compacta property.*) By Veech's Theorem such groups cannot be locally compact. As it turned out, a number of important, non-locally compact Polish groups are extremely amenable. Among them are: the unitary group $U(H)$ of the infinite dimensional separable Hilbert space H (Gromov-Milman [83]); $L(I, \mathbb{T})$, the group of measurable maps from $I = [0, 1]$ to $\mathbb{T}$, with pointwise multiplication, and the topology of convergence in measure (Furstenberg-Weiss, Glasner [98]); $H_+(I)$ and $H_+(\mathbb{R})$, the groups of orientation preserving homeomorphisms of I and $\mathbb{R}$, with the compact-open topology (Pestov [98a]); $\mathrm{Aut}(I, \lambda)$ (resp., $\mathrm{Aut}^*(I, \lambda)$), the groups of measure preserving (resp., measure-class preserving) automorphisms of Lebesgue measure λ on I, with the weak topology (Giordano-Pestov [02]); $\mathrm{Iso}(\mathbf{U})$, the group of isometries of the Urysohn space, with the pointwise convergence topology (Pestov [02])), and $\mathrm{Aut}(\langle \mathbb{Q}, < \rangle)$, the group of automorphisms of the rationals with the usual ordering, with the pointwise convergence topology (Pestov [98a]). For more about extreme amenability, see also Pestov [99],[02a],[02b] and Uspenskij [02].

In case the group G is separable metrizable, we can restate the definition of extreme amenability in terms of metrizable flows only. In other words, a separable metrizable group G is extremely amenable iff every metrizable G-flow has a fixed point. Indeed, if every metrizable G-flow has a fixed point, every minimal metrizable G-flow is a singleton, and thus so is $M(G)$, being an inverse limit of such G-flows.

(E) Except for the case of extremely amenable groups, there were very few cases where the universal minimal flow $M(G)$ was known to be metrizable. Pestov [98a] first computed that for the group $H_+(\mathbb{T})$ of orientation-preserving homeomorphisms of the circle $\mathbb{T}$, the canonical evaluation action on $\mathbb{T}$ is the universal minimal flow. Then Glasner-Weiss [02] computed the universal minimal flow of S_∞, the group of permutations of $\mathbb{N}$ with the pointwise convergence topology. It turns out to be the canonical action on the compact, metrizable space of linear orderings on $\mathbb{N}$. Finally, we have recently received a preprint of Glasner-Weiss [03], which computes the universal minimal flow of the group $H(2^{\mathbb{N}})$ of the homeomorphisms of the Cantor space $2^{\mathbb{N}}$. It is the action of $H(2^{\mathbb{N}})$ on the space of maximal chains of compact subsets of $2^{\mathbb{N}}$, invented by Uspenskij [00].

Our goal in this paper is to study extreme amenability and universal minimal flows of closed subgroups of S_∞, i.e., automorphism groups of countable structures. In particular, we find new examples of extremely amenable groups and also new cases, where the universal minimal flow is metrizable and can be computed.

2 Fraïssé Theory

We will review here some basic ideas of model theory concerning the Fraïssé construction and ultrahomogeneous countable structures. Our main reference here is Hodges [93, Ch. 7]. See also Cherlin [98] and Cameron [90].

A (countable) *signature* is a countable collection $L = \{R_i\}_{i\in I} \cup \{f_j\}_{j\in J}$ of (distinct) *relation* and *function symbols* each of which has an associated number, called its *arity*. The arity $n(i)$ of each relation symbol R_i is a positive integer and the arity $m(j)$ of each function symbol f_j is a non-negative integer. A structure for L is an object of the form

$$\boldsymbol{A} = \langle A, \{R_i^{\boldsymbol{A}}\}_{i\in I}, \{f_j^{\boldsymbol{A}}\}_{j\in J}\rangle,$$

where A is a *non-empty* set, called the *universe* of $\boldsymbol{A}$, $R_i^{\boldsymbol{A}} \subseteq A^{n(i)}$, i.e., $R_i^{\boldsymbol{A}}$ is a $n(i)$-ary relation on A, and $f_j : A^{m(j)} \to A$, i.e., $f_j^{\boldsymbol{A}}$ is an m_j-ary function on A. (When $m(j) = 0$, $f_j^{\boldsymbol{A}}$ is a distinguished element of A.)

Given two structures $\boldsymbol{A}, \boldsymbol{B}$ of the same signature L, a *homomorphism* of $\boldsymbol{A}$ to $\boldsymbol{B}$ is a map $\pi : A \to B$ such that

$$R_i^{\boldsymbol{A}}(a_1, \ldots, a_{n(i)}) \Leftrightarrow R_i^{\boldsymbol{B}}(\pi(a_1), \ldots, \pi(a_{n(i)}))$$

and

$$\pi(f_j^{\boldsymbol{A}}(a_1, \ldots, a_{m(j)})) = f_j^{\boldsymbol{B}}(\pi(a_1), \ldots, \pi(a_{m(j)})).$$

We write also in this case $\pi : \boldsymbol{A} \to \boldsymbol{B}$. (*Caution.* Sometimes in the definition of homomorphism, one only requires the left-to-right implication concerning $R_i^{\boldsymbol{A}}, R_i^{\boldsymbol{B}}$. We will use here only the stronger version above.) If π is also 1-1, it is called a *monomorphism* or *embedding*. Finally, if π is 1-1 and onto it is called an *isomorphism*. If there is an isomorphism from $\boldsymbol{A}$ to $\boldsymbol{B}$, we say that $\boldsymbol{A}, \boldsymbol{B}$ are *isomorphic*, in symbols $\boldsymbol{A} \cong \boldsymbol{B}$. An *automorphism* of $\boldsymbol{A}$ is an isomorphism of $\boldsymbol{A}$ to itself. We denote by $\mathrm{Aut}(\boldsymbol{A})$ the group of automorphisms of $\boldsymbol{A}$.

We will be primarily interested in countable structures $\boldsymbol{A}$ (i.e., the universe A is countable). For countable $\boldsymbol{A}$, the group $\mathrm{Aut}(\boldsymbol{A})$, with the pointwise convergence topology, is Polish, in fact it is a closed subgroup of S_A, the Polish group of permutations of A with the pointwise convergence topology. Conversely, given a closed subgroup $G \subseteq S_A$, there is a signature L and a structure $\boldsymbol{A}_G$ with universe A, so that $\mathrm{Aut}(\boldsymbol{A}_G) = G$. To see this, let for each $n \geq 1, \mathcal{O}_1^n, \mathcal{O}_2^n, \dots$ be the orbits of G on A^n, the action of G on A^n being defined by $g \cdot (a_1, \dots, a_n) = (g(a_1), \dots, g(a_n))$. Let $L = \{R_{n,i}\}_{n \geq 1}$, where each $R_{n,i}$ is an n-ary relation symbol. Define $\boldsymbol{A}_G$ by letting $R_{n,i}^{\boldsymbol{A}_G} = \mathcal{O}_i^n \subseteq A^n$. Then it is easy to check that $\mathrm{Aut}(\boldsymbol{A}_G) = G$. We call $\boldsymbol{A}_G$ the *induced structure* associated to G (see Hodges [93, 4.1.4], where this is called the canonical structure for G - we use however the term canonical for other purposes in this paper).

A substructure $\boldsymbol{B}$ of $\boldsymbol{A}$ has as universe a (non-empty) subset $B \subseteq A$ closed under each $f_j^{\boldsymbol{A}}$, and $R_i^{\boldsymbol{B}} = R_i^{\boldsymbol{A}} \cap B^{n(i)}$, $f_j^{\boldsymbol{B}} = f_j^{\boldsymbol{A}} | B^{m(j)}$. For each $X \subseteq A$, there is a smallest substructure containing X, called the *substructure generated* by X. A substructure is *finitely generated* it is generated by a finite set. A structure is *locally finite* if all its finitely generated substructures are finite. For example, if L is *relational*, i.e., $J = \emptyset$, the substructure generated by X has universe X and so every finitely generated substructure is finite. This is also true if J is finite and each f_j has arity 0.

A structure $\boldsymbol{A}$ is called *ultrahomogeneous* if every isomorphism between finitely generated substructures $\boldsymbol{B}, \boldsymbol{C}$ of $\boldsymbol{A}$ can be extended to an automorphism of $\boldsymbol{A}$. For example, $\langle \mathbb{Q}, < \rangle$, the rationals with the usual order, is an ultrahomogeneous structure. Fraïssé's theory provides a general analysis of ultrahomogeneous countable structures.

Let $\boldsymbol{A}$ be a structure for L. The *age* of A, $\mathrm{Age}(\boldsymbol{A})$ is the collection of all finitely generated structures in L that can be embedded in $\boldsymbol{A}$, i.e., the closure under isomorphism of the collection of finitely generated substructures of $\boldsymbol{A}$. Clearly the class $\mathcal{K} = \mathrm{Age}(\mathcal{A})$ is non-empty, and satisfies the following two properties:

(i) *Hereditary property (HP)*: If $\boldsymbol{B} \in \mathcal{K}$ and $\boldsymbol{C}$ is a finitely generated structure that can be embedded in $\boldsymbol{B}$, then $\boldsymbol{C} \in \mathcal{K}$.

(ii) *Joint embedding property (JEP)*: If $\boldsymbol{B}, \boldsymbol{C} \in \mathcal{K}$, there is $\boldsymbol{D} \in \mathcal{K}$ such that $\boldsymbol{B}, \boldsymbol{C}$ can be embedded in $\boldsymbol{D}$.

When $\boldsymbol{A}$ is moreover ultrahomogeneous, it is easy to see that $\mathcal{K} = \mathrm{Age}(\boldsymbol{A})$ satisfies also the following crucial property:

(iii) *Amalgamation property (AP)*: If $\boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D} \in \mathcal{K}$ and $f : \boldsymbol{B} \to \boldsymbol{C}, g : \boldsymbol{B} \to \boldsymbol{D}$ are embeddings, then there is $\boldsymbol{E} \in \mathcal{K}$ and embeddings $r : \boldsymbol{C} \to \boldsymbol{E}, s : \boldsymbol{D} \to \boldsymbol{E}$ such that $r \circ f = s \circ g$.

We summarize:

Proposition 2.1 *Let $\boldsymbol{A}$ be an ultrahomogeneous structure. Then $\mathcal{K} = \mathrm{Age}(\boldsymbol{A})$ is non-empty, and satisfies HP, JEP and AP.*

If $\boldsymbol{A}$ is countable, then clearly $\mathrm{Age}(\boldsymbol{A})$ contains only countably many isomorphism types. Abusing language, we say that a class $\mathcal{K}$ of structures is *countable* if it contains only countably many isomorphic types.

We now have the following main result of Fraïssé [54].

Theorem 2.2 (Fraïssé) *Let L be a signature and $\mathcal{K}$ a class of finitely generated structures for L, which is non-empty, countable, and satisfies HP, JEP and AP. Then there is a unique, up to isomorphism, countable structure $\boldsymbol{A}$ such that $\boldsymbol{A}$ is ultrahomogeneous and $\mathcal{K} = \mathrm{Age}(\boldsymbol{A})$.*

We call this structure the *Fraïssé limit* of $\mathcal{K}$,

$$\boldsymbol{A} = \mathrm{Flim}(\mathcal{K}).$$

Thus a countable ultrahomogeneous structure is the Fraïssé limit of its age. For example, $\mathrm{Age}(\langle \mathbb{Q}, < \rangle) =$ the class of finite linear orderings $= \mathcal{L}O$ and so $\langle \mathbb{Q}, < \rangle = \mathrm{Flim}(\mathcal{L}O)$.

We note here the following alternative characterization of ultrahomogeneity.

Proposition 2.3 *Let $\boldsymbol{A}$ be a countable structure. Then $\boldsymbol{A}$ is ultrahomogeneous iff it satisfies the following extension property:*

If $\boldsymbol{B}, \boldsymbol{C}$ are finitely generated, $f : \boldsymbol{B} \to \boldsymbol{A}, g : \boldsymbol{B} \to \boldsymbol{C}$ are embeddings, then there is $h : \boldsymbol{C} \to \boldsymbol{A}$ such that $h \circ g = f$.

It follows that if $\boldsymbol{A}$ is countable, $\boldsymbol{B}$ is countable ultrahomogeneous with $\mathrm{Age}(\boldsymbol{A}) \subseteq \mathrm{Age}(\boldsymbol{B})$ and $\boldsymbol{C} \subseteq \boldsymbol{A}$ is finitely generated, then every embedding $f : \boldsymbol{C} \to \boldsymbol{B}$ can be extended to an embedding $g : \boldsymbol{A} \to \boldsymbol{B}$.

In particular, $\boldsymbol{B}$ is *universal* for the class of all countable structures $\boldsymbol{A}$ whose age is contained in that of $\boldsymbol{B}$, i.e., every such $\boldsymbol{A}$ can be embedded in $\boldsymbol{B}$. For example, any countable linear ordering can be embedded in $\langle \mathbb{Q}, < \rangle$.

We will be primarily interested in the case when the classes $\mathcal{K}$ in 2.2 actually consist of *finite* structures and the Fraïssé limit of $\mathcal{K}$ is countably infinite. It will be convenient then to introduce, for later use, the following terminology, where the *cardinality* of a structure $\boldsymbol{A} = \langle A, \dots \rangle$ is the cardinality of its universe A.

Definition 2.4 *Given a signature L, a* Fraïssé class *in L is a class of finite structures in L, which contains structures of arbitrary large (finite) cardinality, is countable, and satisfies HP, JEP and AP. A* Fraïssé structure *in L is a countably infinite structure which is locally finite and ultrahomogeneous.*

Thus the map $\mathcal{K} \mapsto \mathrm{Flim}(\mathcal{K})$ is a bijection between Fraïssé classes and Fraïssé structures (up to isomorphism) with inverse the map $\boldsymbol{A} \mapsto \mathrm{Age}(\boldsymbol{A})$.

We would like to point out here that for G a closed subgroup of S_A, the induced structure $\boldsymbol{A}_G$ is ultrahomogeneous and, since the associated signature is relational, it is locally finite, so it is a Fraïssé structure, provided A is infinite.

Finally, for further reference, we recall the following definition. A class $\mathcal{K}$ of structures for L satisfies the *strong amalgamation property* (SAP) if for any $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathcal{K}$ and embeddings $f : \boldsymbol{A} \to \boldsymbol{B}, g : \boldsymbol{A} \to \boldsymbol{C}$, there is $\boldsymbol{D} \in \mathcal{K}$ and embeddings $r : \boldsymbol{B} \to \boldsymbol{D}, s : \boldsymbol{C} \to \boldsymbol{D}$ with $r \circ f = s \circ g$, such that moreover $r(B) \cap s(C) = r(f(A))$ $(= s(g(A))$.

Similarly, we say that $\mathcal{K}$ satisfies the *strong joint embedding property* (SJEP) if for any $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{K}$ there is $\boldsymbol{C} \in \mathcal{K}$ and embeddings $f : \boldsymbol{A} \to \boldsymbol{C}, g : \boldsymbol{B} \to \boldsymbol{C}$ such that $f(A) \cap g(B) = \emptyset$.

Remark. In retrospect, one can say that Fraissé's construction was anticipated by Urysohn [27], who considered the special case of the class of finite metric spaces with rational distances. He constructed a countable metric space $\boldsymbol{U}_0$ whose completion $\boldsymbol{U}$, known as the *Urysohn space*, is the unique universal ultrahomogeneous (with respect to isometries) complete separable metric space. Note that we can view metric spaces (X, d) as structures in a countable signature $L = \{R_q\}_{q \in \mathbb{Q}}$, R_q binary, identifying (X, d) with $\boldsymbol{X} = (X, R_q^{\boldsymbol{X}})$, where $(x, y) \in R_q^{\boldsymbol{X}} \Leftrightarrow d(x, y) < q$.

3 Structural Ramsey Theory

We will now recall some concepts and results from Ramsey theory, for which we refer the reader to Nešetřil [95], Nešetřil-Rödl [90] and Graham-Rothchild-Spencer [90].

Let $\boldsymbol{A}, \boldsymbol{B}$ be structures in a signature L. We write

$$\boldsymbol{A} \leq \boldsymbol{B}$$

if $\boldsymbol{A}$ can be embedded into $\boldsymbol{B}$. If $\boldsymbol{A} \leq \boldsymbol{B}$, we let

$$\binom{\boldsymbol{B}}{\boldsymbol{A}} = \{\boldsymbol{A}_0 : \boldsymbol{A}_0 \text{ is a substructure of } \boldsymbol{B} \text{ isomorphic to } \boldsymbol{A}\}.$$

For $\boldsymbol{A} \leq \boldsymbol{B} \leq \boldsymbol{C},\ k = 2, 3, \dots,$ we write

$$\boldsymbol{C} \to (\boldsymbol{B})^{\boldsymbol{A}}_k$$

if for any coloring $c : \binom{\boldsymbol{C}}{\boldsymbol{A}} \to \{1, \dots, k\}$ with k colors, there is $\boldsymbol{B}_0 \in \binom{\boldsymbol{C}}{\boldsymbol{B}}$ which is *homogeneous*, in the sense that for some $1 \leq i \leq k$, and all $\boldsymbol{A}_0 \in \binom{\boldsymbol{B}_0}{\boldsymbol{A}}$, $c(\boldsymbol{A}_0) = i$, i.e., $\binom{\boldsymbol{B}_0}{\boldsymbol{A}}$ is monochromatic.

Let now $\mathcal{K}$ be a class of finite structures in a signature L. We say that $\mathcal{K}$ satisfies the *Ramsey property* if $\mathcal{K}$ is hereditary (i.e., satisfies HP) and for any $\boldsymbol{A} \leq \boldsymbol{B}$ in $\mathcal{K},\ k = 2, 3, \dots,$ there is $\boldsymbol{C} \in \mathcal{K}$ with $\boldsymbol{B} \leq \boldsymbol{C}$ such that

$$\boldsymbol{C} \to (\boldsymbol{B})^{\boldsymbol{A}}_k.$$

Note that by a simple induction, we can restrict this condition to $k = 2$.

Let us now mention some examples of classes with the Ramsey property:

(i) Let $L = \{<\}$ and let $\mathcal{LO}$ be the class of finite linear orderings. Then $\mathcal{LO}$ has the Ramsey property, by the classical Ramsey theorem.

(ii) Let $L = \{<, E\}$, $<, E$ binary relation symbols and let $\mathcal{OG}$ be the class of all finite ordered graphs $\boldsymbol{A} = \langle A, <^{\boldsymbol{A}}, E^{\boldsymbol{A}} \rangle$ (i.e., $<^{\boldsymbol{A}}$ is a linear ordering of A and $E^{\boldsymbol{A}}$ is a symmetric, irreflexive relation). Then Nešetřil-Rödl [77, 83] showed that $\mathcal{OG}$ has the Ramsey property.

(iii) Let F be a finite field and let $L = \{+\} \cup \{f_\alpha\}_{\alpha \in F}$ (all function symbols), where $+$ has arity 2 and each f_α is unary. Any vector space over F can be viewed as a structure in this language with $+$ representing addition and f_α scalar multiplication by $\alpha \in F$. A substructure of a vector space is clearly a subspace. Let $\mathcal{V}_F$ be the class of all finite-dimensional vector spaces over F. Clearly $\boldsymbol{A} \leq \boldsymbol{B} \Leftrightarrow \dim \boldsymbol{A} \leq \dim \boldsymbol{B}$. It was shown in Graham-Leeb-Rothchild [72], that $\mathcal{V}_F$ has the Ramsey property.

(iv) Let now $L = \{0, 1, -, \wedge, \vee\}$ (all function symbols), where $0, 1$ have arity 0 and $-, \wedge, \vee$ have arities 1, 2, 2 resp. Any Boolean algebra is a structure in this language (with $-$ representing Boolean complementation). Substructures are again subalgebras. Let $\mathcal{BA}$ be the class of all finite Boolean algebras. Then the so-called Dual Ramsey Theorem of Graham-Rothchild [71] can be equivalently reformulated by saying that $\mathcal{BA}$ has the Ramsey property (see Nešetřil [95, 4.13]).

Finally, let us point out the following connection between the Ramsey property and the amalgamation property, discussed in the previous section (see Nešetřil-Rödl [77, p. 294, Lemma 1]).

Let $\mathcal{K}$ be a class of finite *rigid* (i.e., having no non-trivial automorphisms) structures in a signature L, which is hereditary. If $\mathcal{K}$ has the JEP and the Ramsey property, then $\mathcal{K}$ has the AP. To see this, fix $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathcal{K}$ and embeddings $f : \boldsymbol{A} \to \boldsymbol{B},\ g : \boldsymbol{A} \to \boldsymbol{C}$. By the JEP, find $\boldsymbol{E} \in \mathcal{K}$ in which both $\boldsymbol{B}, \boldsymbol{C}$ can be embedded. Then find $\boldsymbol{D} \in \mathcal{K}$ such that $\boldsymbol{D} \to (\boldsymbol{E})^{\boldsymbol{A}}_4$ and consider the coloring $c : \binom{\boldsymbol{D}}{\boldsymbol{A}} \to \{x : x \subseteq \{\boldsymbol{B}, \boldsymbol{C}\}\}$ defined as follows: Given $\boldsymbol{A}_0 \in \binom{\boldsymbol{D}}{\boldsymbol{A}}, \boldsymbol{B} \in c(\boldsymbol{A}_0) \Leftrightarrow$ there is an embedding $r : \boldsymbol{B} \to \boldsymbol{D}$ with $r \circ f(\boldsymbol{A}) = \boldsymbol{A}_0$, and similarly for $\boldsymbol{C}$. Let $\boldsymbol{E}_0 \in \binom{\boldsymbol{D}}{\boldsymbol{E}}$ be a homogeneous set. Then $c(\boldsymbol{A}_0) = \{\boldsymbol{B}, \boldsymbol{C}\}$, for all $\boldsymbol{A}_0 \in \binom{\boldsymbol{E}_0}{\boldsymbol{A}}$. For such $\boldsymbol{A}_0$, there is $r : \boldsymbol{B} \to \boldsymbol{D}$ with $f \circ r(\boldsymbol{A}) = \boldsymbol{A}_0$ and $s : \boldsymbol{C} \to \boldsymbol{D}$ with $g \circ s(\boldsymbol{A}) = \boldsymbol{A}_0$. So $r \circ f,\ g \circ s$ are isomorphisms of $\boldsymbol{A}$ with $\boldsymbol{A}_0$. Since $\boldsymbol{A}, \boldsymbol{A}_0$ are rigid, it follows that $r \circ f = s \circ g$, so $\boldsymbol{D}, r, s$ verify the amalgamation property for $f : \boldsymbol{A} \to \boldsymbol{B}, g : \boldsymbol{A} \to \boldsymbol{C}$.

In particular, if $\mathcal{K}$ is a non-empty class of rigid finite structures which is countable, contains structures of arbitrarily large cardinality, and satisfies HP and JEP and the Ramsey property, then $\mathcal{K}$ is a Fraïssé class, i.e., the age of a countably infinite ultrahomogeneous structure.

4 Characterizing extremely amenable automorphism groups by a Ramsey property

We will first reformulate the condition that a G-flow has a fixed point in the following manner.

Lemma 4.1 *Let G be a topological group and X a G-flow. Then the following are equivalent:*

(i) The G-flow X has a fixed point.

(ii) For every $n = 1, 2, \ldots,$ and continuous $f : X \to \mathbb{R}^n,\ \epsilon > 0,\ F \subseteq G$ finite, there is $x \in X$, such that $|f(x) - f(g \cdot x)| \leq \epsilon,\ \forall g \in F$, where $|\ |$ refers to Euclidean norm.

Proof. (i) $\Rightarrow$ (ii). This is obvious.

(ii) $\Rightarrow$ (i). We use a compactness argument. For $f : X \to \mathbb{R}^n$ continuous, $\epsilon > 0,\ F \subseteq G$ finite, put

$$A_{f,\epsilon,F} = \{x \in X : \forall g \in F(|f(x) - f(g \cdot x)| \leq \epsilon)\}.$$

Claim. $\bigcap_{f,\epsilon,F} A_{f,\epsilon,F} \neq \emptyset$.

Granting this, fix $x \in \bigcap_{f,\epsilon,F} A_{f,\epsilon,F}$. Then x is a fixed point, since otherwise there is $g \in G$ with $g \cdot x \neq x$, so there is a continuous $f : X \to \mathbb{R}$ with $f(x) = 0, f(g \cdot x) = 1$, thus $x \notin A_{f,1,\{g\}}$.

Proof of the claim. Notice that $A_{f,\epsilon F}$ is closed, so, by compactness, it is enough to show that for any finite collection $(f_j, \epsilon_j, F_j),\ j = 1, \ldots, m$, we have $\bigcap_{j=1}^m A_{f_j,\epsilon_j,F_j} \neq \emptyset$. Put

$$\bar{F} = F_1 \cup \cdots \cup F_m,\ \bar{\epsilon} = \min\{\epsilon_1, \ldots, \epsilon_m\}$$

$$\bar{f} = (f_1, \ldots, f_m) : X \to \mathbb{R}^{n_1+\cdots+n_m},$$

where $f_i : X \to \mathbb{R}^{n_i}$. Then $A_{\bar{f},\bar{\epsilon},\bar{F}} \subseteq \bigcap_{j=1}^m A_{f_j,\epsilon_j,F_j}$. But $A_{\bar{f},\bar{\epsilon},\bar{F}}$ is non-empty by (ii). $\dashv$

We use this to prove the following preliminary characterization.

Proposition 4.2 *Let S_∞ be the Polish group of permutations of $\mathbb{N}$ with the pointwise convergence topology. If $G \leq S_\infty$ is a closed subgroup, then the following are equivalent:*

(i) G is extremely amenable.

(ii) For any open subgroup V of G, every coloring $c : G/V \to \{1, \dots, k\}$, of the set of left-cosets hV of V, and every finite $A \subseteq G/V$, there is $g \in G$ and $1 \leq i \leq k$, such that $c(g \cdot a) = i,\ \forall a \in A$, where G acts on G/V in the usual way $g \cdot hV = ghV$.

Proof. (i) $\Rightarrow$ (ii): Fix V, k, c as in (ii), and consider the shift action of G on $Y = \{1, \dots, k\}^{G/V},\ g \cdot p(x) = p(g^{-1} \cdot x)$, for $p \in Y,\ x \in G/V$. This is a G-flow and $c \in Y$. Let $X = \overline{G \cdot c}$. By (i) find a fixed point $\gamma \in X$. Since G acts transitively on G/V, clearly $\gamma : G/V \to \{1, \dots, k\}$ is a constant function, say $\gamma(a) = i,\ \forall a \in G/V$. Fix now finite $A \subseteq G/H$. Since $\gamma \in \overline{G \cdot c}$, there is $g \in G$ such that $g^{-1} \cdot c|A = \gamma|A$, so $c(g \cdot a) = \gamma(a) = i,\ \forall a \in A$.

(ii) $\Rightarrow$ (i): Clearly (ii) is equivalent to the corresponding statement about the space V/G of right-cosets Vh of V on which G acts as usual by $g \cdot Vh = Vhg^{-1}$, and this is what we will use below. Using 4.1, it suffices to show that if X is a G-flow and $f : X \to \mathbb{R}^n$ is continuous, $\epsilon > 0,\ F \subseteq G$ is finite, then there is $x \in X$, with $|f(x) - f(h \cdot x)| \leq \epsilon,\ \forall h \in F$.

As in the proof of 1.3, there is an open nbhd of 1_G, say V, such that $\forall h \in V \forall x \in X |f(x) - f(h \cdot x)| \leq \epsilon/3$. But, since G is a closed subgroup of S_∞, we can assume that V is an open subgroup of G (see Becker-Kechris [96, 1.5]). Partition now the compact set $f(X) \subseteq \mathbb{R}^n$ into sets $A_1, \dots, A_k$ of diameter $\leq \epsilon/3$.

Fix $x_0 \in X$ and let

$$U_i = \{g \in G : f(g \cdot x_0) \in A_i\}.$$

Put $VU_i = V_i$, so that V_i is the union of right-cosets of V and thus can be viewed as a subset of G/V. Since $\bigcup_{i=1}^k V_i = G/V$, we can find $c : G/V \to \{1, \dots, k\}$ such that $c^{-1}(\{i\}) \subseteq V_i$. So by (ii) there is $1 \leq i \leq k$ and $g \in G$ with $(F \cup \{1_G\})g \subseteq V_i = VU_i$. We will now show that $x = g \cdot x_0$ works.

Indeed, fix $h \in F$. Let $v_1, v_2 \in V$ be such that $v_1 hg \in U_i,\ v_2 g \in U_i$, so $f(v_1 hg \cdot x_0),\ f(v_2 g \cdot x_0) \in A_i$, thus $|f(v_1 hg \cdot x_0) - f(v_2 g \cdot x_0)| \leq \epsilon/3$. But also $|f(v_1 hg \cdot x_0) - f(hg \cdot x_0)| \leq \epsilon/3$ and $|f(v_2 g \cdot x_0) - f(g \cdot x_0)| \leq \epsilon/3$, so $|f(hg \cdot x_0) - f(g \cdot x_0)| \leq \epsilon$ or $|f(x) - f(h \cdot x)| \leq \epsilon$. $\dashv$

Clearly in 4.2 we can restrict V to any local basis at 1_G consisting of open subgroups. In particular, if for each non-empty finite $F \subseteq \mathbb{N}$ we let

$$G_{(F)} = \{g \in G : \forall i \in F(g(i) = i)\}$$

be the *pointwise stabilizer* of F, then as $\{G_{(F)} : \emptyset \neq F \subseteq \mathbb{N},\ F \text{ finite}\}$ is a local basis of 1_G, we can restrict V in 4.2 to be of the form $G_{(F)}$, and moreover it is enough to consider only F in any cofinal under inclusion collection of finite subsets of $\mathbb{N}$.

Remark. By the proof of 4.2, to test extreme amenability of a closed subgroup $G \leq S_\infty$ it is enough to find fixed points in compact invariant subsets of the G-flow $\{1, \dots, k\}^{G/V},\ V \leq G$ open.

For the next result we will need some further notation and terminology. Each $G \leq S_\infty$ acts on the finite subsets of $\mathbb{N}$ in the obvious way

$$g \cdot F = \{g(i) : i \in F\}.$$

For each finite $\emptyset \neq F \subseteq \mathbb{N}$, we let then

$$G_F = \{g \in G : g \cdot F = F\}$$

be the stabilizer of F in this action, i.e., the *setwise stabilizer* of F. Clearly $G_{(F)} \leq G_F$ and $[G_F : G_{(F)}] < \infty$.

The *G-type* of $\emptyset \neq F \subseteq \mathbb{N},\ F$ finite, is the orbit $G \cdot F$ of F. A *G-type* σ is the G-type of some finite nonempty $F,\ \sigma = G \cdot F$. If ρ, σ are G-types, we write

$$\begin{aligned} \rho \leq \sigma \quad & \Leftrightarrow \exists F \in \sigma \exists F' \in \rho (F' \subseteq F). \\ & \Leftrightarrow \forall F \in \sigma \exists F' \in \rho (F' \subseteq F) \\ & \Leftrightarrow \forall F' \in \rho \exists F \in \sigma (F' \subseteq F). \end{aligned}$$

Finally, given a signature $L = \{R_i\}_{i \in I} \cup \{f_j\}_{j \in J}$, we denote by X_L the space of all structures for L with universe $\mathbb{N}$. Thus

$$X_L = \prod_i 2^{(\mathbb{N}^{n_i})} \times \prod_j \mathbb{N}^{(\mathbb{N}^{m_j})}$$

If L is relational, i.e., $J = \emptyset,\ X_L$ is compact (homeomorphic to $2^{\mathbb{N}}$). The group S_∞ acts canonically on X_L as follows: Given $\boldsymbol{A} = \langle \mathbb{N}, \{R_i^{\boldsymbol{A}}\},\ \{f_j^{\boldsymbol{A}}\} \rangle$ we let $g \cdot \boldsymbol{A} = \boldsymbol{B} = \langle \mathbb{N}, \{R_i^{\boldsymbol{B}}\},\ \{f_j^{\boldsymbol{B}}\} \rangle$, where

$$R_i^{\boldsymbol{B}}(a_1, \dots, a_{n(i)}) \Leftrightarrow R_i^{\boldsymbol{A}}(g^{-1}(a_1), \dots, g^{-1}(a_{n(i)}))$$

$$f_j^{\boldsymbol{B}}(a_1, \dots, a_{m(j)}) = f_j^{\boldsymbol{A}}(g^{-1}(a_1), \dots, g^{-1}(a_{m(i)})),$$

so that g is an isomorphism from $\boldsymbol{A}$ to $\boldsymbol{B}$. This action, called the *logic action*, is clearly continuous. In particular, if L is relational, the action of any $G \leq S_\infty$ on X_L is a G-flow. Consider the language $L = \{<\}$, $<$ a binary relation symbol, and denote by LO the compact S_∞-invariant subset of X_L consisting of all linear orderings $\boldsymbol{A} = \langle \mathbb{N}, <^{\boldsymbol{A}} \rangle$ on $\mathbb{N}$. Clearly for any $G \leq S_\infty$, LO is a subflow of the G-flow X_L. We say that G *preserves an ordering* if this subflow has a fixed point, i.e., there is an ordering $\prec$ on $\mathbb{N}$ such that for every $g \in G$, $a \prec b \Leftrightarrow g(a) \prec g(b)$.

We now have:

Proposition 4.3 *Let $G \leq S_\infty$ be a closed subgroup. Then the following are equivalent:*

(i) G is extremely amenable.

(ii) (a) For any finite $\emptyset \neq F \subseteq \mathbb{N}$, $G_{(F)} = G_F$ and (b) For any two G-types ρ, σ with $\rho \leq \sigma$, and every finite coloring $c : \rho \to \{1, \ldots, k\}$, there is $1 \leq i \leq k$ and $F \in \sigma$ such that $c(F') = i$, $\forall F' \subseteq F$, $F' \in \rho$.

(iii) (a)' G preserves an ordering and (b) as in (ii) above.

Proof. (i) $\Rightarrow$ (iii): Consider (a)' first. Since G is extremely amenable and LO is a G-flow, there is a fixed point, i.e., G preserves an ordering.

We next prove (b). Fix $\rho \leq \sigma$, $c : \rho \to \{1, \ldots, k\}$. Say $G \cdot F' = \rho$. Then if $V = G_{(F')}$, we note that $V = G_{(F')} = G_{F'}$ by (a)' and we can identify G/V with $G \cdot F' = \rho$. Applying then (ii) of 4.2, to $V, c, A = \{F'_0 \subseteq F_0 : F'_0 \in \rho\}$, where $F_0 \in \sigma$, we find $1 \leq i \leq k$ and $g \in G$ with $c(g \cdot F'_0) = i$, $\forall F'_0 \in A$. Let $F = g \cdot F_0 \in \sigma$. If $F' \subseteq F$, $F' \in \rho$, then $g^{-1} \cdot F' = F'_0 \subseteq F_0$, and $F'_0 \in \rho$, so $c(g \cdot F'_0) = c(F') = i$.

(iii) $\Rightarrow$ (ii): Clearly, (a)' $\Rightarrow$ (a).

(ii) $\Rightarrow$ (i): We verify (ii) of 4.2 for V of the form $G_{(F)} = G_F$, $\emptyset \neq F \subseteq \mathbb{N}$ finite. If $V = G_F$, then G/V can be identified with $\rho = G \cdot F$. So fix $c : \rho \to \{1, \ldots, k\}$ and $A \subseteq \rho$ finite. Put $\bigcup A = F_0$, $\sigma = G \cdot F_0$. Clearly $\rho \leq \sigma$, so there is $1 \leq i \leq k$ and $g \in G$ such that for all $F' \subseteq g \cdot F_0$ with $F' \in \rho$, we have $c(F') = i$. Thus $c(g \cdot F) = i$, $\forall F \in A$. $\dashv$

We will now use a compactness argument to put this characterization in a final form. It will be convenient first to introduce the following notation.

Definition 4.4 *Let $G \leq S_\infty$. Let $\rho \leq \sigma$ be G-types. If $F \in \sigma$, we put*

$$\binom{F}{\rho} = \{F' \subseteq F : F' \in \rho\}.$$

If $\rho \leq \sigma \leq \tau$ are G-types, we put

$$\tau \to (\sigma)^{\rho}_{k},$$

where $k = 2, 3, \ldots,$ if for every $F \in \tau$ and coloring $c : \binom{F}{\rho} \to \{1, \ldots, k\}$, there is $F_0 \in \binom{F}{\sigma}$, which is homogeneous, i.e., c is monochromatic on $\binom{F_0}{\rho}$: for some $1 \leq i \leq k$, and every $F' \in \binom{F_0}{\rho}$, $c(F') = i$. (Note that this is equivalent to asserting that this is true for some $F \in \tau$.)

We say that G has the Ramsey property *if for every G-types $\rho \leq \sigma$ and every $k = 2, 3, \ldots,$ there is a G-type $\tau \geq \sigma$ with $\tau \to (\sigma)^{\rho}_{k}$.*

We now have

Theorem 4.5 *Let $G \leq S_\infty$ be a closed subgroup. Then the following are equivalent:*

(i) G is extremely amenable.

(ii) (a) G preserves an ordering and (b) G has the Ramsey property.

Proof. (i) $\Rightarrow$ (ii). We have already seen (a). To prove (b), first note that, by a simple induction, it is enough to restrict ourselves in 4.4. to the case $k = 2$. So assume, towards a contradiction, that for some G-types $\rho \leq \sigma$, there is no $\tau \geq \sigma$ with $\tau \to (\sigma)^{\rho}_{2}$. Fix $F_0 \in \sigma$. Then for every finite set $E \supseteq F_0$ there is a coloring $c_E : \binom{E}{\rho} \to \{1, 2\}$ which does not have a homogeneous set $F \in \binom{E}{\sigma}$. Pick an ultrafilter U on the set I of finite non-empty subsets of $\mathbb{N}$ such that for every finite $F \subseteq \mathbb{N}$, $\{E : F \subseteq E\} \in U$. Then for each $D \in \rho$, $\{E \supseteq D \cup F_0 : c_E(D) = 1\} \in U$ or $\{E \supseteq D \cup F_0 : c_E(D) = 2\} \in U$, so put $c(D) = i$ iff $\{E \supseteq D \cup F_0 : c_E(D) = i\} \in U$. This gives a coloring $c : \rho \to \{1, 2\}$. Then by 4.3, (ii) (b), there is $F \in \sigma$ such that c is monochromatic on $\binom{F}{\rho}$, say with value i. If $D \in \binom{F}{\rho}$, $A_D = \{E \supseteq F \cup F_0 : c_E(D) = c(D) = i\} \in U$, so pick $E \in \bigcap_{D \in \binom{F}{\rho}} A_D$. Then $E \supseteq F$ and for each $D \in \binom{F}{\rho}$, $c_E(D) = i$, so $F \in \binom{E}{\sigma}$ is homogeneous for c_E, a contradiction. $\dashv$

(ii) $\Rightarrow$ (i): It is of course enough to verify (b) of 4.3 (ii), which follows trivially from the assumption that G has the Ramsey property. $\dashv$

Remark 4.6 Let $G \leq S_\infty$. We call a set T of G-types *cofinal* if for every G-type ρ there is $\sigma \in T$ with $\rho \leq \sigma$. Then it is not hard to see that Theorem 4.5 still holds if in the definition 4.4 of G having the Ramsey property, we restrict the G-types to be in any given cofinal set of G-types.

We will finally tie-up extreme amenability of automorphism groups with the structural Ramsey theory of §3.

Let L be a signature with a distinguished binary relation symbol $<$ (and perhaps other symbols). An *order structure* for L is a structure $\boldsymbol{A}$ of L in which $<^{\boldsymbol{A}}$ is a linear ordering. If $\mathcal{K}$ is a class of structures of L, we say that $\mathcal{K}$ is an *order class* if every $\boldsymbol{A} \in \mathcal{K}$ is an order structure.

We also recall that up to (topological group) isomorphism the closed subgroups of S_∞ are exactly the same as the automorphism groups of countable structures and also the same as the Polish groups which admit a countable nbhd basis at the identity consisting of open subgroups (see Becker-Kechris [96, 1.5]). So the next result provides a characterization of the groups in this last class that are extremely amenable.

Theorem 4.7 *Let $G \leq S_\infty$ be a closed subgroup. Then the following are equivalent:*

(i) G is extremely amenable.

(ii) $G = \mathrm{Aut}(\boldsymbol{A})$, where $\boldsymbol{A}$ is the Fraïssé limit of a Fraïssé order class with the Ramsey property.

Proof. (i) $\Rightarrow$ (ii): Let $\boldsymbol{A}_G = \langle \mathbb{N}, \dots \rangle$ be the canonical structure for G. As we pointed out in §2 (second paragraph before the last remark) $\boldsymbol{A}_G$ is ultrahomogeneous. Also, since G is extremely amenable, G preserves a linear order $\prec$ on $\mathbb{N}$. Let L be the signature obtained from the signature of $\boldsymbol{A}_G$ by adding a new binary relation symbol $<$. Let $\boldsymbol{A}$ be the expansion of the structure $\boldsymbol{A}_G$ in which $<^A = \prec$. Clearly we have $\mathrm{Aut}(\boldsymbol{A}) = G$, in particular $\boldsymbol{A}$ is still ultrahomogeneous. Note also that the signature of $\boldsymbol{A}_G$ and thus of $\boldsymbol{A}$ is relational, so $\boldsymbol{A}$ is locally finite. Thus $\mathcal{K} = \mathrm{Age}(\boldsymbol{A})$ is a Fraïssé order class. Noting now that, by ultrahomogeneity, a G-type is exactly the collection of all substructures of $\boldsymbol{A}$ isomorphic to a given $\boldsymbol{A}_0 \in \mathrm{Age}(\boldsymbol{A})$, we see that the G having the Ramsey property is equivalent to $\mathrm{Age}(\boldsymbol{A})$ having the Ramsey property, so we are done by 4.5.

(ii) $\Rightarrow$ (i): Since $\boldsymbol{A}$ is the Fraïssé limit of a Fraïssé order class it is a locally finite order structure. This implies that G preserves an ordering and also that the G-types of finite substructures are cofinal in all the G-types. As noted earlier, the G-type of a finite substructure $\boldsymbol{A}_0$ is the collection of all substructures of $\boldsymbol{A}$ isomorphic to $\boldsymbol{A}_0$, so, by 4.5 and 4.6, it is clear that G has the Ramsey property, so it is extremely amenable. $\dashv$

We make explicit, for further reference, the following fact observed in the preceding proof.

Proposition 4.8 *Let $\mathcal{K}$ be a Fraïssé order class and $\boldsymbol{A} = \mathrm{Flim}(\mathcal{K})$. Then the following are equivalent:*
(i) $\mathrm{Aut}(\boldsymbol{A})$ *is extremely amenable.*
(ii) $\mathcal{K}$ has the Ramsey property.

5 Reducts

Let $L = \{R_i\}_{i\in I} \cup \{f_i\}_{j\in J}$ be a signature and $\boldsymbol{A} = \langle A, \{R_i^{\boldsymbol{A}}\}, \{f_j^{\boldsymbol{A}}\}\rangle$ a structure for L. If $L_0 = \{R_i\}_{i\in I_0} \cup \{f_j\}_{j\in J_0}$, with $I_0 \subseteq I,\ J_0 \subseteq J$, so that $L_0 \subseteq L$, we let $\boldsymbol{A}_0 = \boldsymbol{A}|L_0 = \langle A, \{R_i^{\boldsymbol{A}}\}_{i\in I_0},\ \{f_j^{\boldsymbol{A}}\}_{j\in J_0}\rangle$ be the *reduct* of $\boldsymbol{A}$ to L_0. We also call $\boldsymbol{A}$ an *expansion* of $\boldsymbol{A}_0$. If $\mathcal{K}$ is a class of structures in L, we denote by

$$\mathcal{K}|L_0 = \{\boldsymbol{A}|L_0 : A \in \mathcal{K}\},$$

the class of reducts of elements of $\mathcal{K}$, called also the *reduct* of $\mathcal{K}$ to L_0.

We have seen that order classes of structures play a crucial role in extreme amenability of automorphism groups. We will now examine what happens to reducts of such classes, when the ordering is dropped.

Let L be a signature with a distinguished binary relation symbol $<$ and let $L_0 = L \setminus \{<\}$. For $\boldsymbol{A}$ a structure for L, we denote by $\boldsymbol{A}_0$ the reduct of $\boldsymbol{A}$ to L_0 and for any class $\mathcal{K}$ of structures for L, we denote by $\mathcal{K}_0$ the reduct of $\mathcal{K}$ to L_0. Conversely if $\boldsymbol{A}_0 = \langle A_0, \dots\rangle$ is a structure for L_0 and $\prec$ a binary relation as A_0, we denote by $\langle \boldsymbol{A}_0, \prec\rangle = \boldsymbol{A}$ the structure for L whose reduct to L_0 is $\boldsymbol{A}_0$ and $\prec = <^{\boldsymbol{A}}$ (thus also $A = A_0$).

Consider a Fraïssé order class $\mathcal{K}$ in a signature $L \supseteq \{<\}$ with Fraïssé limit $\boldsymbol{F}$. We characterize when $\mathcal{K}_0$ is also a Fraïssé class with limit $\boldsymbol{F}_0 = \boldsymbol{F}|L_0$.

Definition 5.1 *Let L be a signature with $L \supseteq \{<\}$, and put $L_0 = L \setminus \{<\}$. Let $\mathcal{K}$ be a Fraïssé order class in L and put $\mathcal{K}_0 = \mathcal{K}|L_0$. We say that $\mathcal{K}$ is* reasonable *if for every* $\boldsymbol{A}_0 \in \mathcal{K}_0,\ \boldsymbol{B}_0 \in \mathcal{K}_0$*, embedding* $\pi : \boldsymbol{A}_0 \to \boldsymbol{B}_0$*, and linear ordering* $\prec$ *on* A_0 *with* $\boldsymbol{A} = \langle \boldsymbol{A}_0, \prec \rangle \in \mathcal{K}$*, there is a linear ordering* $\prec'$ *on* B_0*, so that* $\boldsymbol{B} = \langle \boldsymbol{B}_0, \prec' \rangle \in \mathcal{K}$ *and* $\pi : \boldsymbol{A} \to \boldsymbol{B}$ *is also an embedding (i.e.,* $a \prec b \Leftrightarrow \pi(a) \prec' \pi(b)$*).*

We now have

Proposition 5.2 *Let* $L \supseteq \{<\}$ *be a signature and* $\mathcal{K}$ *a Fraïssé order class in* L*. Let* $L_0 = L \setminus \{<\},\ \mathcal{K}_0 = \mathcal{K}|L_0,\ \boldsymbol{F} = \mathrm{Flim}(\mathcal{K}),\ \boldsymbol{F}_0 = \boldsymbol{F}|L_0$*. Then the following are equivalent:*

(i) $\mathcal{K}_0$ *is a Fraïssé class and* $\boldsymbol{F}_0 = \mathrm{Flim}(\mathcal{K}_0)$*.*

(ii) $\mathcal{K}$ *is reasonable.*

Proof. (i) $\Rightarrow$ (ii). Let $\boldsymbol{A}_0 \in \mathcal{K}_0,\ \boldsymbol{B}_0 \in \mathcal{K}_0,\ \pi : \boldsymbol{A}_0 \to \boldsymbol{B}_0$ an embedding, and fix a linear ordering $\prec$ of A_0 with $\boldsymbol{A} = \langle \boldsymbol{A}_0, \prec \rangle \in \mathcal{K}$. Then there is an embedding $\varphi : \boldsymbol{A} \to \boldsymbol{F}$, and φ is of course also an embedding $\varphi : \boldsymbol{A}_0 \to \boldsymbol{F}_0$. By the extension property 2.3 (since $\boldsymbol{F}_0$ is ultrahomogeneous), there is an embedding $\psi : \boldsymbol{B}_0 \to \boldsymbol{F}_0$ with $\psi \circ \pi = \varphi$. Let $\prec' = \psi^{-1}(<^{\boldsymbol{F}} |\psi(B_0))$. Then $\boldsymbol{B} = \langle \boldsymbol{B}_0, \prec' \rangle \in \mathcal{K}$, as $\boldsymbol{B}$ is isomorphic to a substructure of $\boldsymbol{F}$, and moreover $\pi : \boldsymbol{A} \to \boldsymbol{B}$ is also an embedding.

(ii) $\Rightarrow$ (i): Clearly $\mathcal{K}_0 = \mathrm{Age}(\boldsymbol{F}_0)$, so to verify that $\mathcal{K}_0$ is a Fraïssé class, we only need to check that $\mathcal{K}_0$ satisfies the AP. Fix $\boldsymbol{A}_0, \boldsymbol{B}_0, \boldsymbol{C}_0 \in \mathcal{K}_0$ and embeddings $f : \boldsymbol{A}_0 \to \boldsymbol{B}_0,\ g : \boldsymbol{A}_0 \to \boldsymbol{C}_0$. Let then $\prec$ be a linear order on A_0 with $\boldsymbol{A} = \langle \boldsymbol{A}_0, \prec \rangle \in \mathcal{K}$. Since $\mathcal{K}$ is reasonable, we can find linear orders $\prec', \prec''$ on B_0, C_0 resp., so that $\boldsymbol{B} = \langle B_0, \prec' \rangle \in \mathcal{K},\ \boldsymbol{C} = \langle \boldsymbol{C}_0, \prec'' \rangle \in \mathcal{K}$ and $f : \boldsymbol{A} \to \boldsymbol{B}, g : \boldsymbol{A} \to \boldsymbol{C}$ are still embeddings. By AP for $\mathcal{K}$ find $\boldsymbol{D} \in \mathcal{K}$ and embeddings $r : \boldsymbol{B} \to \boldsymbol{D},\ s : \boldsymbol{C} \to \boldsymbol{D}$ with $r \circ f = s \circ g$. Let $\boldsymbol{D}_0 = \boldsymbol{D}|L_0$. Clearly $r : \boldsymbol{B}_0 \to \boldsymbol{D}_0, s : \boldsymbol{C}_0 \to \boldsymbol{D}_0$ and we are done.

Finally, we check that $\mathrm{Flim}(\mathcal{K}_0) = \boldsymbol{F}_0$, for which it is enough to verify that $\boldsymbol{F}_0$ has the extension property 2.3. So fix $\boldsymbol{A}_0, \boldsymbol{B}_0 \in \mathcal{K}_0, \pi : \boldsymbol{A}_0 \to \boldsymbol{B}_0$ an embedding, and $\varphi : \boldsymbol{A}_0 \to \boldsymbol{F}_0$ an embedding. Then let

$$\prec = \varphi^{-1}(<^{\boldsymbol{F}} |\varphi(\boldsymbol{A}_0)),$$

so that $\boldsymbol{A} = \langle \boldsymbol{A}_0, \prec \rangle \in \mathcal{K}$ and $\varphi : \boldsymbol{A} \to \boldsymbol{F}$ is an embedding. Since $\mathcal{K}$ is reasonable, let $\prec'$ be a linear ordering on B_0 with $\langle \boldsymbol{B}_0, \prec' \rangle \in \mathcal{K}$ and $\pi : \boldsymbol{A} \to \boldsymbol{B}$ still an embedding. Since $\boldsymbol{F}$ satisfies the extension property, there is an embedding $\psi : \boldsymbol{B} \to \boldsymbol{F}$ with $\psi \circ \pi = \varphi$ and, since clearly also $\psi : \boldsymbol{B}_0 \to \boldsymbol{F}_0$, we are done. $\dashv$

A common way to construct an order class $\mathcal{K}$ in $L \supseteq \{<\}$ is to start with a class $\mathcal{K}_0$ in $L_0 = L \setminus \{<\}$ and take

$$\mathcal{K} = \mathcal{K}_0 * \mathcal{LO} = \{\langle \boldsymbol{A}_0, \prec\rangle : \boldsymbol{A}_0 \in \mathcal{K}_0 \text{ and } \prec \text{ is a linear ordering on } A_0\}.$$

For example, if $\mathcal{K}_0$ is the class of finite graphs, $\mathcal{K}$ is the class of all finite ordered graphs. We now have

Proposition 5.3 *Let $L \supseteq \{<\}$ be a signature and let $L_0 = L \setminus \{<\}$. Let $\mathcal{K}_0$ be a class of structures in L_0 and put $\mathcal{K} = \mathcal{K}_0 * \mathcal{LO}$. Then the following are equivalent:*
(i) $\mathcal{K}$ satisfies the amalgamation property.
(ii) $\mathcal{K}$ satisfies the strong amalgamation property.
(iii) $\mathcal{K}_0$ satisfies the strong amalgamation property.

Proof. Suppose that $\mathcal{K}$ satisfies AP in order to show that $\mathcal{K}_0$ satisfies the strong amalgamation property. Fix $\boldsymbol{A}_0, \boldsymbol{B}_0, \boldsymbol{C}_0 \in \mathcal{K}_0$ and embeddings $f : \boldsymbol{A}_0 \to \boldsymbol{B}_0,\ g : \boldsymbol{A}_0 \to \boldsymbol{C}_0$. There are clearly linear orderings $\prec$ on $A_0, \prec'$ on B_0, and $\prec''$ on C_0 such that if $\boldsymbol{A} = \langle \boldsymbol{A}_0, \prec\rangle,\ \boldsymbol{B} = \langle \boldsymbol{B}_0, \prec'\rangle,\ \boldsymbol{C} = \langle \boldsymbol{C}_0, \prec''\rangle$ (which are all in $\mathcal{K}$), then $f : \boldsymbol{A} \to \boldsymbol{B},\ g : \boldsymbol{A} \to \boldsymbol{C}$ are still embeddings, and $f(A_0) \prec' (B_0 \setminus f(A_0))$ and $(C_0 \setminus g(A_0)) \prec'' f(A_0)$ (where if $\prec\!\prec$ is a linear order on a set X and $Y, Z \subseteq X$, then $Y \prec\!\prec Z \Leftrightarrow \forall y \in Y \forall z \in Z (y \prec\!\prec z)$). By the AP for $\mathcal{K}$, let $r : \boldsymbol{B} \to \boldsymbol{D},\ s : \boldsymbol{C} \to \boldsymbol{D}, \boldsymbol{D} \in \mathcal{K}$, be such that $r \circ f = s \circ g$. If, towards a contradiction, there is $d \in r(B) \cap s(C),\ d \notin r(f(A))$ (where $A = A_0,\ B = B_0,\ C = C_0$), then $r(f(A)) <^{\boldsymbol{D}} d$, since $d \in r(B \setminus f(A))$ and $d <^{\boldsymbol{D}} r(f(A)) = s(g(A))$, since $d \in s(C \setminus g(A))$, which is absurd. So $r(B) \cap s(C) = r(f(A))$.

Now assume that $\mathcal{K}_0$ satisfies strong amalgamation. We verify that $\mathcal{K}$ satisfies strong amalgamation. Let $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathcal{K}$ and let $f : \boldsymbol{A} \to \boldsymbol{B},\ g : \boldsymbol{A} \to \boldsymbol{C}$ be embeddings. Then also $f : \boldsymbol{A}_0 \to \boldsymbol{B}_0,\ g : \boldsymbol{B}_0 \to \boldsymbol{C}_0$ are embeddings, so, by strong amalgamation for $\mathcal{K}_0$, find $r : \boldsymbol{B}_0 \to \boldsymbol{D}_0,\ s : \boldsymbol{C}_0 \to \boldsymbol{D}_0,\ \boldsymbol{D}_0 \in \mathcal{K}_0$, so that $r \circ f = s \circ g$ and $r(B) \cap s(C) = r(f(A))$. Then $r(B) \setminus r(f(A)),\ s(C) \setminus s(g(A)),\ r(f(A))\ (= s(g(A))$ are pairwise disjoint, so clearly there is an order $\prec$ on D_0 such that if $\boldsymbol{D} = \langle \boldsymbol{D}_0, \prec\rangle$, which is in $\mathcal{K}$, then $r : \boldsymbol{B} \to \boldsymbol{D},\ s : \boldsymbol{C} \to \boldsymbol{D}$. $\dashv$

The following is quite obvious:

Proposition 5.4 *Let $L, L_0, \mathcal{K}, \mathcal{K}_0$ be as in 5.3. Then $\mathcal{K}_0$ satisfies the strong joint embedding property iff $\mathcal{K}$ satisfies the strong joint embedding property.*

Clearly it is not true that if $\mathcal{K}$ satisfies the joint embedding property, then $\mathcal{K}_0$ satisfies the strong joint embedding property. Consider, e.g., $L = \{c\} \cup \{<\}$, where c is a 0-ary function symbol, $\mathcal{K}$ = all finite structures in L.

Finally, we note a condition that implies a connection between the Ramsey property for $\mathcal{K}$ and $\mathcal{K}_0$.

Definition 5.5 *Let $L \supseteq \{<\}$ be a signature, $L_0 = L \setminus \{<\}$, $\mathcal{K}$ a class of structures in L and $\mathcal{K}_0 = \mathcal{K}|L_0$. We say that $\mathcal{K}$ is* order forgetful *if for every $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{K}$, letting $\boldsymbol{A}_0 = \boldsymbol{A}|L_0$, $\boldsymbol{B}_0 = \boldsymbol{B}|L_0$, we have*

$$\boldsymbol{A} \cong \boldsymbol{B} \Leftrightarrow \boldsymbol{A}_0 \cong \boldsymbol{B}_0.$$

(Notice that this does not say that any isomorphism of $\boldsymbol{A}_0$ with $\boldsymbol{B}_0$ is also an isomorphism of $\boldsymbol{A}$ with $\boldsymbol{B}$.)

An example of an order forgetful class is the class of finite-dimensional vector spaces $\boldsymbol{V}$ over a finite field F with antilexicographical ordering induced by an ordering of a basis of $\boldsymbol{V}$. This was considered in Thomas [86].

We now have the following fact, which is easy to prove:

Proposition 5.6 *Let $L \supseteq \{<\}$ be a signature, $L_0 = L \setminus \{<\}$ and let $\mathcal{K}$ be a class of finite structures in L which is hereditary. Put $\mathcal{K}_0 = \mathcal{K}|L_0$. If $\mathcal{K}$ is order forgetful, then the following are equivalent:*
(i) $\mathcal{K}$ satisfies the Ramsey property.
(ii) $\mathcal{K}_0$ satisfies the Ramsey property.

6 Extremely amenable automorphism groups

We will now apply the preceding general results to find many new examples of extremely amenable automorphism groups. We will use the following immediate consequence of earlier results.

Theorem 6.1 *Let L be a signature with $L \supseteq \{<\}$ and let $\mathcal{K}$ be a Fraïssé order class in L. Let $\boldsymbol{F} = \mathrm{Flim}(\mathcal{K})$ be the Fraïssé limit of $\mathcal{K}$, so that $\boldsymbol{F}$ is an order structure. Then the following are equivalent:*
(i) $G = \mathrm{Aut}(\boldsymbol{F})$ is extremely amenable.
(ii) $\mathcal{K}$ has the Ramsey property.

Proof. (ii) $\Rightarrow$ (i) follows from 4.7, (ii) $\Rightarrow$ (i). (i) $\Rightarrow$ (ii) is as in the proof of (i) $\Rightarrow$ (ii) of 4.7. $\dashv$

Ramsey theory provides now many examples of $\mathcal{K}$ satisfying the Ramsey property and we use them to produce new examples of extremely amenable groups.

(A) Graphs

Let $L_0 = \{E\}$ be the signature with one binary relation symbol E. Let also $L = \{E, <\}$. A structure $\boldsymbol{A}_0 = \langle A_0, E^{\boldsymbol{A}_0}\rangle$ is a *graph* if $E^{\boldsymbol{A}_0}$ irreflexive and symmetric. An *ordered graph* is a structure $\boldsymbol{A} = \langle \boldsymbol{A}_0, <^{\boldsymbol{A}}\rangle$ for L in which $\boldsymbol{A}_0$ is a graph and $<^{\boldsymbol{A}}$ a linear ordering.

Lachlan-Woodrow [80] classified all Fraïssé classes $\mathcal{K}_0$ of finite graphs. They are exactly the following:

(i) $\mathcal{GR}$ = all finite graphs.

(ii) For $n = 3, 4, \ldots$, $\mathcal{F}orb(K_n)$ = the class of all finite graphs omitting K_n, the complete graph on n vertices (i.e., the class of finite graphs that do not contain K_n as a substructure).

(iii) $\mathcal{EQ}$ = the class of finite equivalence relations.

(iv) For $n = 1, 2, \ldots$, $\mathcal{EQ}_n$ = the class of finite equivalence relations with at most n classes.

(v) For $n = 1, 2, \ldots$, $\mathcal{EQ}^*_n$ = the class of finite equivalence relations, all of whose classes have at most n elements.

(vi) The complement $\overline{\mathcal{K}_0}$ of one of the classes $\mathcal{K}_0$ listed in (ii)-(iv) above, where for any graph $\boldsymbol{A}_0 = \langle A_0, E^{\boldsymbol{A}_0}\rangle$ its *complement* is $\overline{\boldsymbol{A}_0} = \langle A_0, \overline{E^{\boldsymbol{A}_0}}\rangle$, where $(x, y) \in \overline{E^{\boldsymbol{A}_0}} \Leftrightarrow x \neq y$ and $(x, y) \notin E^{\boldsymbol{A}_0}$, and $\overline{\mathcal{K}_0} = \{\overline{\boldsymbol{A}_0} : \boldsymbol{A}_0 \in \mathcal{K}_0\}$.

Remark. Strictly speaking an equivalence relation is not a graph, because it is reflexive. So when we think of an equivalence relation $\boldsymbol{X} = \langle X, R\rangle$ as a graph, we identify it with $\langle X, R \setminus \{(x, x) : x \in X\}\rangle$.

Since the automorphism group of the complement of a given graph $\boldsymbol{A}_0$ is the same as the automorphism group of $\boldsymbol{A}_0$, we do not need to consider the classes of type (vi). For any one of the classes $\mathcal{K}_0$ of type (i)-(iv), let $\mathcal{OK}_0 = \mathcal{K}_0 * \mathcal{LO}$ = the class of finite ordered graphs $\boldsymbol{A} = \langle \boldsymbol{A}_0, <^{\boldsymbol{A}}\rangle$ with $\boldsymbol{A}_0 \in \mathcal{K}_0$. Now for $\mathcal{K}_0$ of type (v) with $n \geq 2$, it is easy to check that $\mathcal{K}_0$ does not have the strong amalgamation property, so $\mathcal{OK}_0$ is not a Fraïssé order class, by 5.3. For $\mathcal{K}_0$ of type (iv) with $n \geq 2$, $\mathcal{OK}_0$ is a Fraïssé order class, whose Fraïssé limit $\boldsymbol{F} = \langle F, E^{\boldsymbol{F}}, <^{\boldsymbol{F}}\rangle$, consists of an equivalence relation $E^{\boldsymbol{F}}$ on F with exactly n infinite classes and in which $<^{\boldsymbol{F}}$ is an ordering isomorphic to the rationals such that every equivalence class is dense. But then it is easy to check that $\mathrm{Aut}(\boldsymbol{F})$ is not extremely amenable, since it acts (continously) on the finite (discrete) space X = the set of $E^{\boldsymbol{F}}$ classes, without fixed point. Finally, notice that $\overline{\mathcal{EQ}_1} = \mathcal{EQ}^*_1$, so we only need to consider $\mathcal{K}_0 = \mathcal{GR}, \mathcal{F}orb(K_n), n = 3, 4, \ldots,$ $\mathcal{EQ}$ and $\mathcal{EQ}_1$.

Each one of the classes $\mathcal{K}_0 = \mathcal{GR}, \mathcal{F}orb(K_n), n = 3, 4, \ldots,$ $\mathcal{EQ}$ and $\mathcal{EQ}_1$, clearly satisfies the strong amalgamation and strong joint embedding properties, so $\mathcal{K} = \mathcal{O}\mathcal{K}_0$ is a Fraïssé order class. $\mathcal{K}$ is reasonable in each case. Finally, Nešetřil and Rödl [77], [83], Nešetřil [89], have shown that each one of these classes $\mathcal{K}$ satisfies the Ramsey property (the case $\mathcal{K}_0 = \mathcal{E}\mathcal{Q}_1$ is of course the classical Ramsey Theorem). Thus if $\boldsymbol{F}$ is the Fraïssé limit of $\mathcal{K}$, then $\mathrm{Aut}(\boldsymbol{F})$ is extremely amenable. We discuss now each case in some more detail:

(i) $\mathcal{K}_0 = \mathcal{G}\mathcal{R}$: Then $\boldsymbol{R} = \mathrm{Flim}(\mathcal{G}\mathcal{R})$ is called the *random graph*. It is natural to call $\boldsymbol{OR} = \mathrm{Flim}(\mathcal{O}\mathcal{G}\mathcal{R})$ the *random ordered graph*. It is of the form $\boldsymbol{OR} = \langle \boldsymbol{R}, <^{\boldsymbol{OR}} \rangle$, where $<^{\boldsymbol{OR}}$ is an appropriate linear order of the random graph, isomorphic to the rationals. Thus we have:

Theorem 6.2 *The automorphism group of the random ordered graph is extremely amenable.*

Of course the automorphism group of the random graph itself is not extremely amenable, since it does not preserve an ordering.

(ii) $\mathcal{K}_0 = \mathcal{F}orb(K_n)$ is called the *K_n-free random graph* and so we call $\boldsymbol{OR}^n = \mathrm{Flim}(\mathcal{O}\mathcal{F}orb(K_n))$ the *random K_n-free ordered graph*. It is of the form $\boldsymbol{OR}^n = \langle \boldsymbol{R}_0^n, <^{\boldsymbol{OR}^n} \rangle$, with $<^{\boldsymbol{OR}^n}$ a linear ordering of the K_n-free random graph, isomorphic to the rationals. Thus we have:

Theorem 6.3 *The automorphism group of the random K_n-free ordered graph is extremely amenable.*

(iii) $\mathcal{K}_0 = \mathcal{E}\mathcal{Q}$: Then $\boldsymbol{F}_0 = \mathrm{Flim}(\mathcal{E}\mathcal{Q})$ is the equivalence relation with infinitely many classes each of which is infinite. So $\boldsymbol{F} = \mathrm{Flim}(\mathcal{O}\mathcal{E}\mathcal{Q}) \cong \langle \mathbb{Q}, E, < \rangle$, where $<$ is the usual ordering on $\mathbb{Q}$ and E is an equivalence relation on $\mathbb{Q}$ with infinitely many classes each of which is dense. Thus we have:

Theorem 6.4 *The automorphism group of the rationals with the usual order and an equivalence relation with infinitely many classes, all of which are dense, is extremely amenable.*

(iv) $\mathcal{K}_0 = \mathcal{E}\mathcal{Q}_1$: Then $\boldsymbol{F}_0 = \mathrm{Flim}(\mathcal{E}\mathcal{Q}_1)$ is clearly the complete graph on a countable infinite set and so, up to isomorphism, $\mathrm{Flim}(\mathcal{O}\mathcal{E}\mathcal{Q}_1) \cong \langle \mathbb{Q}, E, < \rangle$, where E is the complete graph on $\mathbb{Q}$. But the automorphism group of this structure is exactly that of $\langle \mathbb{Q}, < \rangle$, so we have

Theorem 6.5 (Pestov [98a]) *The automorphism group of the rationals with the usual order is extremely amenable.*

In the preceding example the automorphism group of $\boldsymbol{F}_0$ is of course exactly S_∞, which is not extremely amenable as proved in Pestov [98a]. This is clear, as S_∞ cannot preserve an ordering.

Finally, we discuss another order class which can be obtained from $\mathcal{EQ}$. This will also play a role in §8.

Let $L_0 = \{E\}$, $L = \{E, <\}$, where $E, <$ are binary relation symbols. Let $\mathcal{K}$ be the class of structures $\boldsymbol{A} = \langle A, E^{\boldsymbol{A}}, <^{\boldsymbol{A}}\rangle$ in L such that $E^{\boldsymbol{A}}$ is an equivalence relation on A, $<^{\boldsymbol{A}}$ a linear order on A, and for every $a <^{\boldsymbol{A}} b <^{\boldsymbol{A}} c$, if $(a, c) \in E^{\boldsymbol{A}}$, then $(a, b) \in E^{\boldsymbol{A}}$, i.e., every $E^{\boldsymbol{A}}$ class is convex in $<^{\boldsymbol{A}}$. We call a structure in $\mathcal{K}$ a *convexly ordered finite equivalence relation.* Clearly $\mathcal{K}|L_0 = \mathcal{EQ}$. Now it is easy to check that $\mathcal{K}$ is a reasonable, Fraïssé order class. The Fraïssé limit of $\mathcal{K}$ is of the form $\boldsymbol{F} = \langle \boldsymbol{F}_0, <^{\boldsymbol{F}}\rangle$, where $\boldsymbol{F}_0$ is the equivalence relation with infinitely many classes each of which is infinite and $<^{\boldsymbol{F}}$ is an ordering such that each equivalence class is convex and isomorphic to the rationals, and the equivalence classes are also ordered like the rationals. So $\boldsymbol{F}$ up to isomorphism, is the same as $\langle \mathbb{Q}^2, E, <_\ell\rangle$, where $<_\ell$ is the lexicographical ordering on $\mathbb{Q}^2$ and E is the equivalence relation on $\mathbb{Q}^2$ given by $(r, s,)E(r', s') \Leftrightarrow r = r'$.

Theorem 6.6 *The automorphism group of $\mathbb{Q}^2$ with the lexicographical ordering and the equivalence relation $(r, s)E(r', s') \Leftrightarrow r = r'$ is extremely amenable.*

Proof. Put $\boldsymbol{F} = \langle \mathbb{Q}^2, E, <_\ell\rangle$. We will show that $\mathrm{Aut}(\boldsymbol{F})$ is extremely amenable.

Let $I_r = \{r\} \times \mathbb{Q}$, so that $r < s \Leftrightarrow I_r <_\ell I_s$. Let now $\pi \in \mathrm{Aut}(\boldsymbol{F})$. Let $f_\pi : \mathbb{Q} \to \mathbb{Q}$ be defined by $f_\pi(r) = s \Leftrightarrow \pi(I_r) = I_s$. Then clearly $f_\pi \in \mathrm{Aut}(\langle \mathbb{Q}, <\rangle)$. Also let for each $r \in \mathbb{Q}$, $(g_\pi)_r : \mathbb{Q} \to \mathbb{Q}$ be defined by $(g_\pi)_r(s) = t \Leftrightarrow \pi(r, s)) = (f_\pi(r), t)$. Thus again $(g_\pi)_r \in \mathrm{Aut}(\langle \mathbb{Q}, <\rangle)$. Put $\Theta(\pi) = (f_\pi, g_\pi)$, where $g_\pi \in \mathrm{Aut}(\mathbb{Q})^{\mathbb{Q}}$, $g_\pi = \{(g_\pi)_r\}$. Consider the semi-direct product $\mathrm{Aut}(\langle \mathbb{Q}, <\rangle) \ltimes \mathrm{Aut}(\langle \mathbb{Q}, <\rangle)^{\mathbb{Q}}$, where $\mathrm{Aut}(\langle \mathbb{Q}, <\rangle)$ acts on $\mathrm{Aut}(\langle \mathbb{Q}, <\rangle)^{\mathbb{Q}}$ by shift: $g \cdot x(r) = x(g^{-1}(r))$. Then it is easy to check that Θ is a (topological group) isomorphism of $\mathrm{Aut}(\boldsymbol{F})$ with the semidirect product $\mathrm{Aut}(\langle \mathbb{Q}, <\rangle) \ltimes \mathrm{Aut}(\langle \mathbb{Q}, <\rangle)^{\mathbb{Q}}$, so it is enough to check that the latter group is extremely amenable. This follows from the following standard closure properties of extreme amenability and 6.5.

Lemma 6.7 *i) Let G be a topological group and $\mathcal{H}$ an upward-directed, under inclusion, family of extremely amenable subgroups of G such that $\bigcup \mathcal{H}$ is dense in G. Then G is extremely amenable.*

ii) Let G be a topological group, $N \trianglelefteq G$ a closed normal subgroup. If $N, G/N$ are extremely amenable, so is G.

iii) The product of extremely amenable groups is extremely amenable.

Proof. i) Let X be a G-flow, so also an H-flow, for any $H \in \mathcal{H}$. Put $X_H = \{x \in X : \forall h \in H(h \cdot x = x)\}$. Then X_H is compact, non-empty and $\{X_H : H \in \mathcal{H}\}$ has the finite intersection property. So $\bigcap_{H\in\mathcal{H}} X_H \neq \emptyset$, and any $x \in \bigcap_{H\in\mathcal{H}} X_H$ is fixed by an $g \in \bigcup \mathcal{H}$, so by G.

ii) Let X be a G-flow, so also a N-flow. Then $X_N = \{x \in X : \forall g \in N(g \cdot x = x)\}$ is a compact, non-empty subset of X. It follows easily by the normality of N that X_N is G-invariant: Let $x \in X_N, g \in G$. If $h \in N$, then $h \cdot (g \cdot x) = hg \cdot x = g \cdot (g^{-1}hg) \cdot x = g \cdot x$ (as $g^{-1}hg \in N$), so $g \cdot x \in X_N$.

Define now an action of G/N on X_N as follows: $gN \cdot x = g \cdot x$ (this is clearly well-defined). It is easy to check that this is a continuous action, so there is a fixed point $x \in X_N$ which is clearly a fixed point of the G-flow on X.

iii) Suppose that each $G_i, i \in I$, is extremely amenable. Then, by ii), the products $\prod_{i\in I_0} G_i$, $I_0 \subseteq I$ finite, are extremely amenable, and identifying $\prod_{i\in I_0} G_i$ with the subgroup $\prod_{i\in I} G'_i$ of $\prod_{i\in I} G_i$, where $G'_i = G_i$, if $i \in I_0, G'_i = \{1_{G_i}\}$, if $i \notin I_0$, the family $\{\prod_{i\in I_0} G_i : I_0 \subseteq I \text{ finite}\}$ is upwards-directed under inclusion and its union is dense in $\prod_{i\in I} G_i$, so $\prod_{i\in I} G_i$ is extremely amenable. $\dashv$

Corollary 6.8 *The class of convexly ordered equivalence relations satisfies the Ramsey property.*

Proof. By 6.7 and 6.1. $\dashv$

This Ramsey result can be also proved directly (and in fact its unordered version has already been considered in the literature; see Rado [54], Graham-Rothchild-Spencer [90], §5, Theorem 5]), but it seems interesting to reverse the roles here and derive it from an extreme amenability result.

(B) Hypergraphs

Let $L_0 = \{R_i\}_{i\in I}$ be a finite relational signature with R_i of arity $n(i) \geq 1$. A *hypergraph of type* L_0 is a structure $\boldsymbol{A}_0 = \langle A_0, \{R_i^{\boldsymbol{A}_0}\}_{i\in I}\rangle$ for L_0 for which each $R_i^{\boldsymbol{A}_0}$ is irreflexive and symmetric, i.e., $(a_1, \dots, a_{n(i)}) \in R_i^{\boldsymbol{A}_0} \Rightarrow a_1, \dots, a_{n(i)}$ are distinct, and for any permutation π of $\{1, \dots, n(i)\}$, $(a_1, \dots, a_{n(i)}) \in R_i^{\boldsymbol{A}_0} \Rightarrow (a_{\pi(i)}, \dots, a_{\pi(n(i))}) \in R_i^{\boldsymbol{A}_0}$. Thus, in essence, $R_i^{\boldsymbol{A}_0} \subseteq [A_0]^{n(i)}$ = the set of subsets of A_0 of cardinality $n(i)$.

Let $\mathcal{H}_{L_0}$ be the class of finite hypergraphs of type L_0, let $L = L_0 \cup \{<\}$, and let $\mathcal{OH}_{L_0} = \mathcal{H}_{L_0} * \mathcal{LO}$ be the class of structures in L which are finite ordered hypergraphs, i.e., of the form $\boldsymbol{A} = \langle \boldsymbol{A}_0, <^{\boldsymbol{A}} \rangle$, with $\boldsymbol{A}_0$ a finite hypergraph, and $<^{\boldsymbol{A}}$ a linear ordering. It is easy to check that $\mathcal{OH}_{L_0}$ is a reasonable Fraïssé order class (note that $\mathcal{H}_{L_0}$ satisfies strong amalgamation and joint embedding). The Fraïssé limit $\mathrm{Flim}(\mathcal{H}_{L_0}) = \boldsymbol{H}_{L_0}$ is called the *random hypergraph of type* L_0, so we call $\mathrm{Flim}(\mathcal{OH}_{L_0}) = \boldsymbol{OH}_{L_0}$ the *random ordered hypergraph of type* L_0. We have $\boldsymbol{OH}_{L_0} = \langle \boldsymbol{H}_{L_0}, <^{\boldsymbol{OH}_{L_0}} \rangle$, where $<^{\boldsymbol{OH}_{L_0}}$ is an appropriate ordering on H_{L_0} isomorphic to the rationals. Now Nešetřil-Rödl [77], [83] (see also Nešetřil [95]) have shown that $\mathcal{OH}_{L_0}$ satisfies the Ramsey property, so we have

Theorem 6.9 *The automorphism group of the random ordered hypergraph of type L_0 is extremely amenable.*

In case $L_0 = \{E\}$, E a binary relation, $\boldsymbol{OH}_{L_0} = \boldsymbol{OR}$, the random ordered graph, so this generalizes 6.2. As another special case, consider $L_0 = \{R_i\}_{i \in I}$, where each $n(i) = 1$. Then $\boldsymbol{OH}_{L_0} \cong \langle \mathbb{Q}, \{A_i\}_{i \in I}, < \rangle$, where $<$ is the usual ordering on $\mathbb{Q}$ and each $A_i \subseteq \mathbb{Q}$ is dense and co-dense in $\mathbb{Q}$. Thus the automorphism group of the rationals with the usual ordering and a finite family of subsets each of which is dense and co-dense is extremely amenable.

The preceding example can be further generalized.

Definition 6.10 *Let $L_0 = \{R_i\}_{i \in I}$ be a finite relational signature. A hypergraph $\boldsymbol{A}_0 = \langle A_0, \{R_i^{\boldsymbol{A}_0}\}_{i \in I} \rangle$ is called* irreducible *if A_0 has at least two elements and for every $a \neq b$ in A_0 there is $i \in I$ with $n(i) \geq 2$ and $c_1, \dots, c_{n(i)-2} \in A_0$ such that $(a, b, c_1, \dots, c_{n(i)-2}) \in R_i^{\boldsymbol{A}_0}$.*

Let $\mathcal{A}$ be a class of finite irreducible hypergraphs of type L_0. Then $\mathcal{F}orb(\mathcal{A})$ is the class of all finite hypergraphs of type L_0 which omit $\mathcal{A}$, i.e., do not contain a substructure isomorphic to a member of $\mathcal{A}$.

For example, for $L_0 = \{E\}$, E binary, $\mathcal{A} = \{K_n\}$, $n = 3, 4, \dots, \mathcal{F}orb(\mathcal{A}) = \mathcal{F}orb(K_n)$, the class of finite graphs that do not contain the complete graph of n elements as a substructure.

For $\mathcal{A}$ a class of finite irreducible hypergraphs of type L_0, we denote by $\mathcal{OF}orb(\mathcal{A})$ the class of finite ordered hypergraphs of type L_0 that omit $\mathcal{A}$. It is again easy to see that $\mathcal{OF}orb(\mathcal{A})$ is a reasonable Fraïssé class. We can call again $\mathrm{Flim}(\mathcal{F}orb(\mathcal{A}))$ the *random $\mathcal{A}$-free ordered hypergraph of type* L_0. Nešetřil-Rödl [77], [83] (see also Nešetřil [95]) proved that $\mathcal{OF}orb(\mathcal{A})$ satisfies the Ramsey property. So we have

Theorem 6.11 *For each class $\mathcal{A}$ of finite irreducible hypergraphs of type L_0, the automorphism group of the random $\mathcal{A}$-free ordered hypergraph of type L_0 is extremely amenable.*

(C) Vector spaces

We will now consider an example of a different type. Fix a finite field F and consider the signature $L_0 = \{+\} \cup \{f_\alpha\}_{\alpha \in F}$ with $+$ a binary function symbol and f_α a unary function symbol. Vector spaces over F can be viewed as structures in this signature. Let $\mathcal{V}_F$ be the class of finite vector spaces over F. This is a Fraïssé class. Let $L = L_0 \cup \{<\}$, and consider the following order class defined in Thomas [86]. Fix an ordering on F such that the 0 and 1 of the field F are the first two elements in that ordering. If $\boldsymbol{V}_0$ is a finite-dimensional vector space over F of dimension n and B is a basis for $\boldsymbol{V}_0$, then every ordering $b_1 < \cdots < b_n$ of B gives an ordering on V_0 by

$$\alpha_1 b_1 + \cdots + \alpha_n b_n <_{a\ell} \beta_1 b_1 + \cdots + \beta_n b_n \Leftrightarrow$$

$$(\alpha_n < \beta_n) \text{ or } (\alpha_n = \beta_n \text{ and } \alpha_{n-1} < \beta_{n-1}) \text{ or } \ldots$$

i.e., $<_{a\ell}$ is the antilexicographical ordering induced by the ordering of B. A *natural ordering* of V_0 is one induced this way by an ordering of a basis. Let $\mathcal{OV}_F$ be the order class of all $\boldsymbol{V} = \langle \boldsymbol{V}_0, <^{\boldsymbol{V}} \rangle$, such that $\boldsymbol{V}_0$ is finite-dimensional vector space and $<^{\boldsymbol{V}}$ a natural ordering on V_0. Thomas [86] shows that this is a Fraïssé class. Next it is easy to check that $\mathcal{OV}_F$ is reasonable. Now the Fraïssé limit $\boldsymbol{V}_F$ of $\mathcal{V}_F$ is easily seen to be the vector space over F of countably infinite dimension, so if $\boldsymbol{OV}_F$ is the Fraïssé limit of $\mathcal{OV}_F$, then $\boldsymbol{OV}_F = \langle \boldsymbol{V}_F, <^{\boldsymbol{OV}_F} \rangle$, where $<^{\boldsymbol{OV}_F}$ is an appropriate linear order on V_F. Let us call $\boldsymbol{OV}_F$ the $\aleph_0$-*dimensional vector space over F with the canonical ordering.*

Finally $\mathcal{V}_F$ has the Ramsey property as shown in Graham-Leeb-Rothchild [72]. It is easy to see though that $\mathcal{OV}_F$ is order forgetful, according to Definition 5.5. Thus, by 5.6, $\boldsymbol{OV}_F$ has the Ramsey property too.

Thus we have:

Theorem 6.12 *The automorphism group of the $\aleph_0$-dimensional vector space over a finite field with the canonical ordering is extremely amenable.*

Of course the automorphism group of this vector space is not extremely amenable, as it cannot preserve an ordering.

(D) Boolean algebras

Let now $L_0 = \{0, 1, -, \wedge, \vee\}$, where $0, 1$ have arity 0, $-$ has arity 1 and $\wedge, \vee$ have arity 2. Boolean algebras are structures in L_0. Let $\mathcal{BA}$ be the class of finite Boolean algebras. Then it is not hard to check that $\mathcal{BA}$ is a Fraïssé class and its Fraïssé limit is $\boldsymbol{B}_\infty$, the countable atomless Boolean algebra.

We will next define natural orderings on finite Boolean algebras motivated by example **(C)**. Let $\boldsymbol{B}_0$ be a finite Boolean algebra and A its set of atoms. Then every ordering $a_1 < \cdots < a_n$ of A gives an ordering of B_0 as follows: Given $x, y \in B_0$, we can write them uniquely as $x = \delta_1 a_1 \vee \cdots \vee \delta_n a_n,\ y = \epsilon_1 a_1 \vee \cdots \vee \epsilon_n a_n$, where $\delta_i, \epsilon_i \in \{0, 1\}$, and for $\epsilon \in \{0, 1\},\ b \in B_0$,

$$\epsilon b = \begin{cases} b, \text{ if } \epsilon = 1, \\ 0^{\boldsymbol{B}}, \text{ if } \epsilon = 0. \end{cases}$$

(Here and below, we simply write $\vee$ instead of $\vee^{\boldsymbol{B}}$, when the Boolean algebra $\boldsymbol{B}$ is understood.) Then put

$$x <_{a\ell} y \Leftrightarrow (\delta_n < \epsilon_n) \text{ or } (\delta_n = \epsilon_n \text{ and } \delta_{n-1} < \epsilon_{n-1}) \text{ or } \ldots$$

i.e., $<_{a\ell}$ is the antilexicographical ordering induced by the ordering $<$ of the atoms. Again a *natural ordering* of $\boldsymbol{B}$ is one induced this way from an ordering of the set of atoms.

Let $\mathcal{OBA}$ be the order class of all $\boldsymbol{B} = \langle \boldsymbol{B}_0, <^{\boldsymbol{B}} \rangle$, such that $\boldsymbol{B}_0$ is a finite Boolean algebra and $<^{\boldsymbol{B}}$ is a natural ordering of B_0. We now have:

Proposition 6.13 *$\mathcal{OBA}$ is a reasonable Fraïssé order class.*

Proof. First we check that $\mathcal{OBA}$ is reasonable (see Definition 5.1). Let $\boldsymbol{B}_1, \boldsymbol{B}_2$ be two finite Boolean algebras and let $\pi : \boldsymbol{B}_1 \to \boldsymbol{B}_2$ be an embedding. Let $<_{a\ell}$ be a natural ordering on $\boldsymbol{B}_1$ induced by an ordering $a_1 < a_2 < \cdots < a_n$ of the atoms of B_1. Let $\{c_1, \ldots, c_k\}$ be the atoms of $\boldsymbol{B}_2$. Then $\pi(a_1) = \bigvee_{i=1}^{k_1} c_{1i}, \ldots, \pi(a_n) = \bigvee_{i=1}^{k_n} c_{ni}$, where $\{c_{1i}\}_{i=1}^{k_1}, \ldots, \{c_{ni}\}_{i=1}^{k_n}$ is a partition of $\{c_1, \ldots, c_k\}$. Order then the atoms of $\boldsymbol{B}_2$ as follows:

$$c_{11} \prec c_{12} \prec \cdots \prec c_{1k_1} \prec c_{21} \prec \cdots \prec c_{2k_2} \prec \cdots \prec c_{n1} \prec \cdots \prec c_{nk_n},$$

and let $\prec_{a\ell}$ be the induced antilexicographical ordering on $\boldsymbol{B}_2$. Then clearly $\pi : \langle \boldsymbol{B}_1, <_{a\ell} \rangle \to \langle \boldsymbol{B}_2, \prec_{a\ell} \rangle$ is still an embedding.

We next check that $\mathcal{OBA}$ is hereditary. To see this, let $\boldsymbol{B}_2$ be a finite Boolean algebra and $\boldsymbol{B}_1$ a subalgebra. Let $<_{a\ell}$ be a natural ordering on $\boldsymbol{B}_2$ induced by an ordering $a_1 < \cdots < a_n$ of the atoms of $\boldsymbol{B}_2$. Let now

$b_1 <_{a\ell} \cdots <_{a\ell} b_k$ be the atoms of $\boldsymbol{B}_1$. Write $b_i = c_{i1} \vee \cdots \vee c_{ik_i}$, where $c_{i1}, \ldots, c_{ik_1}$ are atoms of $\boldsymbol{B}_2$ and $c_{i1} < \cdots < c_{ik_i}$. Then $c_{ik_i} < c_{jk_j}$ if $i < j$. From this it easily follows that $<_{a\ell} |B_1$ = the antilexicographical ordering induced by $<_{a\ell} |\{b_1, \ldots, b_k\}$, so a substructure of an element of $\mathcal{OBA}$ is also an element of $\mathcal{OBA}$.

Finally, we check that $\mathcal{OBA}$ satisfies the amalgamation property (from which JEP also follows, since the two element Boolean algebra embeds in any Boolean algebra).

Suppose $\boldsymbol{B}$ is a finite Boolean algebra and $b_1 <^{\boldsymbol{B}} \cdots <^{\boldsymbol{B}} b_k$ is an ordering of the atoms of $\boldsymbol{B}$ with induced antilexicographical ordering $<^{\boldsymbol{B}}_{a\ell}$. Let also $\boldsymbol{C}, \boldsymbol{D}$ be finite Boolean algebras with orderings $c_1 <^{\boldsymbol{C}} \cdots <^{\boldsymbol{C}} c_l$, $d_1 <^{\boldsymbol{D}} \cdots <^{\boldsymbol{D}} d_m$ of their atoms, and corresponding induced antilexicographical orderings $<^{\boldsymbol{C}}_{a\ell}, <^{\boldsymbol{D}}_{a\ell}$. Suppose we have embeddings

$$f : \langle \boldsymbol{B}, <^{\boldsymbol{B}}_{a\ell} \rangle \to \langle \boldsymbol{C}, <^{\boldsymbol{C}}_{a\ell} \rangle, \ g : \langle \boldsymbol{B}, <^{\boldsymbol{B}}_{a\ell} \rangle \to \langle \boldsymbol{D}, <^{\boldsymbol{D}}_{a\ell} \rangle.$$

We will find a Boolean algebra $\boldsymbol{E}$ with $m+\ell-k$ atoms and an ordering $<^{\boldsymbol{E}}$ on these atoms, so that, if $<^{\boldsymbol{E}}_{a\ell}$ is the induced antilexicographical ordering, then there are embeddings $r : \langle \boldsymbol{C}, <^{\boldsymbol{D}}_{a\ell} \rangle \to \langle \boldsymbol{E}, \ <^{\boldsymbol{E}}_{a\ell} \rangle$, $s : \langle \boldsymbol{D}, <^{\boldsymbol{C}}_{a\ell} \rangle \to < \boldsymbol{E}, \ <^{\boldsymbol{E}}_{a\ell} \rangle$, such that $r \circ f = s \circ g$. To specify r, s, it is of course enough to define where the atoms of $\boldsymbol{C}, \boldsymbol{D}$ go.

Let $f(b_i) = c_{i1} \vee \cdots \vee c_{ik_i}$, with $c_{i1} <^{\boldsymbol{C}} \cdots <^{\boldsymbol{C}} c_{ik_i}$ atoms in $\boldsymbol{C}$. Then $c_{ik_i} < c_{jk_j}$, if $i < j$. Similarly, $g(b_i) = d_{i1} \vee \cdots \vee d_{i\ell_i}$, where $d_{i1} <^{\boldsymbol{D}} \cdots <^{\boldsymbol{D}} d_{i\ell_i}$ are atoms in $\boldsymbol{D}$ with $d_{i\ell_i} < d_{j\ell_j}$, if $i < j$.

The Boolean algebra $\boldsymbol{E}$ will have atoms $\{\bar{c}_{ij}\}_{1 \leq i \leq k, \ 1 \leq j \leq k_i}$, $\{\bar{d}_{ij}\}_{1 \leq i \leq k, \ 1 \leq j \leq \ell_i}$ all distinct, except that

$$\bar{c}_{ik_i} = \bar{d}_{i\ell_i}, \ 1 \leq i \leq k.$$

We will now define the ordering $<^{\boldsymbol{E}}$ on these atoms and decide where the atoms of $\boldsymbol{C}, \boldsymbol{D}$ go.

We first order $\{\bar{c}_{11}, \ldots, \bar{c}_{1k_1}\} \cup \{\bar{d}_{11}, \ldots, \bar{d}_{1\ell_1}\}$ as follows:

$$\bar{c}_{11} <^{\boldsymbol{E}} \cdots <^{\boldsymbol{E}} \bar{c}_{1k_1}, \bar{d}_{11} <^{\boldsymbol{E}} \cdots <^{\boldsymbol{E}} \bar{d}_{1\ell_1} (= \bar{c}_{1k_1})$$

and extend $<^{\boldsymbol{E}}$ on the rest in an arbitrary way. Using the notation $(a, b] = \{x : a < x \leq b\}$, $(-\infty, a] = \{x : x \leq a\}$ in an arbitrary ordering, we now define

$$r(c_{11}) = \bigvee (-\infty, \bar{c}_{11}], \ r(c_{12}) = \bigvee (\bar{c}_{11}, \bar{c}_{12}], \ldots, r(c_{1k_1}) = \bigvee (\bar{c}_{1,k_1-1}, \bar{c}_{1k_1}],$$

$$s(d_{11}) = \bigvee(-\infty, \bar{d}_{11}], \ldots, r(d_{1\ell_1}) = \bigvee(\bar{d}_{1,\ell_1-1}, \bar{d}_{1,\ell_1}],$$

where, for example $\bigvee(\bar{c}_{11}, \bar{c}_{12}]$ means $a_1 \vee \cdots \vee a_p$, where $a_1, \ldots, a_p$ are the elements of $\{\bar{c}_{11}, \ldots, \bar{c}_{1k_1}\} \cup \{\bar{d}_{11}, \ldots, \bar{d}_{1\ell_1}\}$ in the interval $(\bar{c}_{11}, \bar{c}_{12}]$ according to $<^{\boldsymbol{E}}$. Clearly r, s are order preserving from the atoms of $\boldsymbol{C}, \boldsymbol{D}$, resp., to $<^{\boldsymbol{E}}_{a\ell}$ and $r \circ f(b_1) = s \circ g(b_1) = \bar{c}_{11} \vee \cdots \vee \bar{c}_{1k_1} \vee \bar{d}_{11} \vee \cdots \vee \bar{d}_{1\ell_1}$.

Next we extend $<^{\boldsymbol{E}}$ to $\{\bar{c}_{11}, \ldots, \bar{c}_{1k_1}\} \cup \{\bar{c}_{21}, \ldots, \bar{c}_{2k_2}\} \cup \{\bar{d}_{11}, \ldots, \bar{d}_{1\ell_1}\} \cup \{\bar{d}_{21}, \ldots, \bar{d}_{2\ell_2}\}$ and define $r(c_{2i}), s(d_{2i})$. We simply do that by requiring that $c_{ij} \mapsto \bar{c}_{ij}$ $(i = 1, 2,\ 1 \leq j \leq k_i)$ is order preserving, and also $d_{ij} \mapsto \bar{d}_{ij}$ $(i = 1, 2,\ 1 \leq j \leq \ell_i)$ is order preserving, and define it arbitrarily otherwise. Notice that this guarantees that $\bar{c}_{2k_2} = \bar{d}_{2\ell_2}$ is the largest element, in particular $\bar{c}_{2k_2} > \bar{c}_{1k_1}, \bar{d}_{2\ell_2} > \bar{d}_{1\ell_1}$. We then extend r, s by defining

$$r(c_{21}) = \bigvee(-\infty, \bar{c}_{21}], \ldots, r(c_{2k_2}) = \bigvee(\bar{c}_{2,k_2-1}, \bar{c}_{2k_2}],$$

$$s(d_{21}) = \bigvee(-\infty, \bar{d}_{21}], \ldots, s(d_{2\ell_2}) = \bigvee(\bar{d}_{2,\ell_2-1}, \bar{d}_{2\ell_2}],$$

where these intervals now refer to the ordering $<^{\boldsymbol{E}}$ restricted to

$$\{c_{21}, \ldots, c_{2k_2}\} \cup \{d_{21}, \ldots, d_{2\ell_2}\}.$$

Then r, s are still order preserving and $r \circ f(b_2) = s \circ g(b_2) = \bar{c}_{21} \vee \cdots \vee \bar{c}_{2k_2} \vee \bar{d}_{21} \vee \cdots \vee \bar{d}_{2\ell_2}$. Proceeding this way, we define $<^{\boldsymbol{E}}$ on all the atoms of $\boldsymbol{E}$ and r, s on all the atoms of $\boldsymbol{C}, \boldsymbol{D}$, resp., so that r, s are order preserving on the atoms and $r \circ f(b) = s \circ g(b)$, for any atom b of $\boldsymbol{B}$. Then r, s extend uniquely to embeddings from $\langle \boldsymbol{C}, <^{\boldsymbol{C}}_{a\ell}\rangle$ to $\langle \boldsymbol{E}, <^{\boldsymbol{E}}_{a\ell}\rangle$ and $\langle \boldsymbol{D}, <^{\boldsymbol{D}}_{a\ell}\rangle$ to $\langle \boldsymbol{E}, <^{\boldsymbol{E}}_{a\ell}\rangle$, resp., and $r \circ f = s \circ g$. $\dashv$

Finally, it is clear that $\mathcal{OBA}$ is order forgetful and, since $\mathcal{BA}$ satisfies the Ramsey property by Graham-Rothchild [71] (the Dual Ramsey Theorem), it follows that so does $\mathcal{OBA}$. Let $\boldsymbol{OB}_\infty = \langle \boldsymbol{B}_\infty, <^{\boldsymbol{O}B_\infty}\rangle$ be the Fraïssé limit of $\mathcal{OBA}$, which we call the *countable atomless Boolean algebra with the canonical ordering*. Then we have:

Theorem 6.14 *The automorphism group of the countable atomless Boolean algebra with the canonical ordering is extremely amenable.*

And we conclude this section by providing a characterization of the group $\mathrm{Aut}(\langle \mathbb{Q}, < \rangle)$ in terms of extreme amenability.

Proposition 6.15 *Let $G \leq S_\infty$ be a closed subgroup of S_∞ which acts transitively on $[\mathbb{N}]^n = \{A \subseteq \mathbb{N} : \text{card}(A) = n\}, n = 1, 2, 3 \ldots$. If G is extremely amenable, then there is an ordering $\prec$ on $\mathbb{N}$ with $\langle \mathbb{N}, \prec \rangle \cong \langle \mathbb{Q}, < \rangle$ and $G = \text{Aut}(\langle \mathbb{N}, \prec \rangle)$.*

Proof. Since G preserves an ordering, it follows from 3.11 of Cameron [90] that there is an ordering $\prec$ on $\mathbb{N}$ with $\langle \mathbb{N}, \prec \rangle \cong \langle \mathbb{Q}, < \rangle$, such that $G \leq \text{Aut}(\langle \mathbb{N}, \prec \rangle)$. Since, for each n, G acts transitively on increasing n-tuples in $\prec$, G is dense in $\text{Aut}(\langle \mathbb{N}, \prec \rangle)$, so $G = \text{Aut}(\langle \mathbb{N}, \prec \rangle)$. $\dashv$

(E) Metric spaces

We can view metric spaces (X, d) as structures for the language $L_0 = \{R_q\}_{q \in \mathbb{Q}}$, R_q binary, identifying (X, d) with $\boldsymbol{X} = \langle X, \{R_q^{\boldsymbol{X}}\}_{q \in \mathbb{Q}} \rangle$, where $(x, y) \in R_q^{\boldsymbol{X}} \Leftrightarrow d(x, y) < q$. Let $\mathcal{M}_\mathbb{Q}$ be the class of finite metric spaces with rational distances. Then it is not hard to check that $\mathcal{M}_\mathbb{Q}$ is a Fraïssé class (see, e.g., Bogatyi [02]). Its Fraïssé limit is $\boldsymbol{U}_0$, originally constructed in Urysohn [27], and which we will call the *rational Urysohn space*. Let also $\mathcal{OM}_\mathbb{Q} = \mathcal{M}_\mathbb{Q} * \mathcal{L}O$ be the class of finite ordered metric spaces with rational distances. Since actually $\mathcal{M}_\mathbb{Q}$ satisfies strong amalgamation and joint embedding, it is easy to verify that $\mathcal{OM}_\mathbb{Q}$ is a reasonable Fraïssé order class. Its Fraïssé limit $\text{Flim}(\mathcal{OM}_\mathbb{Q}) = \boldsymbol{OU}_0$ will be called the *ordered rational Urysohn space*. Nešetřil [03], in response to an inquiry of the authors, has shown that $\mathcal{OM}_\mathbb{Q}$ satisfies the Ramsey property, so we have:

Theorem 6.16 *The automorphism group of the ordered rational Urysohn space is extremely amenable.*

This result has an interesting application, which actually was our motivation for looking at the case of metric spaces.

Let $\boldsymbol{U}$ be the so-called *Urysohn space*, see Urysohn [27]. This is the unique, up to isometry, complete separable metric space which contains (up to isometry) all finite metric spaces and is ultrahomogeneous, for isometries. Uspenskij [90] showed that $\text{Iso}(\boldsymbol{U})$, with the pointwise convergence topology, is a universal Polish group, i.e., contains up to isomorphism any Polish group. Note that the topology of the group $\text{Iso}(\boldsymbol{U})$ is that of pointwise convergence on the space $\boldsymbol{U}$ equipped with the metric topology, not the discrete one, unlike the case of $\text{Aut}(\boldsymbol{OU}_0)$. Pestov [02], using quite different techniques than the ones used in our paper, showed that $\text{Iso}(\boldsymbol{U})$ is extremely amenable. This result has several applications. We now use 6.16 to provide a quite different proof of this theorem.

Theorem 6.17 (Pestov [02]) *The group of isometries Iso($\boldsymbol{U}$) of the Urysohn space $\boldsymbol{U}$, with the pointwise convergence topology, is extremely amenable.*

Proof. We start with the following standard fact.

Lemma 6.18 *Let G, H be topological groups and $\pi : G \to H$ a continuous homomorphism with $\pi(G)$ dense in H. If G is extremely amenable, so is H.*

Proof. Let X be an H-flow. Denote by $\alpha : H \times X \to X$ the action. Define now $\tilde{\alpha} : G \times X \to X$ by $\tilde{\alpha}(g, x) = \alpha(\pi(g), x)$. This turns X into a G-flow, so there is a fixed point $x_0 \in X$. Clearly x_0 is a fixed point for the H-flow, since $\pi(G)$ is dense in H. ⊣

Now denote by $\langle \boldsymbol{U}_0, \prec\rangle$ the ordered rational Urysohn space (so that $\boldsymbol{U}_0$ is the rational Urysohn space). Already Urysohn [27] showed that the completion of $\boldsymbol{U}_0$ is $\boldsymbol{U}$, so we view $\boldsymbol{U}_0$ as a dense subspace of $\boldsymbol{U}$. Thus if $g \in \mathrm{Iso}(\boldsymbol{U}_0)$, there is a unique extension $\bar{g} \in \mathrm{Iso}(\boldsymbol{U})$. Since every $g \in \mathrm{Aut}(\langle \boldsymbol{U}_0, \prec\rangle)$ is in particular an isometry of $\boldsymbol{U}_0$, the map $g \mapsto \bar{g}$ is 1–1 from $\mathrm{Aut}(\langle \boldsymbol{U}_0, \prec\rangle)$ into $\mathrm{Iso}(\boldsymbol{U})$ and it is easy to check that it is continuous. It only remains to show that its range is dense in $\mathrm{Iso}(\boldsymbol{U})$ and then use 6.18 and 6.16.

Lemma 6.19 *Let $D \subseteq \mathrm{Iso}(\boldsymbol{U})$. Let d be the metric on $\boldsymbol{U}$. Then D is dense, if the following holds:*

$$(*)\ \forall \epsilon > 0 \forall x_1, \dots, x_n \in U \forall h \in \mathrm{Iso}(\boldsymbol{U})$$

$$\exists x_1', \dots, x_n', y_1', \dots, y_n' \in U \exists g \in D$$

$$(d(x_i, x_i') < \epsilon,\ d(h(x_i), y_i') < \epsilon,\ g(x_i') = y_i',\ i = 1, \dots, n).$$

Proof. To check that D is dense, fix $\epsilon > 0,\ h \in \mathrm{Iso}(\boldsymbol{U}), x_1, \dots, x_n \in U$, in order to find $g \in D$ with $d(g(x_i), h(x_i)) < \epsilon$.

By $(*)$ find $x_1', \dots, x_n', y_1', \dots y_n'$ and $g \in D$ for $\epsilon/2$. Then

$$\begin{aligned} d(g(x_i), h(x_i)) &\leq d(g(x_i), g(x_i')) + d(g(x_i'), h(x_i)) \\ &\leq d(x_i, x_i') + d(y_i', h(x_i)) \\ &\leq \epsilon \end{aligned}$$

⊣

So to check that $\{\bar{g} : g \in \mathrm{Aut}(\langle \boldsymbol{U}_0, \prec \rangle)\}$ is dense in $\mathrm{Iso}(\boldsymbol{U})$, it is enough to show the following.

Lemma 6.20 *Given $x_1, \ldots, x_n, y_1, \ldots, y_n \in U$ such that $x_i \mapsto y_i,\ i = 1, \ldots n$, is an isometry, and given $\epsilon > 0$, there are $x'_1, \ldots, x'_n, y'_1, \ldots y'_n \in U_0$ so that $x'_i \mapsto y'_i$ is an order preserving (with respect to $\prec$) isometry and*

$$d(x'_i, x_i) < \epsilon,\ d(y'_i, y_i) < \epsilon,\ i = 1, \ldots, n.$$

Proof. By induction on n.

$\boldsymbol{n = 1}$: Simply choose $x'_1, y'_1 \in U_0$ with $d(x'_1, x_1) < \epsilon,\ d(y'_1, y_1) < \epsilon$.

$\boldsymbol{n \to n+1}$: Suppose $x_1, \ldots, x_n, x_{n+1}, y_1, \ldots, y_n, y_{n+1} \in U$ are given so that $x_i \mapsto y_i$ is an isometry. By induction hypothesis, find $x'_1, \ldots x'_n, y'_1, \ldots y'_n \in U_0$, so that $x'_i \mapsto y'_i$ is an order preserving isometry and $d(x_i, x'_i) < \epsilon/2,\ d(y_i, y'_i) < \epsilon/2,\ i = 1, \ldots, n$. Let $x^0_{n+1}, y^0_{n+1} \in U_0$ be such that

$$d(x^0_{n+1}, x_{n+1}) < \epsilon/2,\ d(y^0_{n+1}, y_{n+1}) < \epsilon/2.$$

Put $d(x^0_{n+1}, x'_i) = d_i,\ d(y^0_{n+1}, y'_i) = d'_i,\ 1 \le i \le n$. We can of course assume that ϵ is small enough so that $d_i, d'_i > \epsilon$.

Therefore,

$$\begin{aligned} |d_i - d(x_{n+1}, x_i)| = \ & |d(x^0_{n+1}, x'_i) - d(x_{n+1}, x_i)| \\ & \le d(x^0_{n+1}, x_{n+1}) + d(x_i, x'_i) < \epsilon \end{aligned}$$

and

$$|d'_i - d(y_{n+1}, y_i)| < \epsilon,$$

so

$$\begin{aligned} |d_i - d'_i| \ & = |d_i - d(x_{n+1}, x_i) + d(x_{n+1}, x_i) - d(y_{n+1}, y_i) + d(y_{n+1}, y_i) - d'_i| \\ & < 2\epsilon. \end{aligned}$$

Put $e_i = \frac{d_i + d'_i}{2}$, and consider the ordered metric space

$$\langle \{x'_1, \ldots, x'_n, x^0_{n+1}, u\}, d', \prec' \rangle,$$

where $d'(x'_i, x'_j) = d(x'_i, x'_j),\ d'(x'_i, x^0_{n+1}) = d(x'_i, x^0_{n+1}),\ d'(u, x'_i) = e_i$, and $d'(u, x^0_{n+1})$ is any rational number satisfying the inequalities

$$d_i + e_i > 2\epsilon > d'(u, x^0_{n+1}) \ge |d_i - e_i|,\ i = 1, \ldots, n.$$

Notice here that

$$d_i + e_i = \frac{3d_i + d'_i}{2} > 2\epsilon$$

and

$$|d_i - e_i| = \tfrac{|d_i - d'_i|}{2} < \epsilon,$$

so such a number exists. We let $\prec'$ agree with the ordering $\prec$ (of U_0) for $x'_1, \ldots, x'_n,\ x^0_{n+1}$ and $x'_i \prec' u, x^0_{n+1} \prec' u$. We need of course to verify that d' is indeed a metric:

(i) Since $d'(x^0_{n+1}, x'_i) = d_i,\ d'(u, x'_i) = e_i$, we need to check that

$$|d_i - e_i| \leq d'(u, x^0_{n+1}) \leq d_i + e_i,$$

which is given by the definition of $d'(u, x^0_{n+1})$.

(ii) Let $\alpha_{ij} = d(x'_i, x'_j)$. We need to verify that

$$|e_i - e_j| \leq \alpha_{ij} \leq e_i + e_j.$$

We have

$$|d_i - d_j| \leq \alpha_{ij} \leq d_i + d_j,$$

since $d_i = d(x^0_{n+1}, x'_i)$. But also $\alpha_{ij} = d(y'_i, y'_j)$, so we also have

$$|d'_i - d'_j| \leq \alpha_{ij} \leq d'_i + d'_j.$$

Adding and dividing by 2, we get

$$|e_i - e_j| \leq \alpha_{ij} \leq e_i + e_j.$$

So by the properties of $\langle \boldsymbol{U}_0, \prec\rangle$, we can find a point $x'_{n+1} \in U_0$ with $x'_i \prec x'_{n+1},\ i = 1, \ldots n,\ x^0_{n+1} \prec x'_{n+1}$, and $d(x'_{n+1}, x'_i) = e_i,\ d(x'_{n+1}, x^0_{n+1}) = d'(u, x^0_{n+1}) < 2\epsilon$. Similarly we can find $y'_{n+1} \in U_0$ with $y'_i \prec y'_{n+1},\ i = 1, \ldots, n,\ y^0_{n+1} \prec y'_{n+1}$ and $d(y'_{n+1}, y'_i) = e_i,\ d(y'_{n+1}, y^0_{n+1}) < 2\epsilon$. Then $x'_i \mapsto y'_i,\ 1 \leq i \leq n+1$, is an order preserving isometry, and $d(x'_{n+1}, x_{n+1}) \leq d(x'_{n+1}, x^0_{n+1}) + d(x^0_{n+1}, x_{n+1}) < 3\epsilon$ and $d(y'_{n+1}, y_{n+1}) < 3\epsilon$, so the proof is complete. $\dashv$

A result similar to 6.16 can be proved for the ordered integer Urysohn space (where we consider the class of ordered finite metric spaces with integer distances), since Nešetřil [03] has also verified the corresponding Ramsey property. It is also conceivable that one can push those ideas to find a new proof of the result of Gromov and Milman [83] that the unitary group of the infinite-dimensional separable Hilbert space is extremely amenable, as well as a recent strengthening of this result by Pestov [02], who established extreme amenability of the group of (affine) isometries of the same Hilbert space.

7 Universal minimal flows and the ordering property

Consider now a signature $L \supseteq \{<\}$ and put $L_0 = L \setminus \{<\}$. Let $\mathcal{K}$ be a reasonable Fraïssé order class in L and put $\mathcal{K}_0 = \mathcal{K}|L_0$. Then by 5.2 we know that $\mathcal{K}_0$ is a Fraïssé class and if $\boldsymbol{F} = \mathrm{Flim}(\mathcal{K})$, $\boldsymbol{F}_0 = \mathrm{Flim}(\mathcal{K}_0)$, we have $\boldsymbol{F}_0 = F|L_0$. Let $<^{\boldsymbol{F}} = \prec_0$. Put $G_0 = \mathrm{Aut}(\boldsymbol{F}_0)$ and consider the logic action of G_0 on LO, the compact space of linear orderings on $F_0 = F$ (which of course we can identify, if we want, with $\mathbb{N}$). Let $X_{\mathcal{K}}$ be the orbit closure of $\prec_0$, $\overline{G_0 \cdot \prec_0} \subseteq \mathrm{LO}$. We first note the following.

Proposition 7.1 *A linear ordering $\prec$ is in $X_{\mathcal{K}}$ iff for every finite substructure $\boldsymbol{B}_0$ of $\boldsymbol{F}_0$, $\boldsymbol{B} = \langle \boldsymbol{B}_0, \prec |B_0\rangle \in \mathcal{K}$.*

Proof. Assume $\prec \in X_{\mathcal{K}}$ and fix a finite substructure $\boldsymbol{B}_0$ of $\boldsymbol{F}_0$. Then as $\prec \in \overline{G_0 \cdot \prec_0}$, there is $g \in G_0 = \mathrm{Aut}(\boldsymbol{F}_0)$ such that $\prec |B_0 = (g \cdot \prec_0)|B_0$. So if $g^{-1}(\boldsymbol{B}_0) = \boldsymbol{A}_0$, a substructure of $\boldsymbol{F}_0$, and $\boldsymbol{A} = \langle \boldsymbol{A}_0, \prec_0 |A_0\rangle$, which is in $\mathcal{K}$, we have that $g : A_0 \to B_0$ is an isomorphism of $\boldsymbol{A}$ with $\boldsymbol{B} = \langle \boldsymbol{B}_0, \prec |B_0\rangle$, so $\boldsymbol{B} \in \mathcal{K}$.

Conversely, assume that for every finite substructure $\boldsymbol{B}_0$ of $\boldsymbol{F}_0$, $\boldsymbol{B} = \langle \boldsymbol{B}_0, \prec |B_0\rangle \in \mathcal{K}$. Then there is an embedding $\pi : \boldsymbol{B} \to \boldsymbol{F}$. If $\pi(\boldsymbol{B}) = \boldsymbol{A}$, then $\boldsymbol{A}$ is a substructure of $\boldsymbol{F}$ and π is an isomorphism of $\boldsymbol{B}, \boldsymbol{A}$, and thus in particular an isomorphism of $\boldsymbol{B}_0, \boldsymbol{A}_0 = \boldsymbol{A}|L_0$. But $\boldsymbol{B}_0, \boldsymbol{A}_0$ are finite substructures of $\boldsymbol{F}_0$, so, by ultrahomogeneity of $\boldsymbol{F}_0$, there is $g \in \mathrm{Aut}(\boldsymbol{F}_0) = G_0$ extending π^{-1}, so in particular, $\prec |B_0 = (g \cdot \prec_0)|B_0$. Since $\boldsymbol{B}_0$ was arbitrary, this shows that $\prec \in X_{\mathcal{K}}$. ⊣

Definition 7.2 *We call any linear ordering in $X_{\mathcal{K}}$ a $\mathcal{K}$-*admissible *ordering.*

Clearly, $X_{\mathcal{K}}$ is a G_0-flow. We will now derive necessary and sufficient conditions for $X_{\mathcal{K}}$ to be a minimal G_0-flow.

The following concept plays an important role in the Ramsey theory of graphs and hypergraphs, see Nešetřil-Rödl [78], Nešetřil [95, 5.2]. We formulate it here in a general context.

Definition 7.3 *Let $L \supseteq \{<\}$ be a signature, $L_0 = L \setminus \{<\}$, $\mathcal{K}$ a reasonable Fraïssé order class in L and let $\mathcal{K}_0 = \mathcal{K}|L_0$. We say that $\mathcal{K}$ satisfies the* ordering property *if for every $\boldsymbol{A}_0 \in \mathcal{K}_0$, there is $\boldsymbol{B}_0 \in \mathcal{K}_0$ such that for every linear ordering $\prec$ on A_0 and linear ordering $\prec'$ on B_0, if $\boldsymbol{A} = \langle \boldsymbol{A}_0, \prec \rangle \in \mathcal{K}$ and $\boldsymbol{B} = \langle \boldsymbol{B}_0, \prec' \rangle \in \mathcal{K}$, we have $\boldsymbol{A} \leq \boldsymbol{B}$.*

For example, if $\mathcal{K}$ is the class of finite ordered graphs, so that $\mathcal{K}_0 =$ the class of finite graphs, then $\mathcal{K}$ satisfies the ordering property by results of Nešetřil-Rödl (see, e.g., Nešetřil [95, 5.2] or Nešetřil-Rödl [78]).

Now we have

Theorem 7.4 *Let $L \supseteq \{<\}$ be a signature, $L_0 = L \setminus \{<\}$, $\mathcal{K}$ a reasonable Fraïssé order class in L. Let $\mathcal{K}_0 = \mathcal{K}|L_0$, and $\boldsymbol{F} = \mathrm{Flim}(\mathcal{K})$, $\boldsymbol{F}_0 = \mathrm{Flim}(\mathcal{K}_0) = \boldsymbol{F}|L_0$. Let $X_{\mathcal{K}}$ be the set of linear orderings $\prec$ on F $(= F_0)$ which are $\mathcal{K}$-admissible. Let also $G_0 = \mathrm{Aut}(\boldsymbol{F}_0)$. Then the following are equivalent:*

(i) $X_{\mathcal{K}}$ is a minimal G_0-flow.

(ii) $\mathcal{K}$ satisfies the ordering property.

Proof. First let us write down explicitly what (i) means: $X_{\mathcal{K}}$ is a minimal G_0-flow iff for every $\prec \in X_{\mathcal{K}}$ and every $\boldsymbol{A} \in \mathcal{K}$ there is a finite substructure $\boldsymbol{C}_0$ of $\boldsymbol{F}_0$ such that $\boldsymbol{C} = \langle \boldsymbol{C}_0, \prec |C_0 \rangle \cong \boldsymbol{A}$. Indeed this last condition is equivalent to the assertion that $G_0 \cdot \prec$ is dense in $X_{\mathcal{K}}$.

(ii) $\Rightarrow$ (i): Fix $\prec \in X_{\mathcal{K}}$, $\boldsymbol{A} \in \mathcal{K}$ and let $\boldsymbol{A}_0 = \boldsymbol{A}|L_0$. By (ii), find $\boldsymbol{B}_0 \in \mathcal{K}_0$ as in 7.3. We can of course assume that $\boldsymbol{B}_0$ is a substructure of $\boldsymbol{F}_0$. Then we have, since $\boldsymbol{B} = \langle \boldsymbol{B}_0, \prec |B_0 \rangle \in \mathcal{K}$, $\boldsymbol{A} \leq \boldsymbol{B}$. Thus there is a substructure $\boldsymbol{C}$ of $\boldsymbol{B}$ isomorphic to $\boldsymbol{A}$. Clearly, if $\boldsymbol{C}_0 = \boldsymbol{C}|L_0$, $\boldsymbol{C} = \langle \boldsymbol{C}_0, \prec |C_0 \rangle \cong \boldsymbol{A}$ and we are done.

(i) $\Rightarrow$ (ii): Notice first that, in order to verify the ordering property, it is enough to show that for every $\boldsymbol{A} \in \mathcal{K}$ there is $\boldsymbol{B}_0 \in \mathcal{K}_0$ such that for every linear ordering $\prec'_0$ on B_0, if $\boldsymbol{B} = \langle \boldsymbol{B}_0, \prec'_0 \rangle \in \mathcal{K}$, then $\boldsymbol{A} \leq \boldsymbol{B}$. This follows from the JEP for $\mathcal{K}_0$.

So fix $\boldsymbol{A} \in \mathcal{K}$, and for every finite substructure $\boldsymbol{C}_0$ of $\boldsymbol{F}_0$, let

$$X_{\boldsymbol{C}_0} = \{\prec \in X_{\mathcal{K}} : \boldsymbol{A} \cong \langle \boldsymbol{C}_0, \prec |C_0 \rangle\}.$$

Then (i) implies that

$$X_{\mathcal{K}} = \bigcup_{\boldsymbol{C}_0} X_{\boldsymbol{C}_0},$$

so since each $X_{\boldsymbol{C}_0}$ is open, by compactness we have $\boldsymbol{C}_0^1, \dots, \boldsymbol{C}_0^n$ with $X_{\mathcal{K}} = \bigcup_{i=1}^n X_{\boldsymbol{C}_0^i}$, so that $\forall \prec \in X_{\mathcal{K}} \exists 1 \leq i \leq n(\boldsymbol{A} \cong \langle \boldsymbol{C}_0^i, \prec |C_0^i \rangle)$. Let $\boldsymbol{B}_0$ be the (finite) substructure of $\boldsymbol{F}_0$ generated by $\bigcup_{i=1}^n C_0^i$, so that

$$\forall \prec \in X_{\mathcal{K}}(\boldsymbol{A} \leq \langle \boldsymbol{B}_0, \prec |B_0 \rangle).$$

Fix now $\prec_0'$, a linear ordering on B_0, such that $\boldsymbol{B} = \langle \boldsymbol{B}_0, \prec_0' \rangle \in \mathcal{K}$. If we can show that we can extend $\prec_0'$ to a linear ordering $\prec' \in X_{\mathcal{K}}$, then $\boldsymbol{A} \leq \langle \boldsymbol{B}_0, \prec' |B_0 \rangle = \langle \boldsymbol{B}_0, \prec_0' \rangle = \boldsymbol{B}$, and this verifies the ordering property. To find such an extension, consider the ordering $\prec_0 \in X_{\mathcal{K}}$, so that $\boldsymbol{F} = \langle \boldsymbol{F}_0, \prec_0 \rangle$. Then there is a finite substructure $\boldsymbol{D}_0$ of $\boldsymbol{F}_0$ and an isomorphism φ from $\boldsymbol{B}$ to $\boldsymbol{D} = \langle \boldsymbol{D}_0, \prec_0 |D_0 \rangle$. In particular, φ is an isomorphism of $\boldsymbol{B}_0$ with $\boldsymbol{D}_0$, so, since $\boldsymbol{F}_0$ is ultrahomogeneous, there is $g \in \mathrm{Aut}(\boldsymbol{F}_0) = G_0$ extending φ. Then $\prec' = g^{-1} \cdot \prec_0 \in X_{\mathcal{K}}$ and clearly $\prec'$ extends $\prec_0'$. ⊣

We can finally show that $X_{\mathcal{K}}$ is the universal minimal flow of $G_0 = \mathrm{Aut}(\boldsymbol{F}_0)$, when $\mathcal{K}$ has also the Ramsey property.

Theorem 7.5 *Let $L \supseteq \{<\}$ be a signature, $L_0 = L \setminus \{<\}$, $\mathcal{K}$ a reasonable Fraïssé class in L, and let $\mathcal{K}_0 = \mathcal{K}|L_0$ and $\boldsymbol{F} = \mathrm{Flim}(\mathcal{K}), \boldsymbol{F}_0 = \mathrm{Flim}(\mathcal{K}_0) = \boldsymbol{F}|L_0$. Let $G_0 = \mathrm{Aut}(\boldsymbol{F}_0)$ and let $X_{\mathcal{K}}$ be the set of linear orderings of F $(= F_0)$ which are $\mathcal{K}$-admissible. Then if $\mathcal{K}$ has the ordering and Ramsey properties, $X_{\mathcal{K}}$ is the universal minimal flow of G_0.*

Proof. Let $G = \mathrm{Aut}(\boldsymbol{F}) \leq G_0$, a closed subgroup of G_0. By 4.7, G is extremely amenable. Let also $<^{\boldsymbol{F}} = \prec_0$, so that $\boldsymbol{F} = \langle \boldsymbol{F}_0, \prec_0 \rangle$. By definition, $g \cdot \prec_0 = \prec_0$, for all $g \in G$.

We now claim that it is enough to show that if X a metrizable G_0-flow and $x_0 \in X$ is such that $g \cdot x_0 = x_0$, $\forall g \in G$, then there is a homomorphism φ of the minimal G_0-flow $X_{\mathcal{K}}$ to the G_0-flow X with $\varphi(\prec_0) = x_0$. Indeed, let $M(G_0)$ be the universal minimal flow on G_0. To show that $X_{\mathcal{K}}$ is the universal minimal flow, we need to find a homomorphism from $X_{\mathcal{K}}$ to $M(G_0)$. Write $M(G_0) = \lim_{\leftarrow} X_i$, where X_i is a metrizable minimal G_0-flow and let $\pi_{ij} : X_j \to X_i$ for $i \preceq j$, $\pi_i : M(G_0) \to X_i$ be the associated homomorphisms. Now $M(G_0)$ is also a G-flow, so there is $\{x_i^0\} \in M(G_0)$ with $g \cdot \{x_i^0\} = \{x_i^0\}$, for all $g \in G$. Thus $g \cdot x_i^0 = x_i^0$, for all $g \in G$, and all i. It follows that there is a (unique) homomorphism of G_0-flows $\varphi_i : X_{\mathcal{K}} \to X_i$ with $\varphi_i(\prec_0) = x_i^0$. Then $\varphi = (\varphi_i)$ is a homomorphism of $X_{\mathcal{K}}$ to $M(G_0)$ and we are done.

So fix X a metrizable G_0-flow and fix $x_0 \in X$ such that $g \cdot x_0 = x_0$, $\forall g \in G$, which exists since G is extremely amenable. We will find a homomorphism φ of the G_0-flow $X_{\mathcal{K}}$ to the G_0-flow X with $\varphi(\prec_0) = x_0$. First, we let $\varphi(g \cdot \prec_0) = g \cdot x_0$ for $g \in G_0$. This is clearly well defined, since, by definition, the stabilizer of $\prec_0$ in G_0 is exactly G. Consider now any $\prec \in X_{\mathcal{K}}$ and let $\{g_i\}$ be a sequence in G_0 with $g_i \cdot \prec_0 \to \prec$. Then $\{g_i \cdot x_0\}$ is a sequence in the compact metrizable space X, so there is a subsequence $\{i_k\}$ with $g_{i_k} \cdot x_0 \to x \in X$. We put $\varphi(\prec) = x$. To show that this is well-defined, we need to establish the following property:

$(*)$ If $\{g_i\}, \{h_i\}$ are sequences in G_0 and $g_i \cdot \prec_0 \to \prec$, $h_i \cdot \prec_0 \to \prec$ and $g_i \cdot x_0 \to x$, $h_i \cdot x_0 \to x'$, then $x = x'$.

Let d_X be a compatible metric for X. If $x \neq x'$, there is $\epsilon > 0$ such that $d_X(x, x') > \epsilon$. Thus we can assume that $d_X(g_i \cdot x_0,\ h_i \cdot x_0) > \epsilon$, for all i. Let d_r be a right-invariant compatible metric for G_0. Since the action of G_0 on X is continuous, the map $g \mapsto \varphi_g$ from G_0 to $H(X)$, where $\varphi_g(y) = g \cdot y$, is continuous. Now the (compact-open) topology on $H(X)$ is compatible with the metric $d_u(f, g) = \sup_{y \in X} d_X(f(y), g(y))$, so there is $\delta > 0$ such that $d_r(1_{G_0}, g) < \delta \Rightarrow \forall y(d_X(g \cdot y, y) < \epsilon)$, and so $d_r(g, h) < \delta \Rightarrow \forall y(d_X(g \cdot h^{-1} \cdot y, y) < \epsilon) \Rightarrow d_X(g \cdot x_0, h \cdot x_0) < \epsilon$.

We will now choose a standard right-invariant metric for G_0. Without loss of generality, we can assume that $F = F_0 = \mathbb{N}$, so that G_0 is a closed subgroup of S_∞. The following is then a left-invariant compatible metric on S_∞ and thus on G_0 : For $f \neq g$ in S_∞,

$$d_\ell(f, g) = 2^{-k-1}, \text{ where } k \text{ is least with } f(k) \neq g(k).$$

Let

$$d_r(f, g) = d_\ell(f^{-1}, g^{-1})$$

be the corresponding right-invariant compatible metric on S_∞ and G_0.

Next choose a finite substructure $\boldsymbol{A}_0$ of $\boldsymbol{F}_0$ such that for any $f, g \in \mathrm{Aut}(\boldsymbol{F}_0) = G_0$,

$$f|A_0 = g|A_0 \Rightarrow d_\ell(f, g) < \delta.$$

Since $g_i \cdot \prec_0 \to \prec, h_i \cdot \prec_0 \to \prec$, find N large enough so that

$$g_N \cdot \prec_0 |A_0 = h_N \cdot \prec_0 |A_0 = \prec |A_0.$$

Thus for $a, b \in A_0$,

$$g_N^{-1}(a) \prec_0 g_N^{-1}(b) \Leftrightarrow h_N^{-1}(a) \prec_0 h_N^{-1}(b).$$

Let $g_N^{-1} \cdot \boldsymbol{A}_0 = \boldsymbol{B}_0,\ h_N^{-1} \cdot \boldsymbol{A}_0 = \boldsymbol{C}_0$, so that $\boldsymbol{B}_0, \boldsymbol{C}_0$ are finite substructures of $\boldsymbol{F}_0$. Let $\boldsymbol{B} = \langle \boldsymbol{B}_0, \prec_0 |B_0 \rangle,\ \boldsymbol{C} = \langle \boldsymbol{C}_0, \prec_0 |C_0 \rangle$, so that $\boldsymbol{B}, \boldsymbol{C}$ are finite substructures of $\boldsymbol{F}$. Thus $\boldsymbol{B} \cong \boldsymbol{C}$ via the isomorphism $\rho : B \to C$, given by

$$\rho(g_N^{-1}(a)) = h_N^{-1}(a), \quad a \in A_0.$$

Since $\boldsymbol{F}$ is ultrahomogeneous, there is $r \in \mathrm{Aut}(\boldsymbol{F}) = G$ extending ρ, i.e., for $a \in A_0,\ r(g_N^{-1}(a)) = h_N^{-1}(a)$, i.e., $r \circ g_N^{-1}|A_0 = h_N^{-1}|A_0$, thus $d_\ell(r \circ g_N^{-1}, h_N^{-1}) < \delta$, so $d_r(g_N \circ r^{-1}, h_N) < \delta$, and therefore $d_X(g_N \cdot (r^{-1} \cdot x_0), h_N \cdot x_0) < \epsilon$, and since $r^{-1} \cdot x_0 = x_0$ (because $r^{-1} \in G$), we have $d(g_N \cdot x_0, h_N \cdot x_0) < \epsilon$, a contradiction.

Next, we need to verify that $\varphi : X_{\mathcal{K}} \to X$ is continuous. Let $\prec_i \to \prec$ be a converging sequence in $X_{\mathcal{K}}$ in order to show that $\varphi(\prec_i) \to \varphi(\prec)$. We can assume of course that $\varphi(\prec_i)$ converges to some $x \in X$. Let $\{g_i\}$ in G_0 be such that $g_i \cdot \prec_0 \to \prec$ and $g_i \cdot x_0 \to \varphi(\prec)$, and, by definition of $\varphi(\prec_i)$, find $h_i \in G_0$ with $d'(h_i \cdot \prec_0, \prec_i) < \frac{1}{i+1}$ and $d_X(\varphi(h_i \cdot \prec_0), \varphi(\prec_i)) < \frac{1}{i+1}$, where d' is a compatible metric for $X_{\mathcal{K}}$. Thus $h_i \cdot \prec_0 \to \prec$ and $\varphi(h_i \cdot \prec_0) = h_i \cdot \varphi(\prec_0) = h_i \cdot x_0 \to x$. So $g_i \cdot \prec_0 \to \prec, h_i \cdot \prec_0 \to \prec, g_i \cdot x_0 \to \varphi(\prec), h_i \cdot x_0 \to x$, so, by $(*)$, $\lim_i \varphi(\prec_i) = x = \varphi(\prec)$.

Finally, we check that φ is a G_0-map: Let $\prec \in X_{\mathcal{K}}$ and choose g_i in G_0 with $g_i \cdot \prec_0 \to \prec$. Then for $g \in G_0, gg_i \cdot \prec_0 \to g \cdot \prec$, so

$$\varphi(g \cdot \prec) = \lim_i \varphi(gg_i \cdot \prec_0) = \lim_i gg_i \cdot \varphi(\prec_0) = g \cdot \lim_i g_i \cdot \varphi(\prec_0)$$

$$= g \cdot \lim_i \varphi(g_i \cdot \prec_0) = g \cdot \varphi(\prec).$$

. ⊣

8 Calculating minimal flows

We now apply the results in §6, §7 to compute the universal minimal flows of several automorphism groups.

Consider first the classes $\mathcal{OGR}$, of finite ordered graphs, $\mathcal{O}\mathcal{F}orb(K_n)$, $n = 3, 4, \ldots$ of K_n-free finite ordered graphs, $\mathcal{OEQ}_1$ of complete finite ordered graphs, $\mathcal{OH}_{L_0}$ of finite ordered hypergraphs of type L_0, $\mathcal{O}\mathcal{F}orb(\mathcal{A})$ of finite ordered hypergraphs of type L_0 that omit $\mathcal{A}$, where $\mathcal{A}$ is a class of finite irreducible hypergraphs of type L_0, and $\mathcal{OM}_{\mathbb{Q}}$ of finite ordered metric spaces with rational distances.

Each of these classes satisfies the ordering property and this follows easily from the fact (already used above in §6) that each of these classes satisfies the Ramsey property. (The case of $\mathcal{OEQ}_1$ is of course trivial.) This is done via a standard Sierpinski-style of coloring obtained by interpolating two orderings. To see how this is done in the case of $\mathcal{K} = \mathcal{O}\mathcal{F}orb(\mathcal{A})$ consider an $\boldsymbol{A}_0 \in \mathcal{K}_0$ ($\equiv \mathcal{K} \restriction (L \setminus \{<\})$ and a linear ordering $\prec$ of A_0. Recall that in this case $L_0 = L \setminus \{<\}$ is simply a finite sequence $\{R_i\}_{i \in I}$ of relation symbols of various finite arities. We may assume that the signature L_0 contains a binary relation symbol R_0 (or else we expand L_0) and we may assume that $\mathcal{A}$ contains neither 1-element nor 2-element structures. Enlarge $\boldsymbol{A}_0$ to $\boldsymbol{A}_0^*$ by inserting a path of size 3 between every pair of elements of A_0 that are consecutive in the ordering $\prec$. Let $\boldsymbol{B}_0^*$ be obtained from two disjoint copies of $\boldsymbol{A}_0^*$ with no relationship between them, and let $\boldsymbol{B}_0^* = \langle \boldsymbol{B}_0^*, \prec^* \rangle$ be the ordered version of $\boldsymbol{B}_0^*$, where $\prec^*$ agrees with $\prec$ on the one copy of $\boldsymbol{A}_0^*$ and agrees with $\succ$ on the other. Applying the Ramsey property for $\mathcal{K}$ we find $\boldsymbol{C} \in \mathcal{K}$ with

$$\boldsymbol{C} \to (\boldsymbol{B}^*)^2_2$$

(Here 2 in the exponent denotes the ordered 2-element member of $\mathcal{K}$ in which the two points are R_0-related.) Let $\boldsymbol{C}_0$ be the L_0-reduct of $\boldsymbol{C}$. We claim that for every order $\prec'$ on $\boldsymbol{C}_0$, the structure $\langle \boldsymbol{C}_0, \prec' \rangle$ contains a substructure isomorphic to $\langle \boldsymbol{A}_0, \prec \rangle$ and this will clearly finish the checking the ordering property for $\mathcal{K}$. Let $<_C$ be the ordering of $C = C_0$ such that $\boldsymbol{C} = (\boldsymbol{C}_0, <_C)$ and consider the coloring of the R_0-edges of $\boldsymbol{C}$ in two colors according to whether the two orderings $<_C$ and $\prec'$ agree or disagree on a given R_0-edge. Let $\boldsymbol{B}'$ be a homogeneous substructure of $\boldsymbol{C}$ such that $\boldsymbol{B}^* = \langle \boldsymbol{B}_0^*, \prec^* \rangle \cong \boldsymbol{B}' = \langle \boldsymbol{B}_0', <_C |B_0' \rangle$. Then $\langle \boldsymbol{B}_0', \prec' |B_0' \rangle$ contains a substructure isomorphic to $\langle \boldsymbol{A}_0, \prec \rangle$. A similar argument will deduce the ordering property for finite ordered metric spaces with rational distances from the corresponding Ramsey property. It should be mentioned, however, that typically the ordering property for a given class of structures is a result considerably easier to prove from the corresponding Ramsey property and can frequently be proved directly (see Nešetřil-Rödl [78] and Nešetřil [95]).

In each one of the above cases, the space of admissible orderings is of course the space LO of all linear orderings on $\mathbb{N}$ (which we identify with the universe of the Fraïssé limit of each class). Thus, by 7.5, we have

Theorem 8.1 *For each one of the groups below, its universal minimal flow is the space* LO *of linear orderings on* $\mathbb{N}$*, so in particular it is metrizable:*

(i) The automorphism group of the random graph.

(ii) The automorphism group of the random K_n*-free graph.*

(iii) (Glasner-Weiss [02]) S_∞*, the permutation group of* $\mathbb{N}$.

(iv) The automorphism group of the random hypergraph of type L_0*.*

(v) The automorphism group of the random $\mathcal{A}$*-free hypergraph of type* L_0*, where* $\mathcal{A}$ *is a class of irreducible finite hypergraphs of type* L_0*.*

(vi) The isometry group of the rational Urysohn space $\boldsymbol{U}_0$*.*

Consider now the classes of convexly ordered finite equivalence relations, naturally ordered finite-dimensional spaces over a finite field F, and naturally ordered finite Boolean algebras. Each is easily seen to satisfy the ordering property. So we have:

Theorem 8.2 *(i) The automorphism group of the structure* $\langle \mathbb{N}, E\rangle$*, where* E *is an equivalence relation on* $\mathbb{N}$ *with infinitely many classes, each of which is infinite, has as universal minimal flow the space of all convex orderings on* $\mathbb{N}$*, i.e., all orderings on* $\mathbb{N}$ *for which each* E*-class is convex.*

(ii) Let $\boldsymbol{V}_F$ *be the* $\aleph_0$*-dimensional vector space over a finite field* F*. The universal minimal flow of its automorphism group (i.e.,* $\mathrm{GL}(\boldsymbol{V}_F)$*) is the space of all orderings on* V_F*, whose restrictions to finite-dimensional subspaces are natural.*

(iii) Let $\boldsymbol{B}_\infty$ *be the countable atomless Boolean algebra. The universal minimal flow of its automorphism group is the space of all orderings on* B_∞*, whose restrictions to finite subalgebras are natural.*

In particular all these universal minimal flows are metrizable.

The question of whether the universal minimal flow of $\mathrm{GL}(\boldsymbol{V}_F)$ is non-trivial was brought to one of the authors' attention by Pierre de la Harpe.

Note that, by Stone duality, $\boldsymbol{B}_\infty$ can be identified with the algebra of clopen subsets of $2^{\mathbb{N}}$ and that every $g \in \mathrm{Aut}(\boldsymbol{B}_\infty)$ determines and is uniquely determined by a homeomorphism $\sigma(g) \in H(2^{\mathbb{N}})$. In Glasner-Weiss [03] there is another representation of the universal minimal flow of $H(2^{\mathbb{N}})$. They showed that the space $\Phi(2^{\mathbb{N}})$ of all maximal chains of closed subsets of $2^{\mathbb{N}}$, defined by Uspenskij [00], can serve as the universal minimal flow of the group $H(2^{\mathbb{N}})$. The existence of an isomorphism between the space $\mathcal{N}(\boldsymbol{B}_\infty)$, of all orderings of $\boldsymbol{B}_\infty$ whose restrictions to finite subalgebras are natural, and $\Phi(2^{\mathbb{N}})$ is of course a consequence of the uniqueness of the universal minimal flow but we exhibit below an explicit one.

Theorem 8.3. *There exists an (explicit) homeomorphism $\varphi : \Phi(2^{\mathbb{N}}) \to \mathcal{N}(\boldsymbol{B}_\infty)$ such that:*
$\varphi(\sigma(g) \cdot x) = g \cdot \varphi(x)$ *for* $x \in \Phi(2^{\mathbb{N}}),\ g \in \mathrm{Aut}(\boldsymbol{B}_\infty)$.

Proof. Given a maximal chain $\mathcal{F}$ of closed subsets of $2^{\mathbb{N}}$, for every clopen subset A of $2^{\mathbb{N}}$, let

$$F_A = \bigcap\{F \in \mathcal{F} : F \cap A \neq \emptyset\}$$

By the maximality of $\mathcal{F}$, $A \cap F_A$ is a singleton. Note that if A is included in B, then F_A is included in F_B, though they can also be equal. Note however that if A and B are disjoint, then F_A and F_B are different, so for each finite Boolean algebra $\boldsymbol{B}$ contained in $\boldsymbol{B}_\infty$, we have a total ordering of the atoms of $\boldsymbol{B}$ and this induces the antilexicographical ordering on the Boolean algebra $\boldsymbol{B}$. These orderings cohere and produce an ordering $<_{\mathcal{F}}$ of $\boldsymbol{B}_\infty$ which is in $\mathcal{N}(B_\infty)$. Let $\varphi(F) =<_{\mathcal{F}}$. This defines a homeomorphism $\varphi : \Phi(2^{\mathbb{N}}) \to \mathcal{N}(\boldsymbol{B}_\infty)$ having the required property.

$\dashv$

9 Concluding remarks and problems

One of the two main ingredients in our proofs of extreme amenability of automorphism groups are the results of the corresponding structural Ramsey theory. It is therefore natural to pose the following problem.

Problem 9.1. *Find alternate proofs (that use the methods of topological dynamics itself as well as the intrinsic geometry of the acting groups) that the automorphism groups of any of the following structures are extremely amenable:*
(i) The rationals with the usual ordering.
(ii) The random ordered graph.
(iii) The random K_n-tree ordered graph, ($n = 3, 4, \dots$).

(iv) The random $\mathcal{A}$-free ordered hypergraph of type L_0, for any class $\mathcal{A}$ of irreducible finite hypergraphs of type L_0.

(v) The ordered rational Urysohn space.

(vi) The $\aleph_0$-dimensional vector space over a finite field with the canonical ordering.

(vii) The countable atomless Boolean algebra with the canonical ordering.

A positive solution to this problem could enhance the already existing tradition of using the methods of topological dynamics to prove results of Ramsey theory initiated in the 1970's by Furstenberg and Weiss.

We have also seen above that the extreme amenability of the automorphism group of an ultrahomogeneous countable structure is equivalent to a corresponding finite Ramsey-theoretic result. This leads us to the following problem.

Problem 9.2. *In each of the cases (i)–(vii) of Problem 9.1, find the topological dynamics analog of a corresponding infinite Ramsey-theoretic result.*

Some explanations of what we mean by the "corresponding *infinite* Ramsey-theoretic result" are in order. Let us explain this in the case of 9.1 (i), i.e., the rationals with the usual ordering. While the finite Ramsey Theorem is the exact analog of the fact that $\mathrm{Aut}(\langle \mathbb{Q}, <\rangle$ is extremely amenable, the corresponding infinite Ramsey-theoretic result may not be the infinite Ramsey Theorem but the considerably harder result due to D. Devlin [79] (see also Todorcevic [03]) which says the following: There is a sequence $\{t_k\}$ with the following property: For each k, there is a coloring

$$c_k : [\mathbb{Q}]^k \to \{1, 2, \ldots, t_k\}$$

such that for every $X \subseteq \mathbb{Q}$ order-isomorphic to $\mathbb{Q}$ and every $1 \leq t \leq t_k$ there exists $\{x_1, \ldots, x_k\} \subseteq X$ such that $c_k(x_1, \ldots, x_k) = t$. (Here $[A]^k$ denotes the set of k-element subsets of A.) On the other hand, for all positive integers k and ℓ and every

$$c : [\mathbb{Q}]^k \to \{1, 2, \ldots, \ell\}$$

there exist $X \subseteq \mathbb{Q}$ order-isomorphic to $\mathbb{Q}$ such that c takes at most t_k values on $[X]^k$. The sequence $\{t_k\}$ is the well-known sequence of *tangent numbers* (because $t_k = T_{2k+1}$, where $\tan z = \sum_{n=0}^{\infty} T_n z^n/n!$). It starts as $t_1 = 1, t_2 = 2, t_3 = 16, t_4 = 272, \ldots$ (see Knuth-Buckholtz [67]). The corresponding infinite Ramsey-theoretic result about the random (ordered) graph is just emerging (see Pouzet-Sauer [96] and Sauer [03]), though the corresponding infinite Ramsey theory in cases (iii)–(vii) has not yet been developed and a solution to Problem 9.2 might be quite illuminating in this project.

One can ask similar questions for other kinds of extremely amenable topological groups not directly covered by the list 9.1 (i)–(vii). Consider, for example, the Gromov-Milman [83] result that the unitary group $U(\ell_2)$ is extremely amenable. The extreme amenability of $U(\ell_2)$ implies the following property: If f is a uniformly continuous function on the unit sphere S_{ℓ_2} of ℓ_2 with values in some $\mathbb{R}^n$, then for every $\epsilon > 0$ and every compact subset K of S_{ℓ_2} there is $u \in U(\ell_2)$ such that the oscillation of f on $u(K)$ is $< \epsilon$. (See Milman and Schechtman [86], Milman [88] or Gromov [83].) Does there exist a topological dynamics property of $U(\ell_2)$ that has similar implications about ∞-dimensional subsets of S_{ℓ_2} such as for example the unit spheres of infinite-dimensional subspaces of ℓ_2? It should be mentioned that the exact infinite-dimensional analog of the above property of spheres is impossible as demonstrated by Odell and Schlumprecht [94] in their solution of the famous distortion problem for ℓ_2, from which it follows that there exists a uniformly continuous f on S_{ℓ_2} such that f has oscillation 1 in every unit sphere of an infinite-dimensional subspace of ℓ_2. But perhaps a topological dynamics property of $U(\ell_2)$ could throw a new light on the distortion property of the Hilbert space ℓ_2.

In particular, does there exist a property of topological groups, which is true of the automorphism group of the rationals (with the usual ordering), and this implies one of the infinite Ramsey-theoretic results mentioned above, but it is not true of the unitary group of ℓ_2, and this in turn implies the distortion property of ℓ_2?

Finally, in connection with the extreme amenability of $U(\ell_2)$ and the group of isometries of ℓ_2 (see the end of Section 6), we would like to ask whether a new proof of these results can be found based on Ramsey theory, as was done in Section 6, (***E***) for the isometry group of the Urysohn space.

Appendix 1. A new proof of Veech's theorem

Veech's theorem (Theorem 2.2.1 in Veech [77]) is an important result of topological dynamics, asserting that every locally compact group acts freely on a suitable compact space. Alternative proofs of this result can be found in Adams and Stuck [93] (in the second countable case) and in Pym [99].

The latter author notes that his proof is 'really the same,' but it emphasizes different features of the original idea. Here we offer yet another proof of Veech's theorem, which, once again, can be seen as a mere new 'repackaging' of one and the same construction, putting emphasis on different aspects of the construction, this time metric and combinatorial ones, and thus making it, perhaps, a little bit more accessible.

Lemma A1.1. *Let G be a locally compact group, and let $g \in G$, $g \neq e$. There exists a right invariant continuous pseudometric d on G, bounded by 1 and such that $0 < d(e,g) < 1$ and the closure of the open ball of unit radius is compact. (Here e is the identity of G.)*

Proof. Let ν be a left-invariant Haar measure on G. For a $f \in L^2(G,\nu)$ and $h \in H$, define ${}^h f \in L^2(G,\nu)$ via

$$ {}^h f(x) := f(h^{-1}x). $$

The correspondence

$$ G \ni h \mapsto [f \mapsto {}^h f] \in U(L^2(G,\nu)) $$

is a strongly continuous unitary representation of G, the left regular representation.

Case 1: $g^2 \neq e$. Choose a symmetric compact neighborhood of the identity, V, in G, with the property $g, g^2 \notin V^2$, and a function $f \in L^2(G,\nu)$ supported on V and such that $\|f\| = 1$ (here and in the sequel, the norm denotes the L^2-norm). Now let $\phi = f + {}^{g^{-1}}f$. This function is supported on $V \cup g^{-1}V$, and since the supports of f and ${}^{g^{-1}}f$ are disjoint, one has $f \perp {}^{g^{-1}}f$ and $\|\phi\| = \sqrt{2}$.

For each $x, y \in G$, define

$$ \rho(x,y) := \|{}^{x^{-1}}\phi - {}^{y^{-1}}\phi\| \equiv \|\phi - {}^{yx^{-1}}\phi\|. $$

This ρ is a right-invariant continuous pseudometric on G, bounded by $2\sqrt{2}$. Notice that if the translates of ϕ by x^{-1} and by y^{-1} are orthogonal, then $\rho(x,y) = 2$. It follows that, if for some $h \in G$ one has $\rho(e,h) < 2$, then

$$ (V \cup g^{-1}V) \cap (h^{-1}V \cup h^{-1}gV) \neq \emptyset, $$

which is, considering four possible cases, equivalent to

$$ h \in V^2 \cup gV^2 \cup V^2 g \cup gV^2 g. $$

Consequently, the open ball $\mathcal{O}_2(e)$ of radius 2 is contained in the compact set on the r.h.s. above and thus has compact closure.

Notice that

$$\begin{aligned}\rho(e,g) &= \|\phi - {}^{g^{-1}}\phi\| \\ &= \|f + {}^{g^{-1}}f - {}^{g^{-1}}f - {}^{g^{-2}}f\| \\ &= \|f - {}^{g^{-2}}f\| \\ &= \sqrt{2},\end{aligned}$$

because the supports of f and ${}^{g^{-2}}f$ are disjoint.

Now define a new pseudometric by

$$d(x,y) = \frac{1}{2}\min\{\rho(x,y), 2\}.$$

This d satisfies all the requirements of the lemma. Indeed, d is right-invariant, continuous, bounded by 1, the open 1-ball has compact closure, and $0 < d(e,g) = \sqrt{2}/2 < 1$.

Case 2: $g^2 = e$. Let V be a compact symmetric neighborhood of identity with the property $g \notin V^2$. Let f be, as before, of L^2-norm one and supported on V, and let

$$\phi = f + 2 \cdot {}^{g}f.$$

Notice that $\|\phi\| = \sqrt{5}$. Define ρ via

$$\rho(x,y) := \|{}^{x^{-1}}\phi - {}^{y^{-1}}\phi\| \equiv \|\phi - {}^{yx^{-1}}\phi\|.$$

The values of this right-invariant continuous pseudometric are bounded by $2\sqrt{5}$. If $\rho(x,y) < \sqrt{10}$, then by necessity the x^{-1} and y^{-1}-translates of ϕ are not orthogonal, which, as before, implies that the closure of the open ball of radius $\sqrt{10}$ is compact.

One has (remembering that $g^{-1} = g$)

$$\begin{aligned}\rho(e,g) &= \|\phi - {}^{g}\phi\| \\ &= \|f + 2\,{}^{g}f - {}^{g}f - 2f\| \\ &= \|{}^{g}f - f\| \\ &= \sqrt{2}.\end{aligned}$$

Defining a new pseudometric d by

$$d(x,y) = \frac{1}{\sqrt{10}} \min\{\rho(x,y), \sqrt{10}\}$$

completes the argument, noticing that $0 < d(e,g) = 1/\sqrt{5} < 1$.

Veech's Theorem. *Every locally compact group G acts freely on the greatest ambit $\mathcal{S}(G)$.*

Proof. Let G be a locally compact group. Let $g \in G$, $g \neq e$ be an arbitrary element different from identity. We will show that g has no fixed points in the greatest ambit of G.

(A) Using Lemma A1.1, choose a right invariant continuous pseudometric d on G, bounded by 1 and such that $0 < d(e,g) < 1$ and the closure, K, of the open ball $\mathcal{O}_1(e)$ of unit radius formed with regard to d is compact. Notice that both K and K^2 are symmetric compact subsets of G.

(B) Now let $0 < \varepsilon < \frac{1}{13}$ be an arbitrary but fixed positive number satisfying

$$9\varepsilon < d(e,g) < 1 - 4\varepsilon. \tag{1}$$

By force of Zorn's lemma, there exists a maximal subset $A \subseteq G$ with the property that whenever $a, b \in A$ and $a \neq b$, one has $\mathcal{O}_\varepsilon(a) \cap \mathcal{O}_\varepsilon(b) \neq \emptyset$. It is immediate that such an A is also a 2ε-net in G, that is, each element $x \in G$ is at a distance less than 2ε from some point in A.

(C) Define a graph, Γ, whose vertex set is A and such that two vertices, $a, b \in A$, are adjacent if and only if $a \neq b$ and $ab^{-1} \in K^2$. (Equivalently: $ba^{-1} \in K^2$, $a \in K^2 b$, $b \in K^2 a$.)

Let $\varkappa$ denote the cardinality of an arbitrary finite family, γ, of open balls of radius ε covering the compact set K^2. Any family δ of pairwise disjoint open balls of radius ε with centers in K^2 has cardinality not exceeding $\varkappa$, because every mapping assigning to every $B \in \delta$ a ball B' in the family γ containing the center of B is an injection. If $a, b_1, \ldots, b_n \in A$ and a is adjacent to each b_i, then $b_i a^{-1} \in K^2$ for all i and the ε-balls centered at $b_i a^{-1}$, $i = 1, 2, \ldots, n$, are pairwise disjoint (the metric d is right-invariant). It follows that $n \leq \varkappa$, and therefore Γ has finite degree, not exceeding $\varkappa$.

(D) As a consequence, the vertices of Γ can be colored with at most $\varkappa + 1$ colors in such a way that no two adjacent vertices have the same color. Let $A = \bigsqcup_{i=1}^m A_i$, $A_i \neq \emptyset$, be such a coloring, where $m \leq \varkappa + 1$.

(E) For each $i = 1, 2, \dots, m$ define a function $f_i \colon G \to \mathbb{R}$ via

$$f_i(x) := d(x, A_i),$$

the distance from $x \in G$ to A_i. This f_i is a 1-Lipschitz function, bounded by 1. In particular, it is a (bounded) right uniformly continuous function on the topological group G, and thus an element of $\mathrm{RUC}^b(G)$.

(F) Let $a, b \in A$ be such that $d(a, b) < 1$. For some $i = 1, 2, \dots, m$, one has $a \in A_i$ and, since a and b are adjacent, also $b \notin A_i$. Moreover, we claim that a is the only element of A_i at a distance < 1 from b and thus the nearest neighbor to b in A_i. Indeed, assuming that there is a $c \in A_i$ with $c \neq a$ and $d(c, b) < 1$, one has $ba^{-1} \in K$, $cb^{-1} \in K$, and thus $ca^{-1} = cb^{-1}ba^{-1} \in K^2$, meaning that c and a are adjacent, in contradiction with the choice of the coloring.

Taking into account that each function f_j, $j = 1, 2, \dots, m$ is 1-Lipschitz, we conclude that

$$\max_{j=1}^{m} |f_j(a) - f_j(b)| = d(a, b) \text{ whenever } a, b \in A \text{ and } d(a, b) < 1. \tag{2}$$

(G) Define the mapping $f \colon G \to \ell^\infty(m)$, where $\ell^\infty(m) = \mathbb{R}^m$ with the max norm $\|.\|_\infty$, as $f = (f_1, f_2, \dots, f_m)$. The property in the Eq. (2) can now be restated as follows:

$$\forall a, b \in A, \quad (d(a, b) < 1) \Rightarrow \|f(a) - f(b)\|_\infty = d(a, b). \tag{3}$$

(H) Let $x \in G$ be arbitrary. There are $a, b \in A$ such that $d(x, a) < 2\varepsilon$ and $d(gx, b) < 2\varepsilon$. Since $d(gx, x) = d(g, e)$, it follows by the triangle inequality that

$$5\varepsilon < d(a, b) < 1.$$

Eq. (3) implies that

$$\|f(a) - f(b)\|_\infty = d(a, b) > 5\varepsilon,$$

and by the triangle inequality

$$\begin{aligned} \|f(x) - f(gx)\|_\infty &\geq \|f(a) - f(b)\|_\infty - 4\varepsilon \\ &> \varepsilon. \end{aligned}$$

(I) The mapping $f\colon G \to \mathbb{R}^m$, being right uniformly continuous and bounded, admits a unique continuous extension, $\bar{f}$, over the greatest ambit $\mathcal{S}(G)$ of G. By continuity of the extension, the mapping $\bar{f}$ has the property:

$$\forall x \in \mathcal{S}(G), \quad \|f(x) - f(g \cdot x)\|_\infty \geq \varepsilon > 0.$$

In particular, it means that for every $x \in \mathcal{S}(G)$, the elements x and $g \cdot x$ are by necessity different, and the action by g on the greatest ambit is fixed point-free.

Appendix 2. Non-metrizability of the universal minimal flow for locally compact groups

A subset A of a group G is called (*discretely*) *left syndetic* if finitely many left translates of A cover G, that is, there are an $n \in \mathbb{N}_+$ and elements $h_1, h_2, \ldots, h_n \in G$ with $\bigcup_{i=1}^n h_i A = G$.

Let us recall a simple and well-known fact from topological dynamics.

Lemma A2.1. *Let G be a topological group, and let M be a minimal compact G-flow. Let $W \subseteq M$ be a non-empty open subset, and let $\xi \in M$. Then the set*

$$\widetilde{W} := \{g \in G\colon g \cdot \xi \in W\}$$

is discretely left syndetic in G.

Proof. The translates $h \cdot W$, $h \in G$ form an open cover of M, because otherwise there would be a point $\zeta \in M$ whose G-orbit misses W, in contradiction with the assumed minimality of M. Choose finitely many elements, $h_1, h_2, \ldots, h_n \in G$ with the property that $h_i \cdot W$, $i = 1, 2, \ldots, n$, cover M. It remains to notice that for every $h \in G$, $\widetilde{h \cdot W} = h\widetilde{W}$, and so the left translates $h_i\widetilde{W}$, $i = 1, 2, \ldots, n$, cover G.

Theorem A2.2. *The universal minimal flow $M(G)$ of a non-compact locally compact group G is non-metrizable.*

Proof. Let G be a locally compact group. Let U be a neighborhood of identity whose closure is compact. Use Zorn's lemma to choose a maximal subset $X \subseteq G$ with the property that $\{Ux\colon x \in X\}$ is a disjoint family.

According to Pym [99] (the Local Structure Theorem on p. 172), the closure $\overline{X}$ of X in the greatest ambit $\mathcal{S}(G)$ is homeomorphic to βX, the Stone-Čech compactification of the discrete space X. Also, if V is an open subset of G with $\overline{V} \subseteq U$, then the subspace $V \cdot \overline{X}$ is open in $\mathcal{S}(G)$ and homeomorphic with $V \times \overline{X}$ (and consequently with $V \times \beta X$) under the map $(v, \xi) \mapsto v \cdot \xi$, $v \in V$, $\xi \in \overline{X}$. Finally, given any $\xi \in \mathcal{S}(G)$, U and X can be chosen so that $\xi \in \overline{X}$.

Denote by M an isomorphic copy of the universal minimal flow $M(G)$ sitting inside the greatest ambit $\mathcal{S}(G)$. Assume from now on that U and X as above are chosen in such a way that $\overline{X} \cap M \neq \emptyset$. Let also V and V_1 be open neighborhoods of identity in G with the property $V \subseteq \overline{V} \subseteq V_1 \subseteq \overline{V_1} \subseteq U$. Since $V \cdot \overline{X}$ is open in $\mathcal{S}(G)$, it follows that $(V \cdot \overline{X}) \cap M$ is (non-empty and) open in M.

Assume now that M is metrizable, in order to deduce that G is compact.

The closed subspace $(\overline{V} \cdot \overline{X}) \cap M$ of M is also metrizable and compact. The second coordinate projection, proj_2, from $V_1 \cdot \overline{X} \cong V_1 \times \beta X$ to $\overline{X} \cong \beta X$ is continuous, and therefore the image $K = \mathrm{proj}_2((\overline{V} \cdot \overline{X}) \cap M)$ is a compact metrizable subspace of the extremally disconnected space βX.

Since an extremally disconnected space does not contain any nontrivial convergent sequences (see e.g. Engelking [77], Exercise 6.2.G.(a) on p. 456), it follows that K is finite. Consequently, for each $\kappa \in K$ the subset $(V \cdot \kappa) \cap M$ is open in $(V \cdot \overline{X}) \cap M$, and we conclude that for some $\kappa' \in \overline{X}$ the set $W = (V \cdot \kappa') \cap M$ is non-empty and open in $(V \cdot \overline{X}) \cap M$ and therefore in M itself.

Let $\xi \in W = (V \cdot \kappa') \cap M$ be arbitrary. By Lemma A2.1, the set

$$\widetilde{W} = \{g \in G \colon g \cdot \xi \in W\}$$

is discretely left syndetic in G. For some $v \in V$, one has $\xi = v \cdot \kappa'$. The set

$$W^{\ddagger} := \{g \in G \colon gv \cdot \kappa' \in V \cdot \kappa'\}$$

is bigger than $\widetilde{W}$ and therefore also discretely left syndetic. Since the action of G on the greatest ambit $\mathcal{S}(G)$ is free by Veech's theorem, the condition $gv \cdot \kappa' \in V \cdot \kappa'$ is equivalent to $gv \in V$ and, in its turn, implies $g \in V^2$. It follows that $W^{\ddagger} \subseteq V^2$.

The compact set $(\overline{V})^2$ contains V^2 and is therefore discretely left syndetic as well. Consequently, G is compact.

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