Appendix A

PARTITIONS OF COUNTABLE HOMOGENEOUS SYSTEMS

Norbert Sauer, University of Calgary, Department of Mathematics

Appeared as an appendix in the new printing of the book: Roland Fraïssé *Theory of relations, Revised edition*. Studies in Logic and the Foundations of Mathematics, 145. North-Holland Publishing Co., Amsterdam, 2000. ii+451pp.

In this Appendix several increasingly severe notions of divisibility of homogeneous relational systems are introduced. Those divisibility properties are related to the amalgamationtype of the age of the homogeneous relational system. The divisibility of some of the more prominent homogeneous relations is investigated.

A *relational language* $\mathbf{L}$ is a set $\{R_i \mid i \in I\}$ of relation symbols. Each of the relation symbols R_i is associated with a number $n_i \in \omega$, the *arity* of R_i. The set $\{n_i \mid i \in I\}$ is the *arity* of $\mathbf{L}$. A *relational system* S *in the language* $\mathbf{L}$ is a set $\langle S\rangle$, the *base* of S, together with a set $\{R_i^S \mid i \in I\}$ so that $R_i^S \subseteq \langle S\rangle^{n_i}$ for every $i \in I$. If $(x_0, x_1, \ldots, x_{n_i-1}) \in \langle S\rangle^{n_i}$ then we write $R_i^S(x_0, x_1, \ldots, x_{n_i-1}) = +$ if $(x_0, x_1, \ldots, x_{n_i-1}) \in R_i^S$ and we write $R_i^S(x_0, x_1, \ldots, x_{n_i-1}) = -$ if $(x_0, x_1, \ldots, x_{n_i-1}) \notin R_i^S$. The relations R_i^S with base $\langle S\rangle$ are the *components of* S. (1.7.) A relational language $\mathbf{K}$ which is a subset of $\mathbf{L}$ is a *reduct* of $\mathbf{L}$ and $\mathbf{L}$ is then an *expansion* of $\mathbf{K}$.

This Appendix is a continuation of Chapter 12 with more emphasis on relational systems instead of single relations. In order to become familiar with the use of the notions some of Chapter 12 is repeated. The relational system R is *denumerable* if the base of R, that is $\langle R\rangle$ is denumerable. All relational systems and languages considered in this Appendix will be denumerable or finite.

A.1 SOME PROMINENT HOMOGENEOUS RELATIONS

Let S and T be two relational systems in the language $\mathbf{L} = \{R_i \mid i \in I\}$ of arity $\{n_i \mid i \in I\}$. The injection f of $\langle S\rangle$ to $\langle T\rangle$ is an *embedding* of S to T if for all $i \in I$ and $(x_0, x_1, \ldots, x_{n_i-1}) \in \langle S\rangle^{n_i}$

$$R_i^S(x_0, x_1, \ldots, x_{n_i-1}) = + \text{ if and only if } R_i^T(f(x_0), f(x_1), \ldots, f(x_{n_i-1})) = +$$

or equivalently

$$(x_0, x_1, \ldots, x_{n_i-1}) \in R_i^S \text{ if and only if } (f(x_0), f(x_1), \ldots, f(x_{n_i-1})) \in R_i^T.$$

That is f is an embedding of S if it is an embedding of every component of S. If the embedding f of S to T is a surjection then it is an *isomorphism*. The relational system A is a *copy* of the relational system B if it has the same relational language and there is an isomorphism from B to A. If A is a restriction of the relational system C and a copy of B then it is a *copy of B in C*.

For $A \subseteq \langle S \rangle$ the *restriction of S to A*, S/A, is the relational system with the same relational language $\mathbf{L}$ so that $R_i^{S/A} = R_i^S \cap A^{n_i}$. That is the components of S/A are the restrictions of the components of S to A. (1.7.1.) If $u \in \langle S \rangle$ then $S - u$ is the restriction of S to $\langle S \rangle - \{u\}$. The relational system R is an *extension* of S if S is a restriction of R. The *skeleton* of S is the set of restrictions of S to finite subsets of $\langle S \rangle$. A *local automorphism* of S is an isomorphism of an element of the skeleton of S to an element of the skeleton of S. An *automorphism* of S is an isomorphism of S to S.

The relational systems A and B of the same language are *compatible* (1.7.2) if the restriction of A to $\langle A \rangle \cap \langle B \rangle$ is equal to the restriction of B to $\langle A \rangle \cap \langle B \rangle$. Let A and B be compatible and D a relational system of the same language . A function f from $\langle A \rangle \cup \langle B \rangle$ to $\langle D \rangle$ is an *amalgamation* function of A and B if f restricted to $\langle A \rangle$ is an embedding of A into D and if f restricted to $\langle B \rangle$ is an embedding of B into D. The relational system D is an *amalgam* of the relational systems A and B. A set $\mathcal{R}$ of relational systems of the same language is *amalgamable* if for any two compatible relational systems A and B belonging to $\mathcal{R}$ there exists an amalgam D in $\mathcal{R}$. (12.2.) The set $\mathcal{R}$ is *directed under embeddability* if any two elements S and T of $\mathcal{R}$ with $\langle S \rangle \cap \langle T \rangle = \emptyset$ have an amalgam in $\mathcal{R}$. (The set $\mathcal{R}$ is 0-amalgamable, 0 for 0 intersection.)

An *age* (10.2.1) is a set of finite relational systems of the same language closed under isomorphism, restriction and directed under embeddability. Let R be a relational system. The *age of R* is the set of relational systems which are isomorphic to an element of the skeleton of R. (See 10.2) The *representative* of an age $\mathcal{R}$ is a relational system R whose age is $\mathcal{R}$. Every age has a representative. (10.2.3) and clearly the age of a relational system is an age. The relational system S is younger than the relational system R if the age of S is a subset of the age of R. (10.1.3, 10.2.1.) If S is younger than R then R is older than S.

Let $\mathcal{R}$ be an amalgamable age. Let A and B be two elements of $\mathcal{R}$ and $u \in \langle A \rangle$ and f an embedding of $A - u$ into B. Then, it follows from the amalgamation property of $\mathcal{R}$, that there is an extension $D \in \mathcal{R}$ of B so that f has an extension to an embedding of A into B. Hence we obtain the following *mapping extension property of an age:*

Let $\mathcal{R}$ be an amalgamable age with $B \in \mathcal{R}$. If $\{A_0, A_1, A_2, \dots, A_{n-1}\}$ is a finite set of elements of $\mathcal{R}$ with $u_i \in \langle A_i \rangle$ for $i \in n$ then there is an extension $D \in \mathcal{R}$ of B so that every embedding f of $A_i - u_i$ into B has an extension to an embedding of A_i into D.

A.1.1 The mapping extension property

The relational system R has the *mapping extension property* if

for every element A in the age of R and every element $u \in \langle A \rangle$ and every embedding f from $A - u$ to R there exists an extension of f which is an embedding of A to R.

The relational system R has the mapping extension property if and only if for every relational system S younger than R and every embedding f from a

restriction of S to a finite subset of $\langle S \rangle$ to R there exists an embedding from S to R which is an extension of f.

• If for every relational system S younger than R and every embedding f from a restriction of S to a finite subset of $\langle S \rangle$ to R there exists an embedding from S to R which is an extension of f then R has the mapping extension property.

Let R have the mapping extension property. Enumerate the elements of $\langle S \rangle$ setminus the domain of f into a sequence $s_0, s_1, s_2, \ldots$. Extend step by step the function embedding f to $\langle S \rangle \cup \{s_0\}, \langle S \rangle \cup \{s_0\} \cup \{s_1\}, \langle S \rangle \cup \{s_0\} \cup \{s_1\} \cup \{s_2\}, \ldots$ using the mapping extension property. Take the limit of the extensions. •

Let R and S be two relational systems with the same language and the same age and both having the mapping extension property. Every embedding g of an element of the skeleton of R into S can be extended to an isomorphism of R to S.

• Enumerate $\langle R \rangle$ into the ω-sequence $r_0, r_1, r_2, \ldots$ and $\langle\langle S \rangle$ into the ω-sequence $s_0, s_1, s_2, \ldots$ so that $g = \{(r_i, s_i) \,|\, i \in m \in \omega\}$.

An embedding f of a finite restriction of R to S is *complete* up to $n \in \omega$ if for all $i \in n$ the element r_i is in the domain of f and the element s_i is in the image of f. The function g is complete up to m. The mapping extension property for R and S implies that if f is complete up to n then there is an extension f' of f which is complete up to $n+1$. Let $f_0 \subset f_1 \subset f_2 \subset \cdots$ be a sequence so that f_n is a complete embedding up to n. It follows that $\bigcup f_n$ is an isomorphism of R to S. •

A.1.2 The mapping extension property and homogeneity

The relational system R is *homogeneous* if every local isomorphism of R has an extension to an automorphism of R. (12.1.1.)

The relational system R is homogeneous if and only if it has the mapping extension property.

• Suppose that R has the mapping extension property. Let B be an element of the skeleton of R and f a local isomorphism of R. According to A.1.1 the local isomorphism f has an extension to an automorphism of R.

Suppose that R is homogeneous. Let A be a relational system in the age of R, the element u in the base of A and f an embedding from $A - u$ to R. Because A is an element of the age of R there is an embedding g from A to R. Let g_1 be the restriction of g to $|A| - \{u\}$, let h be the local automorphism given by $f \circ g_1^{-1}$, let h' be an extension of the local isomorphism $f \circ g_1^{-1}$ to an automorphism of R. Then $h' \circ g$ restricted to $\langle A \rangle$ is an embedding from A to R which extends f. •

A.1.3 The amalgamation theorem

The age of a homogeneous relational system is closed under isomorphism, restriction, directed under embeddability and it is amalgamable. An amalgamable age has a homogeneous representative. (R. Fraïssé) (12.1.1) (12.2.) (12.2.1.)

• Let H be a homogeneous relational system. The age of H is closed under isomorphism, restriction and directed under embeddability. Let A and B be two compatible elements of the age of H and let C be the common restriction of A and B to $\langle A\rangle \cap \langle B\rangle$. Then C is in the age of H and hence there is an embedding f of C into H. The embedding f has an extension f_A to an embedding of A into H and f has an extension f_B to an embedding of B into H. Then $f_A \cup f_B$ is an amalgamation function.

Let $\mathcal{R}$ be an amalgamable age with language $\mathbf{L}$. We assumed that $\mathbf{L}$ is denumerable. Hence the pairs of the form (A, u) with $A \in \mathcal{R}$ and $u \in \langle A\rangle$ are denumerable into an ω sequence $(A_0, u_0), (A_1, u_1), (A_2, u_2), \dots$. Let $B_0 = A_0$ and B_n an element of the age which embeds D_n and A_n. The relational system D_n is an element of the age so that for every $i \in n$ and every embedding f of $A_i - u_i$ into B_{n-1} there is an extension of f to an embedding of A_i into D_n. (Such an element $D_n \in \mathcal{R}$ exists because of the mapping extension property of the age $\mathcal{R}$. A.1) Then B_{n+1} is an extension of B_n for all $n \in \omega$. Let H be the limit. Then the age of H is $\mathcal{R}$ and H has the mapping extension property and is therefore homogeneous. •

A.1.4 Graphs and hypergraphs

A graph G is a binary symmetric antireflexive relation. The base of G is denoted by $V(G)$ and $E(G) = \{\{x, y\} \subseteq V(G) \,|\, G(x, y) = +\} \subseteq [V(G)]^2$. (As usual in the theory of graphs the base of a graph G is denoted by $V(G)$ instead of $\langle G\rangle$.) The elements of $V(G)$ are the *vertices* of G and the elements of $E(G)$ are the *edges* of G. The vertices x and y of G are *adjacent*, $x \sim y$, if $\{x, y\}$ is an edge of G. A graph G is *complete* if $E(G) = [V(G)]^2$. By K_n we denote the *complete graph* with $n \in \omega + 1$ vertices.

A k-uniform hypergraph G is a relation of arity k so that $G(x_0, x_1, \dots, x_{k-1}) = +$ implies $x_i \neq x_j$ for all $i \neq j$ in k and $G(x_{\pi(0)}, x_{\pi(1)}, \dots, x_{\pi(k-1)}) = +$ for all permutations π of k. The base of G is denoted by $V(G)$. The set of *hyperedges* of G is the set $E(G) = \{\{x_0, x_1, \dots, x_{k-1}\} \,|\, G(x_0, x_1, \dots, x_{k-1}) = +\} \subseteq [V(G)]^k$. The hypergraph G is *complete* if $\bigcup_{A \in E(G)} [A]^2 = [V(G)]^2$.

Let $\mathbf{L} = \{E_0, E_1, E_2, \dots, E_{n-1}\}$ be a binary relational language. The relational structure G with language $\mathbf{L}$ is a graph with *n-types of edges* if all of the relations $E_0^G, E_1^G, E_2^G, \dots, E_{n-1}^G$ are graph relations. We write $E_i(G)$ for the set of edges of type i, with $i \in n$. Analogously we define a k-uniform hypergraph with several types of edges.

Let the arities of the relation symbols E_i in $\mathbf{L}$ be arbitrary numbers. A relational system G with language $\mathbf{L}$ is a hypergraph with several types of edges if each of the relations E_i^G is a hypergraph relation. The hypergraph G with several types of edges is *complete* if $\bigcup_{A \in \bigcup_{i \in n} E_i(G)} [A]^2 = [V(G)]^2$; that is if for every two elements of $V(G)$ there is a hyperedge of G which contains both.

A.1.5 The rationals and the Rado graph

The age of the chain $\mathbf{Q}$ is the set of all finite chains. Clearly if f is an embedding of a finite chain A to $\mathbf{Q}$ and if B is a one element extension of A then there is an extension of f to an embedding of B. Hence $\mathbf{Q}$ is a homogeneous relation by A.1.2.

Another important example of a homogeneous relation is the *Rado graph*. It is constructed similar to the rich relation in 10.3.1 and the relational system having the mapping extension property in A.1.3. The Rado graph is the union of finite graphs $A_0 \subset A_1 \subset A_2 \subset A_3 \subset \cdots$ where A_0 is the graph with exactly one vertex. For every subset $F \subseteq V(A_i)$ there is a vertex $a \in V(A_{i+1})$ which is adjacent to all vertices in F and not adjacent to the vertices in $V(A_i) - F$. It follows from the construction that the age of the Rado graph is the set of all finite graphs and that the Rado graph has the mapping extension property. Hence the Rado graph is a homogeneous relation. The defining property of the Rado graph is:

The Rado graph is the graph in which for any two finite subsets $F_1 \subseteq F$ of vertices there exists a vertex which is not in F, adjacent to all vertices in F_1 and to no vertex in $F - F_1$.

• It follows from the construction that the Rado graph has this property. If a graph has this property then it has the mapping extension property. •

A.1.6 The K_n-free homogeneous graphs H_n

The graph G is *K_n-free* if there is no embedding from K_n into G. The subset $T \subseteq V(G)$ is K_n-free if there is no embedding from K_n to the restriction of G to T.

The *K_n-free homogeneous graph H_n* is constructed in a similar way as the Rado graph. It is the union of finite graphs $A_0 \subset A_1 \subset A_2 \subset A_3 \subset \cdots$ where A_0 is the graph with exactly one vertex. For every K_{n-1}-free subset $S \subseteq V(A_i)$ there is a vertex $a \in V(A_{i+1})$ which is adjacent to all vertices in S and not adjacent to the vertices in $V(A_i) - S$. It follows from the construction that the age of the graph H_n is the set of all finite K_n-free graphs and that H_n has the mapping extension property. Hence the graph H_n is a homogeneous relation. The defining property of H_n is:

The K_n-free homogeneous graph H_n is the graph in which for every finite subset F of vertices and every K_{n-1}-free subset $F_1 \subseteq F$ there exists a vertex which is not in F, adjacent to all vertices in F_1 and to no vertex in $F - F_1$.

• It follows from the construction that the Graph H_n has the property above. If a graph has the property above then it has the mapping extension property. •

Let $\mathbf{L} = \{E_i \mid i \in n\}$ be a relational language and $\mathcal{T}$ a set of finite complete hypergraphs with language $\mathbf{L}$. A hypergraph G with several types of edges whose language is $\mathbf{L}$ is *$\mathcal{T}$-free* if there is no embedding from any of the hypergraphs of $\mathcal{T}$ into G. The $\mathcal{T}$-free homogeneous hypergraph $H_{\mathcal{T}}$ is constructed in a similar way as the K_n-free graph H_n. The hypergraph A_0 is the hypergraph with exactly one vertex. For every possible way of extending A_i, without creating one of the forbidden hypergraphs of $\mathcal{T}$, by a single vertex we add this vertex to A_i. The hypergraph A_{i+1} consists of the extension of A_i to all of those additional vertices.

A.1.7 Known characterizations of homogeneous relational systems

The complement of a graph G has the same set of vertices as G and two vertices are adjacent in the complement if and only if they are not adjacent in G. (The complement of a graph is the negation of the graph viewed as a relation as defined in 1.7.4.) Note that a graph G is homogeneous if and only if its complement is homogeneous. The complement of the Rado graph is again the Rado graph.

More generally let R be a relational system and S a set of components of R. The relational system R' derived from R by negating the components in S is called *essentially equivalent* to R. If R and R' are essentially equivalent then R is a homogeneous relational system if and only if R' is a homogeneous relational system. (The automorphism group of R is the same as the automorphism group R'.)

The relation which consists of denumerably many copies of K_n or of finitely many or denumerably many copies of K_ω is a homogeneous relation. The Rado graph, the K_n-free graphs the graphs which consist of copies of K_n or K_ω and the complements of those graphs are all the homogeneous graphs. [Woodrow]

The homogeneous tournaments are characterized in [Lachlan1], the homogeneous partial orders are characterized in [Schmerl], the homogeneous directed graphs are characterized in [Cherlin], the countable stable structures which are homogeneous for relational systems with finitely many components are characterized in [Lachlan2]. As every finite relational system is stable also all finite homogeneous graphs together with other finite homogeneous systems are characterized in [Lachlan2]. All the finite homogeneous graphs where exhibited earlier in [Gardiner]. The pentagon for example is a finite homogeneous graph.

A.2 VARIOUS TYPES OF AMALGAMATION

Strong amalgamation has already been defined in 12.2. The definition given below is equivalent. Free amalgamation is also called independent amalgamation.

A.2.1 Strong amalgamation

Let $\mathcal{R}$ be a set of relational systems with language $\mathbf{L}$. The set $\mathcal{R}$ is said to be *strongly amalgamable*, (12.2), if for any two compatible relational systems A, B belonging to $\mathcal{R}$ there exists an amalgam $D \in \mathcal{R}$ which is an extension of A and B. That is if there is an amalgamation function which is the identity on $\langle A\rangle \cup \langle B\rangle$. Or equivalently if there is a one to one amalgamation function whose image is an element of $\mathcal{R}$. Let $K_\omega + K_\omega$ be the graph which consists of two disjoint copies of K_ω. The age of $K_\omega + K_\omega$ is strongly amalgamable. The set of all finite chains is strongly amalgamable with $\mathbf{Q}$ as representative. The set of all finite posets is strongly amalgamable. (1.7.3) The *universal homogeneous poset* is the homogeneous representative of the set of finite posets. The homogeneous graph which consists of denumerably many disjoint triangles is not strongly amalgamable.

A.2.2 Free amalgamation

Let A and B be two compatible relations of the same language $\mathbf{L} = \{R_i \mid i \in I\}$. The *free amalgam* of A and B is the extension D of A and B, with base $\langle A\rangle \cup \langle B\rangle$ so that $R_i^D = R_i^A \cup R_i^B$ for all $i \in I$.

The set $\mathcal{R}$ is said to be *freely amalgamable* if the free amalgam of any two compatible relational systems A, B belonging to $\mathcal{R}$ is again an element of $\mathcal{R}$. The set of all finite graphs which do not embed the complete graph K_n is a freely amalgamable age. The representative of the set of all finite graphs which do not embed the complete graph K_n is the K_n-free homogeneous graph H_n. A freely amalgamable age is strongly amalgamable. The age of $K_\omega + K_\omega$ is strongly but not freely amalgamable. The set of all finite chains and the set of all finite posets are examples of strongly but not freely amalgamable ages.

Let $\mathcal{T}$ be a set of finite complete hypergraphs with several types of edges with relational language $\mathbf{L}$. Let $\mathcal{R}$ be the set of finite hypergraphs A of the language $\mathbf{L}$ so that there is no embedding of an element of $\mathcal{T}$ to A. Then $\mathcal{R}$ is a freely amalgamable age. The homogeneous hypergraph which is the homogeneous representative of $\mathcal{R}$ is denoted by $H_{\mathcal{T}}$.

A.2.3 Amalgamation over a particular relational system

The set $\mathcal{R}$ is *amalgamable over the relational system* C if any two compatible relational systems A and B belonging to $\mathcal{R}$ whose intersection is C have an amalgam in $\mathcal{R}$. The set $\mathcal{R}$ is *n-amalgamable* if any two compatible relational systems A and B belonging to $\mathcal{R}$ whose intersection has n elements have an amalgam in $\mathcal{R}$. Hence a set $\mathcal{R}$ of relational systems is 0-amalgamable if and only if it is directed under embeddability. (10.2.) Of interest in relation to divisibility properties are amalgamable ages which are strongly 0-amalgamable, freely 0-amalgamable, strongly or freely 1-amalgamable and so on.

A.2.4 Bounds, complete relational systems

The relational system C with language $\mathbf{L} = \{R_i \mid i \in I\}$ and arity $\{n_i \mid i \in I\}$ is *complete* if for any two elements $y, z \in \langle C\rangle$ there is an $i \in I$ and an n_i-tupel $(x_0, x_1, x_2, \ldots, x_{n_i-1}) \in R_i^S$ so that $\{x, y\} \subseteq \{x_0, x_1, x_2, \ldots, x_{n_i-1}\}$.

Let $\mathcal{R}$ be a set of finite relational systems all having the same language . A *bound* of $\mathcal{R}$ is a finite relational system C of that same language which is not in $\mathcal{R}$ but every proper restriction of C is an element of $\mathcal{R}$. If $\mathcal{R}$ is closed under restriction and isomorphism it is completely determined by its bounds. That is, for a given set $\mathcal{C}$ of finite relational systems of the same language there is one and only one set $\mathcal{R}$ of finite relational systems closed under restriction and isomorphism whose set of bounds is $\mathcal{C}$. The set $\mathcal{R}$ is then the set of all finite relational systems of the given language which do not embed any one of the elements of $\mathcal{C}$. If we want to be economic we will also require that no element of C has an embedding into any other element of C.

Let $\mathcal{C}$ be a set of finite relational systems all of the same language which are pairwise incomparable under embedding. Let $\mathcal{R}$ be the set of finite relational systems whose bound is $\mathcal{C}$. The set $\mathcal{R}$ is the set of *$\mathcal{C}$-free* relational systems of the given language denoted by $\mathrm{Forb}(\mathcal{C})$. If $\mathcal{C}$ is a finite set then the $\mathcal{C}$-free set $\mathcal{R}$ of relational systems is a universal class. (See 5.10.)

A.2.5 Characterization of freely amalgamable ages

Let $\mathcal{C}$ be a set of finite relational systems having the same language $\mathbf{L} = \{R_i \mid i \in I\}$ of arity $\{n_i \mid i \in I\}$ so that no relational system in $\mathcal{C}$ can be embedded into another relational system of $\mathcal{C}$. The $\mathcal{C}$-free set $\mathrm{Forb}(\mathcal{C})$ of relational systems is a freely amalgamable age if and only if every element of $\mathcal{C}$ is complete.

• Let every element of $\mathcal{C}$ be complete. Clearly $\mathrm{Forb}(\mathcal{C})$ is closed under isomorphism and restriction. Let A and B be two compatible elements of $\mathrm{Forb}(\mathcal{C})$. Let D be the free amalgam of A and B. Assume for a contradiction that there is an embedding from an element $C \in \mathcal{C}$ into D. We may assume that C is a restriction of D. As both A and B do not embed C there are two elements y and z belonging to $\langle D\rangle$ so that $y \in \langle A\rangle - \langle B\rangle$ and $z \in \langle B\rangle - \langle A\rangle$. There is an $i \in I$ and n_i-tupel $(x_0, x_1, \ldots, x_{n_i-1})$ of elements in the base of C so that $(x_0, x_1, \ldots, x_{n_i-1}) \in R_i^C$ and $\{x, y\} \subseteq \{x_0, x_1, \ldots, x_{n_i-1}\}$. It follows that $(x_0, x_1, \ldots, x_{n_i-1}) \in R_i^D$ because C is a restriction of D and hence because D is the free amalgam of A and B that $(x_0, x_1, \ldots, x_{n_i-1}) \in R_i^A$ or $(x_0, x_1, \ldots, x_{n_i-1}) \in R_i^B$. But neither one of those two cases can occur because $\{x_0, x_1, \ldots, x_{n_i-1}\}$ contains an element which is not in $\langle A\rangle$ and an element which is not in $\langle B\rangle$.

Assume $\mathrm{Forb}(\mathcal{C})$ is a freely amalgamable age and that there is a relational system $A \in \mathcal{C}$ and elements $y \in \langle A\rangle$ and $z \in \langle A\rangle$ so that for every $i \in I$ and n_i-tupel $(x_0, x_1, \ldots, x_{n_i-1})$ the set $\{x, y\}$ is not a subset of $\{x_0, x_1, \ldots, x_{n_i-1}\}$. The relational systems $A-y$ and $A-z$ are compatible and elements of $\mathrm{Forb}(\mathcal{C})$. The relational system A is the free amalgam of $A - y$ and $A - z$. Hence $A \in \mathrm{Forb}(\mathcal{C})$, a contradiction. •

Let A and B be two relational systems with language $\mathbf{L} = \{R_i \mid i \in I\}$ and $\langle A\rangle = \langle B\rangle$. The relational system A is *thinner* than the relational system B if $R_i^A \subseteq R_i^B$ for all $i \in I$. Let A be thinner than B and C a complete relational system with language $\mathbf{L}$. If there is no embedding from C into B then there is no embedding from C into A. It follows that if A is an element of the age of the relational system T and T is thinner than the relational system S and the age of S is freely amalgamable then every element in the age of T is also an element in the age of S.

Let S with language $\mathbf{L} = \{R_i \mid i \in I\}$ be a relational system whose age is freely amalgamable and $J \subseteq I$. Let T with language $\mathbf{K} = \{R_i \mid i \in J\}$ and the same base as S be such that $R_i^T = R_i^S$ for all $i \in J$. Then the age of T is freely amalgamable.

• Let T' be a relational system with language $\mathbf{L}$ and $R_i^{T'} = R_i^S$ for all $i \in J$ and $R_i^{T'} = \emptyset$ for all $i \in I - J$. Then T' is a thinning of S. Let A and B be two compatible elements of the age of T'. Then they are also compatible elements of the age of S. Hence they have a free amalgam D in the age of S. We have $R_i^D = R_i^A \cup R_i^B$ for all $i \in I$. Hence D is in the age of T'. It follows that the age of T' is freely amalgamable. But so then of course is the age of T. •

A.2.6 More homogeneous relational systems with freely amalgamable age

The first theorem in A.2.5 implies that there is large number of homogeneous structures. If $\mathcal{C}$ is a set of finite complete relational systems with the same language so that no relational system in $\mathcal{C}$ can be embedded into another relational system of $\mathcal{C}$ then the homogeneous representative of $\mathrm{Forb}(\mathcal{C})$ is denoted by $H_{\mathcal{C}}$. ($\mathrm{Forb}(\mathcal{C})$ is the set of relational systems R with language $\mathbf{L}$ so that no element of C can be embedded into R.) Let L be the binary relation consisting of a singleton base, say $\{x\}$, so that $L(x,x) = +$. Let R be the binary relation with base $\{x, y\}$ so that $R(x,x) = R(y,y) = R(y,x) = -$ and $R(x,y) = +$. Then the Rado graph is the homogeneous structure $H_{\{L,R\}}$. The K_n-free homogeneous graph, which in the context of graphs is just denoted by H_n, is then the homogeneous structure $H_{\{L,R,K_n\}}$. The set of finite graphs with two types of edges is an age. The homogeneous representative of this age is the *universal graph with two types of edges*.

Let A_n be the graph with two types of edges and base n so that $E_1(A_n) := \{\{i, i+1\} \mid i \in n\}$ and $E_2(A_n) := [n]^2 - E_1(A_n)$. The graph A_n is complete and if $n \neq m$ then there is no embedding from A_n into A_m. Hence for every subset $\mathcal{C} \subseteq \{A_n \mid n \in \omega\}$ there exists the homogeneous graph $H_{\mathcal{C}}$ with two types of edges. It follows that there are continuum many homogeneous graphs with two types of edges.

Henson [Henson] has constructed an infinite set of finite tournaments not one of which can be embedded into another. Hence there are continuum many homogeneous directed graphs. Similarly there are continuum many three uniform hypergraphs and so on.

A.2.7 The cyclic order

From the complete characterizations obtained so far it seems that for every language there are the homogeneous relations of the type $H_{\mathcal{C}}$ with $\mathcal{C}$ a set of finite complete relations and then some special homogeneous relations whose age has a more complicated amalgamation. Here is just one example of such a special homogeneous relation.

The *cyclic order*. The cyclic order has as base any countable dense subset of the unit circle with the property that no two points make an angle of $2\pi/3$ at the centre of the circle. Two vertices will be adjacent if the acute angle they make at the centre of the circle is less than $2\pi/3$.

A.2.8 Orbits

Let H be a homogeneous relational system and $\mathrm{aut}(H)$ the automorphism group of H. If A is a finite subset of $\langle H\rangle$ then $\mathrm{aut}(H)_A$ is the subgroup of all elements $g \in \mathrm{aut}(H)$ so that $g(a) = a$ for all $a \in A$; the stabilizer subgroup of $\mathrm{aut}(H)$ stabilizing the set A. The elements $x, y \in \langle H\rangle - A$ are in the same *orbit* of A if there is a $g \in \mathrm{aut}(H)_A$ so that $g(x) = y$. The relation of being in the same orbit of A is an equivalence relation on $\langle H\rangle - A$. The equivalence classes of this equivalence relation are called the *orbits of A*. If X is an orbit of A then A is the *base* of X which we denote by $F(X)$. An *orbit of H* is an orbit of A for some finite subset A of $\langle H\rangle$. It follows that if H does not have any unary components then $\langle H\rangle$ is the only orbit of the empty set.

Let T be an orbit of H and x and y two elements of T. Then, according to the definition of orbit, there is a local isomorphism f so that $f(x) = y$ and $f(a) = a$ for all elements $a \in F(T)$. This implies that the *type* of x over $F(T)$ is the same as the type of y over $F(T)$. If for example H is a graph then a vertex a in $F(T)$ is adjacent to x if and only if it is adjacent to y. Hence, given F, an orbit T of F is completely determined by the subset $F_1 \subseteq F$ of vertices in F which are adjacent to one and hence to all of the elements of T. If H is a three uniform hypergraph then two vertices a and b in $F(T)$ form a hyperedge with x if and only if they form a hyperedge with y.

Many notions for homogeneous relational structures dependent only on the automorphism group. For example orbit. Given a group G of permutations of a set E which is closed under adherence (12.3.4) then there exists a homogeneous relational system whose group of automorphisms is G. (12.3.5) This group theoretic aspect of homogeneous structures has been investigated in [Cameron 1], [Cameron2], [Cameron3], [Cameron4] and [Macpherson].

A.2.10 Universal homogeneous relations

Let T be a relational system with ω as base and language $\mathbf{L} = \{R_i \mid i \in I\}$ with corresponding arities $\{n_i \mid i \in I\}$. The *square* S of T has base $\langle S\rangle = \{(x, y) \mid x \in \omega \wedge y \in \omega\} = \langle S\rangle \times \langle S\rangle$.The interpretation of the relationsymbols is

$$\begin{aligned} R_i^D = &\{(x_0, y_0), (x_1, y_1), (x_2, y_2), \dots, (x_{n_i-1}, y_{n_i-1}) \mid \\ &\Big(\forall i, j \in n_i (x_i = x_j) \wedge R_i^T(y_0, y_1, \dots, y_{n_i-1})\Big) \vee \\ &\Big(x_i \neq x_j \text{ for all } i \neq j \wedge R_i^T(x_0, x_1, \dots, x_{n_i-1})\Big)\}. \end{aligned}$$

A homogeneous relational system U is called *universal* if the age of the square of U is equal to the age of U. Examples of universal relations are the Rado graph, also called the universal graph, the universal tournament which is the homogeneous representative of the set of all finite tournaments, the universal directed graph which is the representative of the set of all finite directed graphs, the universal three uniform hypergaph which is the representative of the set of all three uniform hypergraphs, and so on.

A.3 CUTTING FINITE PIECES FROM HOMOGENEOUS SYSTEMS

A.3.1 Inexhaustible ages of homogeneous relational systems

Let H be a homogeneous relational system. If the age of H is inexhaustible (10.6.2) then there exists an inexhaustible representative (10.6) of the age of H which is embeddable into H. Hence for every finite subset $A \subseteq \langle H \rangle$ the age of H restricted to $\langle H \rangle - A$ is equal to the age of H. We will then say that H is *age inexhaustible*. If H is age inexhaustible then clearly the age of H is strongly 0-amalgamable. If the age of H is strongly 0-amalgamable then the age of H is inexhaustible according to 10.6.3.

A.3.2 Orbits of the empty set and inexhaustible ages

Let H be a homogeneous relational structure. Every orbit in H of the empty set is infinite if and only if H is age inexhaustible. [Pouzet1]

• If $K \subseteq \langle H \rangle$ is a finite orbit of the empty set then the restriction of H to $\langle H \rangle - K$ cannot embed the restriction of H to K.

Assume that every orbit of the empty set is infinite. It suffices to prove that if A is a finite subset of $\langle H \rangle$ then there are infinitely many pairwise disjoint subsets $A = A_0, A_1, A_2, \ldots$ of $\langle H \rangle$ so that the restriction of H to A is isomorphic to the restriction of H to A_i for every $i \in \omega$. The assertion is certainly true if A is a singleton set $\{a\}$. We proceed by induction on the number of elements in $\langle A \rangle$.

Let $a \in A$ and $B = A - a$. Assume that $A = A_0, A_1, A_2, \ldots, A_{n-1}$ is a maximal sequence of pairwise disjoint subsets of $\langle H \rangle$ so that the restriction of H to A is isomorphic to the restriction of H to A_i for every $i \in \omega$. Let $a = a_0, a_1, a_2, \ldots, a_{n-1}$ be the corresponding sequence of a's and $B = B_0, B_1, B_2, \ldots, B_{n-1}$ the corresponding sequence of B's. Assume that there are infinitely many pairwise disjoint subsets of $\langle H \rangle$ so that the restriction of H to B is isomorphic to the restriction of H to each of those subsets. Only finitely many of those subsets have a non empty intersection with the set $\bigcup_{i \in n} B_i$. Hence there is an infinite extension of the sequence $B = B_0, B_1, B_2, \ldots, B_{n-1}$ to a sequence $B = B_0, B_1, B_2, \ldots, B_{n-1}, B_n, B_{n+1}, B_{n+2} \ldots$ of pairwise disjoint sets so that the restriction of H to B is isomorphic to the restriction of H to B_i for every $i \in \omega$.

Let f_i be an extension of the local isomorphism of B_0 to B_i to an automorphism of H. The automorphism f_i maps the set $\{a_0, a_1, \ldots, a_{n-1}\}$ onto itself because of the maximality of the sequence $A = A_0, A_1, A_2, \ldots, A_{n-1}$. Hence there is $j \in n$ so that $f_i(a) = a_j$ for infinitely many $i \in \omega$. That is a_j forms with infinitely many pairwise disjoint copies of B a copy of A.

Let g be an automorphism of H which maps a_j into $\langle H \rangle - \{a_0, a_1, \ldots, a_{n-1}\}$. Then $g(a_j)$ forms with infinitely many pairwise disjoint copies of B a copy of A. That is there are infinitely many pairwise disjoint sets $C_0, C_1, \ldots$ so that H restricted to $\{g(a_j)\} \cup C_i$ forms a copy of A for every $i \in \omega$. This is a contradiction to the maximality of the sequence $A = A_0, A_1, A_2, \ldots, A_{n-1}$ because only finitely many of the sets C_i intersect the set $\bigcup_{i \in n} B_i$. •

A.3.3 Inexhaustible homogeneous systems and inexhaustible ages

The proof of the next theorem is from [Zhu], the theorem follows directly from general results in [Pouzet1]

A homogeneous relational system H with finite language $\mathbf{L}$ which is age inexhaustible is inexhaustible.

• Assume that the base of H is ω and that I_n is the restriction of H to n. Let $A \subseteq \langle H \rangle$ be finite. To prove that there is an embedding k of H into the restriction of H to $\langle H \rangle - A$.

Let F_n be the set of local isomorphisms from I_n to $\langle H \rangle - A$. Because the age of H is inexhaustible the set F_n is not empty. For f and g in F_n let $f \overset{n}{\sim} g$ iff $(f \circ g^{-1}) \cup \mathrm{id}_A$ is an isomorphism from the image of g union A to the image of f union A. The relation $\overset{n}{\sim}$ is an equivalence relation on F_n. The equivalence relation $\overset{n}{\sim}$ has finitely many classes because the language $\mathbf{L}$ of H is finite. (If H has finite language then there are only finitely many pairwise non isomorphic extensions of disjoint copies of A and H_n.) Let P_n be the set of equivalence classes of $\overset{n}{\sim}$ and P the union of the P_n. Let F be the union of the F_n and $\sim = \bigcup_{n \in \omega} \overset{n}{\sim}$.

For f and g in F put $g \leq f$ if $(f \circ g^{-1}) \cup \mathrm{id}_A$ is an embedding from the image of g union A into the image of f union A. If $f' \sim f$ and $g' \sim g$ and $g \leq f$ then $g' \leq f'$. Hence $\leq$ induces a partial order on P. The partial order $(P; \leq)$ is denumerable, every element has a finite cover and it is well founded. Hence $(P; \leq)$ contains according to Kőnigs Lemma an infinite chain $C = \{C_0 < C_1 < C_2 < \cdots\}$ which contains an element from every level of P.

Let $g_0 \in C_0$. Assume $g_0 \subseteq g_1 \subseteq g_2 \subseteq \cdots \subseteq g_{n-1}$ with $g_i \in C_i$ for all $i \in n$. If $f \in C_n$ then $g_{n-1} < f$ and hence $h = (f \circ g_{n-1}^{-1}) \cup \mathrm{id}_A$ is a local isomorphism which can be extended to an automorphism l of H. Then $l^{-1}f \sim f$ because $(l^{-1}f)f^{-1} = l^{-1}/\mathrm{im}(f)$ is an isomorphism and $l^{-1}/A = \mathrm{id}_A$. Because $g_{n-1} \subseteq l^{-1}f$ the embedding $g_n = l^{-1}f$ extends the sequence $g_0 \subseteq g_1 \subseteq g_2 \subseteq \cdots \subseteq g_{n-1}$ as required. Hence there is an infinite sequence $g_0 \subseteq g_1 \subseteq g_2 \subseteq \cdots$ with $g_i \in F_i$ for all $i \in \omega$. Then $k = \bigcup_{n \in \omega} g_n$ is an embedding of H into the restriction of H to $\langle H \rangle - A$. •

It follows that

A homogeneous relational system with finite language is inexhaustible if and only if its age is strongly 0-amalgamable.

and

If the homogeneous relational system H has finite language $\mathbf{L}$ and none of the relation symbols in $\mathbf{L}$ is unary then H is inexhaustible.

A.3.4 An example with infinitely many components

Example: The base of H is ω. The elements $x, y \in \omega$ are in the relation τ_n if and only if $|x - y| = n$. This is a homogeneous structure, the translations, the reflection $x \mapsto -x$ and their compositions are the only isomorphisms even between finite isomorphic substructures. The automorphism group of H acts transitively and there is no proper substructure of H isomorphic to H. Hence H is not inexhaustible. The restriction of H to the non negative integers is an inexhaustible relational system.

Hence H is age inexhaustible but not inexhaustible. The only orbit of the empty set is $\langle H\rangle$, the automorphism group is transitive. The relational system H and its restrictions to subsets of $\langle H\rangle$ are the only representatives of the age of H. Hence 10.6.2 (4) fails in this case. Nevertheless the age of H does have an inexhaustible representative.

A.3.5 Strongly inexhaustible homogeneous relational systems

The relational system R is *strongly inexhaustible* if for every finite $F \subseteq \langle R\rangle$ the restriction of R to $\langle R\rangle - F$ is isomorphic to R. The homogeneous relational system H has the *strong embedding property* if for every A in the age of H and $a \in A$ and embedding f from $A-a$ into H there are infinitely many embeddings from A into H which are an extension of f.

Let H be a homogeneous relational system. The age of H is strongly amalgamable if and only if H has the strong embedding property if and only if H is strongly inexhaustible. [ElSauer1]

• If the age of H is strongly amalgamable then H has the strong embedding property: Assume that there is an element A in the age of H with $a \in A$ and an embedding f from $A-a$ into H which has only finitely many extensions $f_0, f_1, \dots, f_{n-1}$ to A which are embeddings. The number n is larger than 0 because H is homogeneous. Let g be the function with domain $\langle A\rangle$ so that g restricted to $\langle A\rangle - a$ is f restricted to $\langle A\rangle - a$ and $g(a) = a$. Let B be the relational structure with $\langle B\rangle = g(\langle A\rangle - a) \cup \{a\}$ so that g is an isomorphism from A to B. Let C be the restriction of H to $f(\langle A\rangle - a) \cup \{f_0(a), f_1(a), \dots, f_{n-1}(a)\}$. The relational systems B and C are compatible and in the age of H. Hence there exists a common extension of B and C to an element D in the age of H. The identity map on $\langle C\rangle$ has an extension h to an embedding which maps a into H with $h(a) \notin \{f_0(a), f_1(a), \dots, f_{n-1}(a)\}$. Put $f_n = h \circ g$. Hence H has the strong embedding property.

If the age of H has the strong embedding property then H is strongly inexhaustible: Assume that F is a finite subset of $\langle H\rangle$. The restriction of H to $\langle H\rangle - F$ will be isomorphic to H if it has the embedding property for the age of H. Let A be an element in the age of H and $a \in \langle A\rangle$ and f an embedding from $A-a$ into the restriction of H to $\langle H\rangle - F$. The embedding f has infinitely many extensions to an embedding of A into H. Not all of them can map a into F. Hence H is strongly inexhaustible.

If H is strongly inexhaustible then the age of H is strongly amalgamable: Let A and B be two compatible elements in the age of H. We may assume without loss of generality that A is a restriction of H. Let C be the common restriction of A and B to $\langle A\rangle \cap \langle B\rangle$. The identity map is an embedding from C into the restriction of H to $\langle H\rangle - (\langle A\rangle - \langle C\rangle)$

which is isomorphic to H. Hence there is an extension f of the identity map on C to an embedding of B into the restriction of H to $\langle H\rangle - (\langle A\rangle - \langle C\rangle)$. Let g be the function with domain $\langle A\rangle \cup f(\langle B\rangle)$ so that g is the identity on $\langle A\rangle$ and equal to f on $\langle B\rangle - \langle A\rangle$. Let D be the image of g. It follows that D is a strong amalgam of A and B. Hence the age of H is strongly amalgamable. •

A.4 PARTITIONS OF HOMOGENEOUS RELATIONAL SYSTEMS

The relational system R is *indivisible* if whenever $\langle R\rangle = X \cup Y$ then R has an embedding into its restriction to X or an embedding into its restriction to Y. (6.8.) The relational system R is *weakly indivisible* if whenever $\langle R\rangle = X \cup Y$ and the age of R restricted to X is a proper subset of the age of R then R has an embedding into its restriction to Y. The relational system is *age indivisible* if whenever $\langle R\rangle = X \cup Y$ then the age of the restriction of R to X or the age of the restriction of R to Y is equal to the age of R. A set $\mathcal{R}$ of relational systems of the same language is a *Ramsey class* if for element A of $\mathcal{R}$ there is an element B of $\mathcal{R}$ so that if $\langle R\rangle = X \cup Y$ then A embeds into the restriction of R to X or into the restriction of R to Y. The next Theorem implies that R is age indivisible if and only if the age of R is a Ramsey class.

A.4.1 Age indivisible homogeneous relational systems

Let R be a relational system then:

The system R has property $P(1,n)$ if for every partition of $\langle R\rangle$ into n classes $C_0, \dots, C_{n-1}$ the age of the restriction of R to one of the classes is equal to the age of R.

The system R has property $P(2,n)$ if for every partition of $\langle R\rangle$ into n classes the union of the ages of the restrictions of R to those classes is equal to the age of R.

The system R has property $P(3,n)$ if for every A in the age of R and for every partition of the base $\langle B\rangle$ of every B in the age of R into n classes $C_0^B, C_1^B, C_2^B, \dots, C_{n-1}^B$ there is an element B in the age of R and an $i \in n$ so that there is an embedding from A into the restriction of B to C_i^B.

The system R has property $P(4,n)$ if for every A in the age of R there is a B in the age of R so that for every partition of $\langle B\rangle$ into n classes there is an embedding of A into the restriction of B to one of the classes of the partition.

Let R be a relational system and R in the age of R and $n \geq 2$ then $P(1,n)$ implies $P(2,n)$ implies $P(3,n)$ implies $P(4,n)$ implies $P(4,2)$ implies $P(4,n)$ implies $P(1,n)$. [ElSauer1]

• Obviously $P(1,n)$ implies $P(2,n)$. Assume $P(2,n)$ and let for every B in the age of R a partition $(C_i^B; i \in n)$ of $\langle B\rangle$ be given. Assume that the base of R is ω and let I_m be the restriction of R to $m \in \omega$. Let $\mathcal{U}$ be an ultrafilter on ω which contains all of the cofinal subsets of ω. Let $\prod_{\mathcal{U}} I_m$ be the ultraproduct of the I_m. The relational system R is isomorphic to the restriction of $\mathcal{U}$ the to the set of constant sequences. We may assume therefore that R is a restriction of the ultraproduct . Partition $\langle R\rangle$ into classes $(C_i; i \in n)$ with $x \in C_i$ if and only if the set of indices m so that $x \in C_i^{I_m}$ is an element of $\mathcal{U}$. There is an $i \in n$ so that A has an embedding into the restriction of R to C_i. Hence there is an $m \in \omega$ so that A has an embedding into the restriction of I_m to $C_i^{R_m}$.

Let A be an element in the age of R and assume that not $P(4, n)$. Then for every B in the age of R there is a partition $(C_i^B; i \in n)$ of $\langle B\rangle$ so that A can not be embedded into the restriction of B to any of the classes of the partition. Then not $P(3, n)$. Hence $P(3, n)$ implies $P(4, n)$ and clearly $P(4, n)$ implies $P(4, 2)$.

In order to prove that $P(4, 2)$ implies $P(4, n)$ it suffices to prove that $P(4, n)$ implies $P(4, 2n)$. Assume $P(4, n)$ and let A be an element of the age of R. Let B be an element of the age of R so that for every partition of $\langle B\rangle$ into n classes there is an embedding of A into the restriction of B to one of the classes. Let C be an element of the age of R so that for every partition of $\langle C\rangle$ into two classes B has an embedding into the restriction of C into one of the classes. Then for every partition of $\langle C\rangle$ into $2n$ classes there is an embedding of A into the restriction of C to one of the two classes.

Finally $P(4, n)$ implies $P(1, n)$. Let $(C_i; i \in n)$ be a partition of $\langle R\rangle$ into n classes. Assume that for every $i \in n$ there is A_i in the age of R but not in the age of the restriction of R to C_i. Because each A_i has an embedding into R there is an element A in the age of R so that there is an embedding from A_i into A for every $i \in n$. Let B in the age of R be such that for every partition of $\langle B\rangle$ into n classes A has an embedding into the restriction of R to one of the classes. There is an embedding f from B into R. The function f induces a partition of $\langle B\rangle$ into n classes $(C_i^B; i \in n)$ via $x \in C_i^B$ if $f(x) \in C_i$. The system A has an embedding into the restriction of B to one of the classes C_i^B and hence an embedding into C_i. •

A.4.2 Divisibility and strong amalgamation

If the relational system R is indivisible then it is weakly indivisible and if it is weakly indivisible then it is age indivisible. In the case that H is a homogeneous relational system there does not seem to be any relationship between those three properties of H and the property of strong amalgamation of its age. The age of the homogeneous graph $K_\omega + K_\omega$ which consists of two disjoint copies of K_ω has strong amalgamation but is not age indivisible.

There is an indivisible homogeneous system H which does not have strong amalgamation. In [Lachlan2] the following homogeneous relational system is described. Let $\langle H\rangle := [S]^2$ for some denumerable set S. The relational system H has a binary relation $B(\ ,\ )$ and a ternary relation $T(\ ,\ ,\ )$ with $B(a, b) = +$ if a and b have an element of S in common and $T(a, b, c) = +$ if a, b and c form a triangle. Then H is homogeneous. The amalgamation of the age of H is not strong because if two triangles have an edge in common they must have the third point in common. If $A \cup B = [S]^2 = \langle H\rangle$ then there is an infinite subset $T \subseteq S$ so that $[T]^2 \subseteq A$ or $[T]^2 \subseteq B$ by Ramseys theorem. Hence H is indivisible.

A.4.3 Weak indivisibility and free 1-amalgamation

On the other hand it follows from the next theorem that if the age of the homogeneous relational system H is freely 1-amalgamable then H is weakly indivisible. [ElSauer1]

Let H be a homogeneous relational system no component of which is unary and whose age is freely 1-amalgamable. Then H is weakly indivisible.

• Because the age of H is freely 1-amalgamable it is also freely 0-amalgamable.

Given an element A of the age of H so that $\langle A\rangle\cap\langle H\rangle = \emptyset$ denote by $A+H$ the free amalgam of A and H. That is the common extension of A and H so that $R(x_1, x_2, \dots, x_n) = -$ for every component R of H if $\langle A\rangle \cap \{x_1, x_2, \dots, x_n\} \neq \emptyset$ and $\langle H\rangle \cap \{x_1, x_2, \dots, x_n\} \neq \emptyset$. The age of H is freely amalgamable and hence it is equal to the age of $A + H$. Hence $A + H$ has an embedding into H. It follows that to every finite subset F of H there is a copy H^F of H in H so that for every finite $F' \subseteq H/H^F$ the relational system $H/(F \cup F')$ is a free amalgam of H/F and H/F'.

The homogeneous system H is *weakly j-indivisible* if for every partition of H into two sets X and Y for which there is no embedding from H into X the age of H/Y contains all elements of the age of H whose base consists of at most j elements. We proceed by induction and proof that if H is weakly j-indivisible then it is weakly $j+1$-indivisible. Let A be an element of the age of H whose base has $j + 1$ elements and $a \in \langle A\rangle$.

Assume that $X \cup Y = \langle H\rangle$ and $X \cap Y = \emptyset$ and that the base of H is ω. Denote by I_n the restriction of H to n. If there is no embedding from H into H/X then there is an $n \in \omega$ and an isomorphism g from I_n to H/X which does not have an extension h which is an embedding of I_{n+1} into H/X. Let F be the base of the image of g. The partition (X, Y) induces a partition $(X \cap \langle H^F\rangle, Y \cap \langle H^F\rangle)$ of the base of H^F. The homogeneous relational system H can not be embedded into the restriction of H to $X \cap \langle H^F\rangle$ otherwise H would have an embedding into H/X. Hence, using induction, there is a subset L in $Y \cap H^F$ so that there is an isomorphism f of $A - a$ to H/L.

Let b be an element which is not in the base of H. Let A' be the relational system with base $L \cup \{b\}$ so that the extension f' of f with $f'(a) = b$ is an isomorphism of A. Let I'_n be the relational system with base $F \cup \{n\}$ so that the extension g' of g with $g'(n) = b$ is an isomorphism of I_{n+1}. Then, because no component of H is unary, the relational systems I'_n and A are compatible and hence can be freely amalgamated to the amalgam B. (The age of H is freely 1-amalgamable.) The identity map on $F \cup L$ must have an extension to an embedding h of B. If $h(b) \in X$ then $h \circ g'$ would be an extension of g to an embedding of I_{n+1} into H. Hence $h(b) \in Y$ This implies that $h \circ f'$ is an extension of f to an embedding of A into Y. •

The argument in this proof can easily be adapted to homogeneous relational systems whose age is not freely 1-amalgamable but which are essentially equivalent to a homogeneous relational systems whose age is freely 1-amalgamable. For example. Let H be the homogeneous relational system which has a binary component which is a graph and a ternary component which is a 3-uniform hypergraph. There is exactly one boundary element B, the 3-uniform hypergraph which consists of two hyperedges having exactly two elements in common while there are none of the binary graph edges in B. Negating the binary edges of the graph we obtain a freely amalgamable age. The corresponding homogeneous representative is weakly indivisible according to the previous theorem. Of course then the original homogeneous system, before negation of the edges, is also weakly indivisible.

A.4.4 Free amalgamation and age indivisibility

A relational system R without unary components, not necessarily homogeneous, whose age is freely amalgamable is age indivisible.

• Let H be the homogeneous representative of the age of R. Then H is weakly indivisible hence age indivisible and hence the age of H, that is the age of R, is a Ramsey class. (A.4.1) Using A.4.1 again it follows that R is age indivisible. •

A.4.5 Necessary condition for indivisibility

Let R and S be two relational systems of the same language . The relational system R *precedes* the relational system S, $R \preceq S$, if there exists a partition of $\langle R\rangle$ into finitely many sets $C_0, C_1, C_2, \dots, C_{n-1}$ so that there is an embedding of R/C_i into S for every $i \in n$. The set $\mathcal{R}$ of relational systems of the same language satisfies the *chain condition* if for any two elements R and S of $\mathcal{R}$ at least one of $R \preceq S$ or $S \preceq R$ holds.

If the homogeneous relational system H is indivisible then the restrictions of H to its orbits satisfy the chain condition.[ElSauer2]

• Assume that the base of H is ω. Let T be an orbit of H with base F. For $x \in \omega = \langle H\rangle$ and $F' \subseteq \omega$ we write $x \geq_T F'$ to mean that x is a larger number than any of the numbers in F' and that there is a local isomorphism f of F' to F so that one and hence every extension of f to an automorphism of H maps x into T. For each number x let $\mathrm{dom}_T(x)$ be the set F' of finite subsets of $\langle H\rangle$ so that $x \geq_T F'$.

Assume that H is indivisible and let T be an orbit with base F. Let S be an orbit with base D. We wish to prove that either $T \preceq S$ or $S \preceq T$. If S is finite then $S \preceq T$ and we are done. Assume that S and T are infinite. The finite subsets of $\langle H\rangle = \omega$ are totally ordered by the lexicographic ordering. (The subset A is larger than the subset B in this lexicographic ordering if the smallest element which is not in the intersection of A and B belongs to A.)

Let $X_{T,S}$ be the set of all elements x of $\langle H\rangle$ so that the lexicographically largest set in $\mathrm{dom}_T(x)$ is lexicographically larger than the lexicographically largest set in $\mathrm{dom}_S(x)$ and neither $\mathrm{dom}_T(x)$ nor $\mathrm{dom}_S(x)$ is empty. Let $X_{S,T}$ be the set of all elements x of $\langle H\rangle$ so that the lexicographically largest set in $\mathrm{dom}_S(x)$ is lexicographically larger than the lexicographically largest set in $\mathrm{dom}_T(x)$ and neither $\mathrm{dom}_T(x)$ nor $\mathrm{dom}_S(x)$ is empty. Let X_T be the set of all elements x of $\langle H\rangle$ so that $\mathrm{dom}_S(x)$ is empty. Let X_S be the set of all elements x of $\langle H\rangle$ so that $\mathrm{dom}_T(x)$ is empty. Clearly $X_T \cup X_S \cup X_{T,S} \cup X_{S,T} = \langle H\rangle$.

Because $\langle H \rangle$ contains elements x for which $\mathrm{dom}_T(x) \neq \emptyset$ and elements x for which $\mathrm{dom}_S(x) \neq \emptyset$ every copy of H in H contains elements x for which $\mathrm{dom}_T(x) \neq \emptyset$ and elements x for which $\mathrm{dom}_S(x) \neq \emptyset$. Hence there is no embedding from H into H/X_T or into H/X_S. Therefore we may assume without loss of generality that there is a copy H^* of H with $\langle H^* \rangle \subseteq X_{T,S}$. Let $F^* \subseteq \langle H^* \rangle$ so that there is an isomorphism f from H/F to H/F^*. Extend f to an embedding g from $H/(T \cup F)$ into H^*. Denote the image of T under g by T^*.

There are only finitely many elements of T^* which are not larger than all of the elements of F^*. If x is in T^* and larger than all of the elements of F^* then $x \geq_T F^*$. Because such an x is also an element of $X_{T,S}$ it follows that $x \geq_S D'$ for some copy D' of D. Furthermore all of those copies of D are lexicographically smaller than F^*. Hence there are only finitely many of those copies of D. Label x with the smallest such copy of D. This labeling partitions the elements of T^* which are larger than all of the elements of F^* into finitely many classes so that the restriction of H to any one of those classes has an embedding into S. Hence $T^* \preceq S$ and therefore $T \preceq S$. •

A.4.6 Orbits of homogeneous systems whose age is freely amalgamable

Let H be a homogeneous relational system with language L and automorphism group $\mathrm{aut}(H)$. Let T be an orbit of H with base F. There exists a homogeneous relational system H^T whose language K is an expansion of L and whose base is T and automorphism group is $\mathrm{aut}(H)_F$. If the age of H is freely amalgamable then the age of H^T is freely amalgamable. If every component of H has arity at most two then H/T is equal to H^T.

• Let f be a function from a subset M of F into $n \in \omega$. Assume that M has m elements. The n-tupel $(x_0, x_1, \ldots, x_{n-1})$ *agrees* with f if $x_{f(y)} = y$ for all $y \in M$. If $(x_0, x_1, \ldots, x_{n-1})$ agrees with f then the $(n-m)$-tupel $(x_0, x_1, \ldots, x_{n-1})_f$ is obtained from $(x_0, x_1, \ldots, x_{n-1})$ by removing all of the elements of the form $x_{f(a)}$ for $a \in M$ and retaining the order of the remaining elements.

For all $m < n \in \omega$, for every n-ary component R of H and for every function f from a subset of F of size m into n let R_f be the $(n-m)$-ary relation with base T so that $R_f(a_0, a_1, \ldots, a_{n-m-1})$ is positive if and only if there is an n-tupel $(x_0, x_1, \ldots, x_{n-1})$ which agrees with f and $(x_0, x_1, \ldots, x_{n-1})_f = (a_0, a_1, \ldots, a_{n-m-1})$ and $R(x_0, x_1, \ldots, x_{n-1}) = +$. Note that if the arity of R_f is one and R_f is positive on some element of T then R_f is positive on all elements of T because $\mathrm{aut}(H)_F$, the stabilizer group of F, is transitive on T.

Let H^T be the relational structure with base T whose components are of this form R_f and not unary. Every component of H/T is a component of H^T. (The relation R does not change if f is the empty function.) If every component of H has arity at most two then every component in H^T which is not a component of H/T has arity at most one. Hence in this case H/T is equal to H^T except for unary relations which are positive on every element of T. We disregard such unary relations.

It follows from the construction of H^T that every local isomorphism of H^T has an extension to a local isomorphism g of H so that $g(x) = x$ for all $x \in F$. The local isomorphism g has an extension to an automorphism of H which is an element of $\mathrm{aut}(H)_F$. Hence the stabilizer group $\mathrm{aut}(H)_F$ is the automorphism group of H^T and every local isomorphism of H^T has an extension to an automorphism of H^T. It follows that H^T is a homogeneous relational system.

Assume that the age of H is freely amalgamable. Let A and B be two compatible elements in the age of H^T and let C be the common restriction of A and B to the intersection of their bases. Let f be an embedding of A into H^T and A^f the copy of A in H^T which is the image of f. Let g be an embedding of B into H^T and B^g the copy of B in H^T which is the image of g. (Note that f and g are embeddings which preserve the relations of H together with the additional relations in H^T.) Let C^f be the copy of C in H^T which is the image of C under f and C^g the copy of C in H^T which is the image of C under g. The function $f \circ g^{-1}$ is a local H^T isomorphism of C^g to C^f. That is it has an extension to a local isomorphism of H which fixes all of the elements of F. Hence it can be extended to an automorphism h of H^T, that is an automorphism of H which fixes the elements of F. (The function h is an element of $\mathrm{aut}(H)_F$.) Let B' be the image of B^g under h. Let A_F be the restriction of H to $\langle A^f\rangle \cup F$. Let B_F be the restriction of H to $\langle B'\rangle \cup F$ and L_F the restriction of H to $\langle C^f\rangle \cup F$. There exists a copy $\overline{B}_F$ of B_F so that $\langle A_F\rangle \cap \langle \overline{B}_F\rangle = \langle L_F\rangle$ and $\langle B_F\rangle \cap \langle A_F\rangle = \emptyset$. Then A_F and $\overline{B}_F$ are compatible. Hence they have a free amalgam D. •

Example: Let B be the complete three-uniform hypergraph with vertices $V(B) = \{a, b, 0, 1, 2\}$ and edges $E(B) = \{\{a, b, 0\}, \{a, b, 1\}, \{a, b, 2\}, \{a, 0, 1\}, \{0, 1, 2\}\}$. Let H_B be the homogeneous B-free three-uniform hypergraph. The age of H_B is freely amalgamable by A.2.5. Let a', b' be two vertices of H_B and T the orbit of vertices x in $V(H_B)$ so that $\{x, a', b'\}$ is a hyperedge of H_B. The set T is then an orbit of the set $F = \{a', b'\}$.

The homogeneous relational system H^T has three components G/T, G_a and G_b. The component G_a is a graph so that $x, y \in T$ are adjacent in G_a if $\{x, y, a\}$ is a hyperedge of H. The component G_b is a graph so that $x, y \in T$ are adjacent in G_b if $\{x, y, b\}$ is a hyperedge of H. It follows from the previous argument that H^T is a homogeneous relational system whose age is freely amalgamable. The boundary of H^T is the relational system R with components $(H/T)'$, G'_a and G'_b. The component $(H/T)'$ is a three uniform hypergraph and the components G'_a and G'_b are graphs. The base of R consists of three elements, say $\{r, s, t\}$. The set $\{r, s, t\}$ is the only hyperedge of the three-uniform component and $E(G'_a) = \{r, s\}$ and $E(G'_b) = \emptyset$.

The age of H/T is freely amalgamable according to the previous theorem. But H/T is not homogeneous as it does not satisfy the mapping condition. Let $0'$ and $1'$ be two elements of T so that $\{a', 0', 1'\}$ is a hyperedge of H_B. Let A with $V(A) = \{0'', 1'', 2''\}$ and $E(A) = \{\{0'', 1'', 2''\}\}$ be a three uniform hypergraph. Then A is in the age of G^T. Let f with $f(0'') = 0'$ and $f(1'') = 1'$ an embedding of $A - 2''$ into H/T. The embedding f does not have an extension to an embedding of A into H/T.

The age of H/T is freely amalgamable and has therefore a homogeneous representative. Every element in the age of H has an embedding into H/T. Hence the homogeneous representative of the age of H/T is H_B. According to A.4.4 the relational system H/T is age indivisible.

A.4.7 Determining the orbits from the boundary of the age

Let H be a homogeneous relational system whose age is freely amalgamable and the arity of each of the components of H is two. Let T be an orbit of H with base F. Then according to A.4.6 the restriction of H to T is a homogeneous relational system whose age is freely amalgamable. According to A.2.5 the boundary of the age of H/T consists of complete relational systems.

Let A be an element of the boundary of the age of H/T. The set $\langle A\rangle$ must have at least two elements otherwise T would be empty. Assume without loss that $\langle A\rangle \subseteq T$. (Of course H restricted to $\langle A\rangle$ must be different from A. Just the elements of A are elements of T.) Let B be the relational system of the same language as H with base $A \cup F(T)$ so that A is a restriction of B and which has the property that for every $a \in \langle A\rangle$ the restriction of B to $F(T) \cup \{a\}$ is equal to the restriction of H to $F(T) \cup \{a\}$. No component of B is positive on any argument unless required to be positive by those conditions. Because A is not in the age of H/T the relational system B is not in the age of H. Let $C \subseteq F(T)$ have a minimal number of elements so that the restriction of B to $\langle A\rangle \cup C$ is not in the age of H. It follows that $\langle A\rangle \cup C$ is in the boundary of the age of H and hence is complete. Let D be the restriction of B to $\langle A\rangle \cup C$.

The base of D is partitioned into $\langle A\rangle$ and C. For any two elements a and b in $\langle A\rangle$ there is a local isomorphism of D which fixes every element of C and maps a to b. That is the type of a over C equals the type of b over C. If D is a relational system and the base of D is partitioned into the sets A and C so that for every pair of vertices a and b of A there is a local isomorphism of D which fixes every element of C and maps a to b we say that the partition (A, C) is *type preserving*.

Let (A, C) be a type preserving partition of the boundary element D of the age of H so that $\langle A\rangle \geq 2$. Then D/C is in the age of H and hence there is a finite subset $F \subseteq \langle H\rangle$ so that there is an isomorphism f from D/C to H/F. Let $a \in A$ and T be the orbit of all elements x of H so that f has an extension to an isomorphism from $D/(\{a\} \cup C)$ which maps a to x. It follows that the restriction of D to A is in the boundary of H/T.

Assume that $D_0, D_1, D_2, \dots$ is a finite or infinite sequence of boundary elements of H. Each D_i having the type preserving partition (A_i, C_i) with $\langle A_i\rangle \geq 2$. Let a_i be an element of A_i for every i and C'_i the restriction of D_i to $\{a_i\} \cup \langle C_i\rangle$. Every D_i/C'_i is in the age of H. Assume without loss that $a_i = a_j$ for all i and j. Then the C'_i are pairwise compatible and hence there is a free amalgam of the D_i/C'_i which is in the age of H. We proceed as before and obtain an orbit T of H whose set of boundaries is the set $\{D_i/A_i \mid i\}$. (Well, not quite. There might be D_i/A_i which have an embedding into another D_j/A_j. In such a case just remove the relational system D_j/A_j.)

It is now easy to see how the type preserving partitions of the boundary of the age of H determine the sets of boundaries of the orbits of H and hence how the orbits of H are related under the relation younger. If T and S are two orbits of H then H/S will be younger than H/T if every element in the boundary of H/T is an element in the boundary of H/S. For every element A in the boundary of T there is an element D in the boundary of H and a type partition $(\langle A\rangle, C)$ of D so that A is $D/\langle A\rangle$. If H/S is younger than H/T there must be an element B in the boundary of H/S to that B has an embedding into

A. Hence there must by an element D' in the boundary of H which has a type preserving partition of the form $(\langle B\rangle, C')$.

Let H be a homogeneous relational system all of whose components have arity two and whose age is freely amalgamable. Then the orbits of H satisfy the chain condition if and only if for any two sequences $D_0, D_1, D_2, \ldots$ and $D'_0, D'_1, D'_2, \ldots$ of elements in the boundary of H which have type preserving partitions $(A_0, C_0), (A_1, C_1), (A_2, C_2), \ldots$ and $(A'_0, C'_0), (A'_1, C'_1), (A'_2, C'_2), \ldots$ respectively there either is for every one of the i's a j so that D_i/A_i has an embedding into D'_j/A'_j or for every one of the j's there is an i sot that D'_j/A'_j has an embedding into D_i/A_i.

• It follows from the previous discussion that for any two orbits S and T of H either H/T is younger than H/S or H/S is younger than H/T if and only if for any two sequences $D_0, D_1, D_2, \ldots$ and $D'_0, D'_1, D'_2, \ldots$ of elements in the boundary of H which have type preserving partitions $(A_0, C_0), (A_1, C_1), (A_2, C_2), \ldots$ and $(A'_0, C'_0), (A'_1, C'_1), (A'_2, C'_2), \ldots$ respectively there either is for every one of the i's a j so that D_i/A_i has an embedding into D'_j/A'_j or for every one of the j's there is an i sot that D'_j/A'_j has an embedding into D_i/A_i.

The restriction of H to an orbit T is again a homogeneous relational system whose age is freely amalgamable. (A.4.6). Hence if H/S is younger than H/T then there is an embedding from H/S into H/T. Hence in this case H/S precedes H/T, that is $H/S \preceq H/T$. •

In the case that the age of H is a universal class, that is the boundary has finitely many elements, this theorem provides an effective checking if the orbits of H satisfy the chain condition. Every element of the boundary is finite and hence has only finitely many type preserving partitions.

Another consequence of the previous discussion is that it is not difficult to construct homogeneous structures whose restrictions to their orbits form any given denumerable partial order under embedding. In particular it is possible to construct for every denumerable or finite partial order P a set $\mathcal{T}$ of tournaments so that the ages of the orbits of the $\mathcal{T}$-free homogeneous directed graph $H_{\mathcal{T}}$ form a partial order under $\subseteq$ which is isomorphic to P.

A.4.8 Indivisibility and the ages of the orbits

If H is a homogeneous relational system whose age is freely amalgamable and if T is an orbit of H then H restricted to T is age indivisible.

• The age of the homogeneous relational system H^T is freely amalgamable by A.4.6. Every component of H/T is a component of H^T by A.4.6. Then the age of H/T is freely amalgamable by A.2.5 which in turn implies that H/T is age indivisible by A.4.4. •.

Let H be an indivisible homogeneous relational system whose age is freely amalgamable. Then the restrictions of H to the orbits of H form a total quasiorder under the relation of younger.

• Let S and T be two orbits of H. Because H is freely amalgamable both S and T are age indivisible. Because H is indivisible $H/S \preceq H/T$ or $H/T \preceq H/S$ by A.4.5. We may assume without loss that $H/S \preceq H/T$ and hence that there is a partition of S into finitely many classes $(C_0, C_1, C_2, \dots, C_{n-1})$ so that H/C_i has an embedding into H/T for every $i \in n$. Because H/S is age indivisible there is an $i \in n$ so that the age of S is equal to the age of S/C_i. Hence the age of H/S is a subset of the age of H/T. •

Problem: Is there an indivisible homogeneous relational system H which is indivisible but one of its orbits is not indivisible? How about this if the age of H is freely amalgamable? If the age of H is freely amalgamable and every component of H has arity at most two?

Problem: Is there a homogeneous relational system whose restriction to its orbits satisfies the chain condition and which is not indivisible? If the age is freely amalgamable? If the age of H is freely amalgamable and every component has arity at most two?

A.4.9 Examples, restrictions to all orbits are isomorphic to each other

The age of the homogeneous K_n-free graph H_n is freely amalgamable. Because the arity of H_n is two the restriction of H_n to any orbit is again a homogeneous graph which is freely amalgamable, hence again one of the homogeneous graphs of the form H_m with $m \leq n$. For every $m \leq n$ there is an orbit of H_n which is isomorphic to H_m. It follows that the homogeneous graphs H_n form a chain of type ω under embedding and the orbits of each H_n form a subchain of this chain. Certainly then the orbits of H_n satisfy the chain condition.

Within the class of graphs with two types of edges, E_1 and E_2, let K_3^1 be the graph with base $3 = \{0, 1, 2\}$, $E_1(K_3^1) = [3]^2$ and $E_2(K_3^1) = \emptyset$. Let K_3^2 be the graph with base 3, $E_2(K_3^1) = [3]^2$ and $E_1(K_3^1) = \emptyset$. Let $\mathcal{R} := \{K_3^1, K_3^2\}$ and H the $\mathcal{R}$-free homogeneous relational system. The elements of R are complete hence the age of H is freely amalgamable. Let x be an element of $\langle H \rangle$. The set $\{x\}$ has three orbits. The set T_0 of vertices which are not adjacent to x, the set T_1 of vertices which are via an edge of E_1 adjacent to x and the set T_2 of vertices which are via an edge of E_2 adjacent to x. Note that $E_1(H) \cap [T_1]^2 = \emptyset$ and that $E_2(H) \cap [T_2]^2 = \emptyset$. The age of H/T_1 is the set of all finite graphs which have edges only of type E_2. Hence because H/T_1 is homogeneous it is the Rado graph with edges entirely from E_2. Similarly H/T_2 is the Rado graph with edges entirely from E_1. Hence the ages of H/T_1 and H/T_2 are not comparable by set inclusion. It follows that H is not indivisible.

If the restriction of the homogeneous relational system H to all of its orbits is isomorphic to H, then the orbits of H satisfy the chain condition. Examples of such homogeneous relational systems are the Rado graph, the universal tournament, the universal directed graph, the universal three uniform hypergraph, the universal n-uniform hypergraph for any $n \in \omega$, the rational numbers as an order structure, the universal partial order etc.

Example: Let P be the graph with two types of edges so that $E_1(P)$ forms a pentagon and $E_2(P)$ forms a pentagon and $E_1(P) \cap E_2(P) = \emptyset$ and $V(P) = 5$. Let H_P be the P-free homogeneous relational system. Because P does not have a type preserving partition the restriction of H_P to any of its orbits is isomorphic to H_P. On the other hand it is not difficult to see that H_P is not universal.

If the restriction of the homogeneous relational system H to any of its orbits is isomorphic to H then H is indivisible.

• Let ω be the base of H and I_n the restriction of H to n. Let (A, B) be a partition of ω. Assume that H can not be embedded into the restriction of H to A. Then for some $n \in \omega$ there is an embedding f of I_n into the restriction of H to A which does not have an extension to an embedding of I_{n+1} into the restriction of H to A. Let X be the set of all elements $g(n)$ so that g is an extension of f to I_{n+1}. Then X is an orbit of H and $X \subseteq B$. Hence there is an embedding of H into the restriction of H to B. •

A.4.10 Indivisibility of universal relational systems

The proof of the next theorem follows by an argument similar to the one in 10.3.3.

If H is a universal relational system then H is indivisible.

If H is the Rado graph or the universal three uniform hypergraph or the universal tournament or the universal directed graph etc. then the age of the square of H is equal to the age of H. Hence all such homogeneous relational systems are indivisible.

The k-uniform hypergraph G is *l-covering* if every element of $[\langle G\rangle]^l$ is a subset of some hyperedge of G. Every l-covering hypergraph with $l \geq 2$ is complete. Hence if $\mathcal{T}$ is a set of finite l-covering k-uniform hypergraphs with $l \geq 3$ then Forb($\mathcal{T}$) is freely amalgamable and we obtain the $\mathcal{T}$-free homogeneous hypergraph $H_{\mathcal{T}}$. Because the age of the square of $H_{\mathcal{T}}$ is equal to the age of $H_{\mathcal{T}}$ the homogeneous hypergraph $H_{\mathcal{T}}$ is indivisible. [ElSauer2]

A.4.11 Indivisibility of the homogeneous K_n-free graphs

It is proven in [Rodl] that the triangle free homogeneous graph is indivisible and in [ElSauer3] that if $\mathcal{T}$ is a finite set of tournaments then the $\mathcal{T}$-free homogeneous graph $H_{\mathcal{T}}$ is indivisible if and only if it satisfies the chain condition.

The K_n-free graph H_n is indivisible for every $n \in \omega$ [ElSauer4]

• Let H be the K_n-free homogeneous graph for some fixed $n \in \omega$. Let ω be the set of vertices of H. If X is an orbit of H then $F(X)$ is the base of X and $F_1(X) \subseteq F(X)$ is the set of elements of $F(X)$ which are adjacent to the elements of X. (If x and y are two elements of X then, according to the definition of orbit, there is an automorphism of H which maps x to y and fixes every element of $F(X)$. Hence if $a \in F_1(X)$ then it is adjacent to every element of X and if $a \in F(X) - F_1(X)$ then it is not adjacent to any element of X.) The set $V(\text{H}) = \omega$ is the orbit with $F(\text{H}) = \emptyset$.

Because the arity of H is two and H has free amalgamation the restriction of H to the orbit X is again a homogeneous system with free amalgamation. (A.4.6). If m is the largest number so that K_m embeds into $F_1(X)$ then $n-m$ is the smallest number so that K_{n-m} does not embed into X. Hence the restriction of H to X is isomorphic to H_{n-m}. If the restriction of H to the orbit X is isomorphic to H_r then r is the *rank* of the orbit X. If X has rank two then no two vertices of X are adjacent. There is no orbit of rank one. (H_1 is the empty graph.)

The orbit X is a continuation of the orbit Y if $F_1(Y) \subseteq F_1(X)$ and if $F(Y)-F_1(Y) \subseteq F(X)-F_1(X)$. It follows that if X is a continuation of Y then $X \subseteq Y$. The orbit X is a *refinement* of the orbit Y if X is a continuation of Y and the rank of X is equal to the rank of Y. Clearly a refinement of a refinement is a refinement. If the rank of Y is larger than two then X is a *restriction* of Y if it is a continuation of Y and the rank of X is equal to the rank of Y minus one. If the rank of Y is equal to two then every refinement of Y is also a restriction of Y. A restriction of a refinement of an orbit is a restriction of that orbit.

The orbits X and Y are *compatible* if for all $a \in F(X) \cap F(Y)$, $x \in X$ and $y \in Y$ the vertex a is adjacent to x if and only if it is adjacent to y. The meet of two compatible orbits X and Y is the orbit Z with $F(Z) = F(X) \cup F(Y)$ and $F_1(Z) = F_1(X) \cup F_1(Y)$. Notice that if the orbit X is a continuation of the orbit Y then X and Y are compatible and Y is the meet of X and Y.

Claim 1: Let X and Y be compatible orbits.

i If no vertex in $F_1(X)-F(Y)$ is adjacent to a vertex of $F_1(Y)-F(X)$ then the rank of the meet of X and Y is the maximum of the ranks of X and Y.

ii If $a \in X$ then the orbit Z with $F(Z) = F(X) \cup \{a\}$ and $F_1(Z) = F_1(X) \cup \{a\}$ is a restriction of X.

iii If $F(X)-F(Y) = \{a\}$ and a is not adjacent to any vertex in $F(Y)-F(X)$ and the rank of X is less than or equal to the rank of Y then the rank of the meet of X and Y is equal to the rank of Y.

iv Let B be a finite set with $F(X) \cap B = \emptyset$ then the orbit Z with $F(Z) = F(X) \cup B$ and $F_1(Z) = F_1(X)$ is a refinement of X.

v For every number l there is a refinement T of X all of whose elements are larger than l and not adjacent to any element in $l - F(X)$.

Proof: i: Let Z be the meet of X and Y. The rank of Z is $n-m$ where m is the largest number so that K_m embeds into $F_1(Z)$. There is no subset in $F_1(X) \cup F_1(Y)$ which induces a complete subgraph and which contains elements of $F_1(X)-F_1(Y)$ and elements of $F_1(Y)-F_1(X)$. Hence every complete subgraph of $F_1(Z)$ is either entirely in $F_1(X)$ or entirely in $F_1(Y)$.

ii. If the largest complete subgraph of $F_1(X)$ has size m then the largest complete subgraph of $F_1(Z)$ has size $m+1$.

iii. A special case of i.

iv. The largest complete subgraph of $F_1(X)$ is equal to the largest complete subgraph of $F_1(Z) = F_1(X)$.

v. Let $A = l \cup F(X)$ and T the orbit with $F(T) = A$ and $F_1(Y) = F_1(X)$. Then according to iv. the orbit T is a refinement of the orbit X.

∎

We will prove that if Blue and $\omega -$ Blue is a partition of $\omega = V(\mathrm{H})$ into blue and red elements then there is an embedding of H into Blue or into $\omega -$ Blue. If X is an orbit of H we write blue(X) to mean that the orbit X contains infinitely many blue elements and red(X) to mean that the orbit X contains infinitely many red elements.

If $A, B \subseteq \omega$ and $l \in \omega$ the relation $l < A$ means that every element in A is larger than l and $B < A$ means that every element of B is smaller than every element of A.

The formula $\phi_{\mathrm{Blue}}(X)$ is

For all $l \in \omega$ there exists a refinement Y of X with $l < F_1(Y) - F_1(X)$
so that for all refinements Z of Y with $F(Y) < F_1(Z) - F_1(Y)$
there exists a restriction R of Z with $F(Z) < F_1(R) - F_1(Z)$ (1)
so that $\phi_{\mathrm{Blue}}(R)$ and blue(R).

Claim 2: $\neg\phi$Blue(X) implies $\phi_{\omega-\mathrm{Blue}}(X)$.

Proof of Claim: We negate (1) formally to obtain $\neg\phi_{\mathrm{Blue}}(X)$:

There is an $l' \in \omega$ so that for all refinements Y' of X with $l' < F_1(Y') - F_1(X)$
there exists a refinement Z' of Y' with $F(Y') < F_1(Z') - F_1(Y')$
so that for all restrictions R' of Z' with $F(Z') < F_1(R') - F_1(Z')$ (2)
not $\phi_{\mathrm{Blue}}(R)$ or only finitely many blue elements in R.

We want to prove that (2) implies

For all $l \in \omega$ there exists a refinement Y of X with $l < F_1(Y) - F_1(X)$
so that for all refinements Z of Y with $F(Y) < F_1(Z) - F_1(Y)$
there exists a restriction R of Z with $F(Z) < F_1(R) - F_1(Z)$ (3)
so that not $\phi_B(R)$ and red(R).

In order to prove (3) from (2) let $l \in \omega$ be given and l' the number produced by formula (2).

Let Y' be a refinement of X with $l' \leq \max(l, l') < F_1(Y') - F_1(X)$ and so that there is an element $y \in F_1(Y') - F(X)$. (Such an orbit Y' exists by Claim 1.iii.) Let Z' be the refinement of Y' produced by formula (2) when given the orbit Y'. Then $F(Y') < F_1(Z') - F_1(Y')$ which implies that $\max(l, l') < F_1(Z') - F_1(X)$. (Note that $F(X) \subseteq F(Y') \subseteq F(Z')$ and that $F_1(X) \subseteq F_1(Y') \subseteq F_1(Z')$. If $a \in F_1(Z') - F(X)$ and $a \in F(Y')$ then $a \in F_1(Y')$ and hence $\max(l, l') < a$ because $\max(l, l') < F_1(Y') - F_1(X)$.

In particular $\max(l,l') < y$. If $a \in F(Z') - F(Y')$ then $\max(l,l') < y < a$ because $F(Y') < F_1(Z') - F_1(Y')$.)

Let $Y = Z'$ produced by formula (2). Then $l \leq \max(l,l') < F_1(Z') - F_1(X) = F_1(Y) - F_1(X)$. Let Z be a refinement of Y with $F(Y) < F_1(Z) - F_1(Y)$. Let R be a restriction of Z with $F(Z) < F_1(R) - F_1(Z)$.

Then $F(Z') < F_1(R) - F_1(Z')$. (If $a \in F_1(R) - F_1(Z')$ and $a \in F(Z)$ then $a \in F_1(Z)$ and $F(Z') < a$ because $F(Z') < F_1(Z) - F_1(Z')$. If $a \in F_1(R) - F(Z)$ then $\max(F(Z') \leq \max(F(Z)) < a$.) Because R is a restriction of Z which is a refinement of $Y = Z'$ it follows that R is a restriction of Z'. Hence formula (2) stipulates that not $\phi_{\text{Blue}}(R)$ or only finitely many blue elements in R which in turn implies that not $\phi_{\text{Blue}}(R)$ and red(R). Because not red(R) implies $\phi_{\text{Blue}}(R)$. Rename $\neg\phi_{\text{Blue}}(X)$ to $\phi_{\omega-\text{Blue}}(X)$

∎

It follows from Claim 2 that it suffices to prove that if $\phi_{\text{Blue}}(\omega)$ then there is an embedding of H into the restriction of H to Blue. Assume $\phi_{\text{Blue}}(\omega)$.

The formula for $\phi_{\text{Blue}}(X)$ produces for every $l \in \omega$ a refinement Y of X with $l < F_1(Y) - F_1(X)$ so that .... as described in (1). We denote that property of Y by $\psi_{\text{Blue}}(Y)$. It follows that if $\psi_{\text{Blue}}(Y)$ and Z is a refinement of Y with $F(Y) < F_1(Z) - F_1(Y)$ then $\psi_{\text{Blue}}(Y)$. Also, if $\phi_{\text{Blue}}(X)$ and $l \in \omega$ there exists a refinement Y of X with $l < F_1(Y) - F_1(X)$ and $\psi_{\text{Blue}}(Y)$. If $\psi_{\text{Blue}}(Y)$ then there is a restriction R of Y so that $\ell < F_1(R) - F_1(Y)$ for all $\ell \in \omega$ and $\psi_{\text{Blue}}(R)$. (This is so because we can first choose a refinement of Y which is higher than ℓ and then the restriction given by the formula will be even higher.)

For $v \in \omega$ let I_v be the restriction of H to v. The subset J of ω having v elements is an *initial segment of length* v if the order preserving map from v to J is an isomorphism of I_v. The initial segment J' of length u is an *extension* of the initial segment J of length v if $v < u$ and every element of J is smaller than every element of $J' - J$.

Let J be an initial segment. We denote by $\mathcal{J}$ the set of subsets $L \subseteq J$ so that K_{n-1} can not be embedded into L. The element $n \in L \in \mathcal{J}$ is *critical* if the largest complete subgraph in $L \cap n$ is strictly smaller than the largest complete subgraph in $n + 1 \cap L$. We declare -1 to be a critical element of $\emptyset$. Given that $n \neq -1$ is a critical element of L denote by n^- the largest critical element of L smaller than n. If L and I are two subsets of J we denote by $L \sqcap I$ the set of all $x \in L \cap I$ so that $y \in L$ if and only if $y \in I$ for all $y \leq x$. (The longest common initial segment of L and I.) Note that if r is a critical element of $L \sqcap I$ then it is a critical element of L and a critical element of I. If $a \in J$ then $\Gamma(a)$ is the set of elements of J which are strictly smaller than a and adjacent to a.

For every element $L \in \mathcal{J}$ denote by O_L^J the orbit with $F(O_L^J) = J$ and $F_1(O_L^J) = L$. If n is a critical element of L denote by $O_L^{J,n}$ the orbit with $F(O_L^{J,n}) = J \cap (n+1)$ and $F_1(O_L^{J,n}) = L \cap (n+1)$.

An *assignment of* J associates with every $L \in \mathcal{J}$ and every critical element n of L a refinement $X_L^{J,n}$ of the orbit $O_L^{J,n}$ with $\psi_{\text{Blue}}(X_L^{J,n})$ and a restriction $R_L^{J,n}$ of $X_L^{J,n}$ with $\psi_{\text{Blue}}(R_L^{J,n})$. If $n \neq -1$ then $X_L^{J,n}$ is the meet of R_L^{J,n^-} and $O_L^{J,n}$. Let $\Delta_L^{J,n} := F_1(R_L^{J,n}) - F_1(X_L^{J,n})$. It follows that $F_1(R_L^{J,n}) - J$ is the union of the $\Delta_L^{J,r}$ over all critical elements $r \leq n$ of L and that $F_1(X_L^{J,n}) - J$ is the union of the $\Delta_L^{J,r}$ over all critical elements $r < n$ of L.

The initial segment $J \subseteq$ Blue together with an assignment is *well chosen* if for every $L, I \in \mathcal{J}$ and critical element n of L and m of I:

i. If n is a maximal critical element of L then the orbits $X_L^{J,n}$ and O_L^J are compatible and have the same rank. The rank of the meet of the compatible orbits $X_L^{J,n}$ and O_L^J is equal to the rank of the orbit $X_L^{J,n}$ equal to the rank of the orbit O_L^J.

ii. If r is a critical element of $L \sqcap I$ then $\Delta_L^{J,r} = \Delta_I^{J,r}$ and $R_L^{J,r} = R_I^{J,r}$ and $X_L^{J,r} = X_I^{J,r}$. If $x \in \Delta_L^{J,n} \cap \Delta_I^{J,m}$ then $n = m$ is a critical element of $L \sqcap I$.

iii. If $a > n$ are two elements of J then $a > F(R_L^{J,n})$ for all $L \in \mathcal{J}$. Also $\Delta_L^{J,n} > b$ for every $b < n$.

iv. If an element a in J is adjacent to an element $x \in \Delta_L^{J,n}$ with $a > n$ then n is a critical element of $\Gamma(a)$ and $n \in \Gamma(a) \sqcap L$.

Let J be a well chosen initial segment with assignment $F_L^{J,n}$ and $R_L^{J,n}$ for all $L \in \mathcal{L}$ and all critical elements n of L. Let $\ell \in \omega$ so that $J < \ell$ and $F(R_L^{J,n}) < \ell$ for all $L \in \mathcal{J}$ and critical elements n of L. (ℓ will be used as a variable which will be assigned larger and larger values.)

The emptyset is a well chosen initial segment because $\phi(\omega)$ and hence there is a refinement $X_\emptyset^{\emptyset,-1}$ of ω with $\psi_{\text{Blue}}(X_\emptyset^{\emptyset,-1})$. The orbit $X_\emptyset^{\emptyset,-1}$ has a restriction $R_\emptyset^{\emptyset,-1}$. The proof will be complete if for every $v \in \omega$ and every well chosen initial segment J of length v there is a well chosen extension J' of length $v + 1$.

Because H has the mapping extension property there is an extension f of the order map from I_v to J to an embedding of I_{v+1}. Let B be the set of elements of J which are adjacent to $f(v)$. Then K_{n-1} does not embed into B hence $B \in \mathcal{L}$. Note that $f(v) \in O_B^J$ and hence every element of O_B^J together with J forms an extension of J to an initial segment of length $v + 1$. We have to find an appropriate $a \in O_B^J \cap$ Blue so that $J \cup \{a\}$ is well chosen.

Let n be the largest critical element of B. Let P be the meet of the orbits O_B^J and $X_B^{J,n}$ then, according to condition i., the rank of P is equal to the rank of O_B^J is equal to the rank of $X_B^{J,n}$. Hence P is a refinement of both orbits and $\psi_{\text{Blue}}(P)$ because $\psi_{\text{Blue}}(X_B^{J,n})$. This argument will be used repeatedly. One also has to check that $F(X_B^{J,n}) < F_1(P) - F_1(X_B^{J,n})$. This is the case because of condition iii. To demonstrate the main line of argument more clearly we will not write this out again.

Using Claim 1.v. let T be a refinement of P so that all of the elements of T are larger than ℓ and not adjacent to any element in $\ell - P$. Then $\psi_{\text{Blue}}(T)$ and hence T contains infinitely many blue elements. Let a be such a blue element. Note that if a is adjacent to an element in $x \in F_1(R_L^{J,r}) - J$ for some $L \in \mathcal{J}$ and critical element r of L then $x \in F_1(X_B^{J,n})$.

Claim 3: Let $L \in \mathcal{J}$ and n a maximal critical element of L. Let K be a complete subgraph in $F_1(R_L^{J,n}) \cup B \cup \{a\}$ so that $a \in K$. Then there is a complete subgraph K_1 in $B \cup \{a\}$ of the same size as K.

Proof: Let x be the largest element in $K - L$. Let $K' := \{y \in K - \{a\} \mid x < y\}$ and $K'' := \{y \in K \mid y \leq x\}$. Then $K'' \subseteq F_1(R^J_{L-\{a\}})$ and also $K'' \subseteq F_1(X^{J,n}_B)$ due to the choice of a and the ordering requirement in the second part of condition iii. Let $y \in K'' - J$. It follows that $y \in F_1(R^{J,n}_{L-\{a\}})$ and hence $y \in F_1(X^{J,n}_B)$. Hence $y \in \Delta^{J,n'}_{L-\{a\}} \cap \Delta^{J,r}_B$ for some critical elements n' of $L - \{a\}$ and critical element r of B. It follows from condition ii. that $n' = r$ is a critical element of $(L - \{a\}) \cap B$ and $X^{J,r}_{L-\{a\}} = X^{J,r}_B$. It follows that there is a smallest critical element r of B so that $K'' \subseteq F_1(X^{J,r}_B)$ and there is some element $y \in K'' \cap \Delta^{J,r}_B$.

Because $X^{J,r}_B$ is a refinement of $O^{J,r}_B$ it follows that $B \cap r + 1$ contains a complete subgraph S of as many elements as there are in K'' and because r is a critical element of B we may assume that $r \in S$. Every element $z \in K'$ is adjacent to y and hence, according to condition iv., is adjacent to every element in S. The element a is adjacent to every element in $S \subseteq B$. Hence $\{a\} \cup K' \cup S$ induces a complete subgraph of as many vertices as there are in K.

▮

We have to prove that $J \cup \{a\} := J_a$ is well chosen. Assign a new value larger than a to the variable ℓ. Let $\mathcal{J}_a$ be the set of subsets $L \subseteq J_a$ so that K_{n-1} does not have an embedding into L. We have to define the orbits $X^{J_a,n}_L$ and $R^{J_a,n}_L$ for all $L \in \mathcal{L}_a$ and critical elements n of L.

Let $L \in \mathcal{L}_a$ and $n \neq a$ a critical element of L. In this case let $X^{J_a,n}_L = X^{J,n}_{L-\{a\}}$ and let $R^{J_a,n}_L = R^{J,n}_{L-\{a\}}$.

If $L \in \mathcal{L}_a$ and a is a critical element of L let n be the largest critical element of $L-\{a\}$. It follows that the rank of the orbit $O^{J_a}_L$ is one smaller than the rank of the orbit $O^J_{L-\{a\}}$ which is equal to the rank of the orbit of $X^{J,n}_{L-\{a\}}$, condition i., which is one larger than the rank of the orbit $R^{J,n}_{L-\{a\}}$. Hence the ranks of the orbits $O^{J_a}_L$ and $R^{J,n}_{L-\{a\}}$ are equal.

The orbits $O^J_{L-\{a\}}$ and $R^{J,n}_{L-\{a\}}$ are compatible hence the orbits $O^{J_a}_L$ and $R^{J,n}_{L-\{a\}}$ are compatible because of the choice of a to be large enough and certainly not in the intersection of $F(O^{J_a}_L)$ and $F(R^{J,n}_{L-\{a\}})$. Claim 3 implies that the rank of the meet M of the orbits $O^{J_a}_L$ and $R^{J,n}_{L-\{a\}}$ is equal to their common rank. Hence M is a refinement of $R^{J,n}_{L-\{a\}}$ and hence $\psi_{\text{Blue}}(M)$. Let $X^{J_a,a}_L := M$ and let $R^{J_a,a}_L$ be a restriction of $X^{J_a,a}_L$ so that $\ell < F_1(R^{J_a,a}_L) - F_1 X^{J_a,a}_L)$ and $\psi_{\text{Blue}}(R^{J_a,a}_L)$. Reset the number ℓ so that $\ell > F_1(R^{J_a,a}_L) - F_1 X^{J_a,a}_L)$ and repeat the process for all elements $L \in \mathcal{J}$.

To check that this assignment of J_a is well chosen is straight forward. Claim 3 has to be used again to verify condition i. $(F_1(X^{J,n}_L) \subseteq F_1(R^{J,n}_L).)$ •

A.5 COLORING COPIES OF RELATIONAL SYSTEMS

The previous section dealt with partitions of the set of elements of homogeneous relational systems. In this chapter we are going to discuss partitions of the set of edges of homogeneous graphs and digraphs. In principle we are interested in partitions of sets of other restrictions of homogeneous relational systems, like complete subgraphs etc, but outside of exactly formulating the problem very little is known. We do not have to consider the case where finitely many finite restrictions are removed from a homogeneous relational system as then only finitely many elements are removed. A case we have already studied.

A.5.1 Sets of copies and the Ramsey arrow

A *copy* of the relational system R in the relational system S is a restriction of S which is isomorphic to R. The set of copies of R in S is denoted by

$$\binom{R}{S}$$

The relational structure R is *S-indivisible* if for every partition (A, B) of $\binom{R}{S}$ there is a copy R^* of R in R such that

$$\binom{R^*}{S} \subset A \quad \text{or} \quad \binom{R^*}{S} \subset B.$$

The fact that R is S-indivisible is also expressed by writing $R \to (R)^S_2$.

The complete graph K_ω is edge-indivisible according to Ramseys theorem.

A.5.2 An obstruction to edge-indivisibility of graphs

Let G be a graph with ω as set of vertices. Associate with every vertex $n \in \omega$ the (0,1)-chain $\sigma(n) = (\sigma(n)_0, \sigma(n)_1, \sigma(n)_2, \ldots, \sigma(n)_{n-1})$ where $\sigma(n)_i = 1$ if and only if n and i are adjacent. For $n \neq m$ let $\sigma(n) < \sigma(m)$ if $\sigma(n)_i = 0$ for the smallest i with $\sigma(n)_i \neq \sigma(m)_i$ or if $n < m$ and $\sigma(n)_i = \sigma(m)_i$ for all $i \in n$. (Lexicographic ordering.)

Let A be the set of all edges $\{n, m\}$ of G so that if $n < m$ then $\sigma(n) < \sigma(m)$. Let B be the set of all the other edges of G. The edges in A are called *up edges* and the edges in B are called *down edges*. This partition of the edges of G into up and down edges is called *lexicographic partition* of the edges of G.

Let $K_{\omega,\omega}$ be the complete bipartite graph, that is the graph on ω as set of vertices in which two vertices are adjacent if and only if they have different parity. It was noticed in [Posa] that

If $K_{\omega,\omega}$ has an embedding into the graph G then G is not edge-indivisible.

• Assume that ω is the set of vertices of G and (A, B) is the lexicographic partition of the edges of G with A the set of up edges and B the set of down edges. Assume that there is an embedding of $K_{\omega,\omega}$ into G.

It suffices to prove that there is no copy $K^*_{\omega,\omega}$ of $K_{\omega,\omega}$ in G such that all of the edges of $K^*_{\omega,\omega}$ are in A or all of the edges of $K^*_{\omega,\omega}$ are in B.

Assume for a contradiction that there is a copy $K^*_{\omega,\omega}$ of $K_{\omega,\omega}$ all of whose edges are in A. Let $L \cup R = V(K^*_{\omega,\omega})$ with $L \cap R = \emptyset$ and every edge of $K^*_{\omega,\omega}$ contains a vertex of L and a vertex of R. Let n be the smallest number in R. For $x \in R$ denote by $\gamma(x)$ the restriction of the sequence $\sigma(x)$ to n. ($\gamma(x)$ consists of the first n elements of $\sigma(x)$.) Let $x \in R$ have maximal $\gamma(x)$ for all elements in S. Let $r \in L$ with $r > x$. Then $\gamma(x) \leq \gamma(r)$. Let $y \in R$ be larger than r. Then $\gamma(r) \leq \gamma(y)$ and hence $\gamma(x) = \gamma(r) = \gamma(y)$. But r is adjacent to n and y is not. Hence $\sigma(r) > \sigma(y)$ in contradiction that all of the edges of $K^*_{\omega,\omega}$ are elements of A.

Assume next that there is a copy $K^*_{\omega,\omega}$ of $K_{\omega,\omega}$ all of whose edges are in B. Let $L \cup R = V(K^*_{\omega,\omega})$ with $L \cap R = \emptyset$ and every edge of $K^*_{\omega,\omega}$ contains a vertex of L and a vertex of R. Let n be the smallest number in R. For $x \in L$ denote by $\gamma(x)$ the restriction of the sequence $\sigma(x)$ to n. Let $x \in L$ have minimal $\gamma(x)$ for all elements in L. Let $r \in R$ with $r > x$. Then $\gamma(x) \geq \gamma(r)$. Let $y \in L$ be larger than r. Then $\gamma(r) \geq \gamma(y)$ and hence $\gamma(x) = \gamma(r) = \gamma(y)$. But r is not adjacent to n and y is. Hence $\sigma(r) < \sigma(y)$ in contradiction that all of the edges of $K^*_{\omega,\omega}$ are elements of B. •

Let $\mathbf{Q}$ be the rationals as order structure. There is a partition (A, B) of $[\mathbf{Q}]^2$ so that for every copy $\mathbf{Q}^*$ of $\mathbf{Q}$ in $\mathbf{Q}$ the set $[\mathbf{Q}^*]^2$ has a non empty intersection with $[A]^2$ and with $[B]^2$. That is

$$\neg\big(\mathbf{Q} \to (\mathbf{Q})_2^{\{a,b\}}\big).$$

• Enumerate $\mathbf{Q}$ into the ω-sequence $r_0, r_1, r_2, \ldots$. let $A := \{\{r_i, r_j\} \mid \text{if } r_i < r_j \text{ then } i < j\}$ and $B = [\mathbf{Q}]^2 - A$. If there would be a copy $\mathbf{Q}^*$ of $\mathbf{Q}$ in $\mathbf{Q}$ so that $[\mathbf{Q}^*]^2 \subseteq A$ then $\mathbf{Q}$ would be order isomorphic to ω. Similarly there is no copy $\mathbf{Q}^*$ of $\mathbf{Q}$ in $\mathbf{Q}$ so that $[\mathbf{Q}^*]^2 \subseteq [\mathbf{Q}]^2 - A$. •

A.5.3 Weak indivisibility, canonical partitions

It follows that neither the Rado graph nor any of the graphs H_n are edge-indivisible and the rationals are not pair-indivisible.

Let

$$R \to (R)^S_{r\setminus l}$$

mean that for every partition of $\binom{R}{S}$ into r parts there is a copy R^* of R in R so that $\binom{R^*}{S}$ has a non empty intersection with at most l parts of the partition.

The relational structure R is weakly S-indivisible if for every partition (X, Y) of $\binom{R}{S}$, either for every element A of the age of R there is a copy A^* in R with $\binom{A^*}{S} \subseteq X$ or there is a copy R^* of R in R with $\binom{R^*}{S} \subseteq Y$.

The partition $(P_0, P_1, \ldots, P_{n-1})$ of the set $\binom{R}{S}$ is *canonical* if

i. for every copy R^* of R in R the set $\binom{R^*}{S}$ has a non empty intersection with every part of the partition,

ii. for every $i \in n$ and every partition of the set P_i into two parts X and Y there is an embedding f from R into R so that if $S' \in P_j$ for some $j \in n$ then the image of S' under f is again an element of P_j and either all of the images of the elements of P_i under f are in X or all of the images of the elements of P_i under f are in Y.

It follows that if $\binom{R}{S}$ has a canonical partition into l parts then $R \to (R)^S_{r\setminus l}$ for every $r \in \omega$ and the number l is minimal under the condition that $R \to (R)^S_{r\setminus l}$ for every $r \in \omega$. If $\binom{R}{S}$ has a canonical partition into two parts (X, Y) and for every element A in the age of R there are copies A^* and A^{**} of A in R so that $\binom{A^*}{S} \subseteq X$ and $\binom{A^{**}}{S} \subseteq Y$ then R is weakly S-indivisible. For a more detailed discussion see [Sauer1].

A.5.4 Partitioning pairs of rationals

The following is an unpublished result of Fred Galvin. It follows also from a much more general result in [Devlin] which has a proof of almost hundred pages.

Let $r_0, r_1, r_2, \ldots$ be an enumeration of the rationals $\mathbf{Q}$ and $P \subseteq [\mathbf{Q}]^2$ be those elements $\{r_i, r_j\}$ of $[\mathbf{Q}]^2$ so that if $r_i < r_j$ then $i < j$. Then $(P, [\mathbf{Q}]^2 - P)$ is a canonical partition of $[\mathbf{Q}]^2$.

• We have already shown (A.5.2) that for every copy $\mathbf{Q}^*$ of $\mathbf{Q}$ in $\mathbf{Q}$ the set $[\mathbf{Q}^*]^2$ has a non empty intersection with P and with $[\mathbf{Q}]^2 - P$. That is the partition $(P, [\mathbf{Q}]^2 - P)$ satisfies condition i. for a canonical partition of the two element subsets of the rationals.

In order to establish condition ii. we will prove that if $L \subseteq P$ and $\overline{L} := P - L$ then there is an embedding f from $\mathbf{Q}$ into $\mathbf{Q}$ so that if $f(r_i) = r_{i'}$ and $f(r_j) = r_{j'}$ and $i < j$ then $i' < j'$ and for which either $f(P) \subseteq L$ or $f(P) \subseteq \overline{L}$.

For $a \in \mathbf{Q}$ let $\Gamma_L(a) := \{b \in \mathbf{Q} \mid \{a, b\} \in L\}$. The set $D \subseteq \mathbf{Q}$ is *large* if there is an open interval of $\mathbf{Q}$ in which D is dense. For $A \subseteq \mathbf{Q}$ denote by large(A) the set of large subsets of A. If A and B are subsets of $\mathbf{Q}$ then $A < B$ means that every element of A is smaller than every element of B. Let ψ_L be the following formula:

$$\begin{aligned}\psi_L : =&\exists U \in \text{large}(\mathbf{Q})\, \forall V \in \text{large}(U) \exists A,B \in \text{large}(V)\, (A < B)\\ &\forall A' \in \text{large}(A)\, \forall B' \in \text{large}(B)\\ &\{a \in A' \colon \Gamma_L(a) \cap B' \in \text{large}(B')\} \in \text{large}(A').\end{aligned}$$

The last two lines of the formula ψ_L are denoted by $\gamma_L(A, B)$, that is

$$\begin{aligned}\gamma_L(A, B) :=&\forall A' \in \text{large}(A)\, \forall B' \in \text{large}(B)\\ &\qquad \{a \in A' \colon \Gamma_L(a) \cap B' \in \text{large}(B')\} \in \text{large}(A').\end{aligned}$$

Then

$$\psi_L = \exists U \in \text{large}(\mathbf{Q})\, \forall V \in \text{large}(U) \exists A,B \in \text{large}(V)\, (A < B)\big(\gamma_L(A, B)\big).$$

Note that

$$\gamma_L(A,B) \text{ and } A' \in \text{large}(A) \text{ and } B' \in \text{large}(B) \text{ implies } \gamma_L(A',B'). \tag{1}$$

Because $\neg\psi_L$ implies $\psi_{\overline{L}}$ we may assume ψ_L. The large set U given by ψ_L contains a copy of $\mathbf{Q}$ and hence we may assume that $U = \mathbf{Q}$. It follows that every large subset V of $\mathbf{Q}$ contains two large subsets $A < B$ so that $\gamma_L(A,B)$. Hence, because of (1), every large subset of $\mathbf{Q}$ contains three large subsets $A < C < B$ so that $\gamma_L(A,B)$.

Let I_n be the set $r_0, r_1, r_2, \dots, r_{n-1}$. The subset $X = \{x_0, x_1, x_2, \dots, x_{n-1}\}$ of $\mathbf{Q}$ is an *initial segment* if the function f which maps r_i to x_i is a local isomorphism of $\mathbf{Q}$. For every initial segment $X = \{x_0, x_1, x_2, \dots, x_{n-1}\}$ let π be the permutation of n so that $x_{\pi(0)} < x_{\pi(1)} < x_{\pi(2)} < \cdots < x_{\pi(n-1)}$.

The initial segment $X = \{x_0, x_1, x_2, \dots, x_{n-1}\}$ is *well chosen* if there is a large set V_i^n for every $i \in n+1$ so that for all $i, k \in n$, $j \in n+1$ with $i < j$:

i. $V_i^n < \{x_{\pi(i)}\} < V_{i+1}^n$.
ii. $\gamma(V_i^n, V_j^n)$.
iii. $\{x_{\pi(i)}\} \times V_j^n \subseteq L$.
iv. If $\{x_{\pi(i)}, x_{\pi(k)}\} \in R$ then $\{x_{\pi(i)}, x_{\pi(k)}\} \in L$.
v. If $x_i = r_{i'}$ and $x_j = r_{j'}$ and $i' < j'$ then $i < j$.

The empty initial segment is well chosen. Hence if there is a well chosen extension for every well chosen initial segment we are done. Let $X = \{x_0, x_1, x_2, \dots, x_{n-1}\}$ be a well chosen initial segment. Let $v \in n$ be the smallest number so that $r_n < r_{\pi(v)}$ if such a number exists and otherwise let $v = n$.

There are large subsets $A_0 < C < B_0$ of V_v^n so that $\gamma_L(A_0, B_0)$. Because $\gamma_L(C, B)$ there is a large subset $C_0 \subseteq C$ so that $\Gamma_L(a) \cap B$ is a large set for all $a \in C_0$. Because $\gamma_L(C_0, V_{v+1}^n)$ there is a large subset $C_1 \subseteq C_0$ so that $\Gamma_L(a) \cap V_{v+1}^n$ is a large set for all $a \in C_1$. There is a large subset $C_2 \subseteq C_1$ so that $\Gamma_L(a) \cap V_{v+2}^n$ is a large set for all $a \in C_2$. Finally there is a large subset $C_n \subseteq C$ so that for all $a \in C_n$ and all $v < i \leq n$ the intersection of $\Gamma_L(a)$ with V_i^n is large. Let $x_n \in C_n$ to obtain the extension Y of the initial segment X. (Because there are infinitely many choices for x_n we can make sure that condition v. is satisfied.) Because of condition iii. condition iv. is satisfied for Y.

For $i < v$ let $V_i^{n+1} = V_i^n$, let $V_v^{n+1} = A$, let $V_{v+1}^{n+1} = B \cap \Gamma_L(a)$ and let $V_{v+1+i}^{n+1} = V_{v+i}^n$ for all $i \in n - v$. It is now easy to check that Y is a well chosen initial segment. •

A.5.4 Some additional partitioning results

Assume that ω is the set of vertices of the Rado graph R. Let (A,B) be the lexicographic partition of the edges of the Rado graph. Because there is an embedding of $K_{\omega,\omega}$ into R there is no copy R^* of the R in R so that all of the edges of R^* are in A or all of the edges of R^* are in B. It is proven in [PouSauer] that for every partition of (X,Y) of A there is an order preserving embedding f from R into R so that all of the up edges of $f(R)$ are in X or all of the up edges of $f(R)$ are in Y. And the same for any partition of B. That is the lexicographic partition of the edges of the Rado graph is a canonical partition. The proof of this is a more involved version of the proof in A.5.4

The Rado graph is weakly edge indivisible

• We have to prove that for every finite graph G there is a copy G^* of G in the Rado graph R so that all edges of G^* are in the set A of up edges and that there is a copy G^{**} of G in R so that all of the edges of G^{**} are in the set D of down edges.

Every finite graph has a linear ordering of its vertices so that all of the edges are up edges. This can be seen by induction. Put an arbitrary vertex lowest. Then use induction to order the vertices not adjacent to this lowest vertex and put the ordering of the vertices adjacent to the lowest vertex above.

Every graph H whose vertices can be ordered into a sequence $v_0, v_1, v_2, \dots$ so that all of its edges are up edges has an order preserving embedding f into the Rado graph R so that every edge of H is mapped to an up edge of R. Assume that embedding f has already been defined for all v_i with $i \in n \in \omega$. We wish to extend it by finding an appropriate image of $f(v_n)$. Let F be the numbers less than or equal to $f(n-1)$ and $F_1 := \{f(v_i) \mid i \in n \,\text{and}\, v_n \,\text{is adjacent to}\, v_i\}$. Let T be the orbit in R with $F(T) = F$ and $F_1(T) = F_1$. This orbit has infinitely many elements, (A.3.5) and hence an element $f(v_n)$ larger than $f(n-1)$.

A similar argument shows that every finite graph has an embedding into the Rado graph which maps all of its edges to down edges. •

A very similar situation exists for the triangle free homogeneous graph H_3. Let ω be the set of vertices of H_3. It is proven in [Sauer2] that the lexicographic partition of the edges of the triangle free homogeneous graph H_3 is a canonical partition of the edges. The prove is again along the lines of the proof in A.5.4 but contains some considerable technical difficulties which make it seem unlikely that this type of proof can be extended to prove analogous results for the other graphs H_n.

Let (X, Y) be a lexicographic partition of the edges of H_3. It can be seen by an argument similar to the one for the Rado graph that for every finite triangle free graph G there is a copy G^* of G in H_3 so that all edges of G^* are in the set X and that there is a copy G^{**} of G in H_3 so that all of the edges of G^{**} are in the set Y. [Sauer2]. Hence it follows that

The triangle free homogeneous graph H_3 is weakly edge-indivisible.

The problem of partitioning $[\mathbf{Q}]^n$, the n-element subsets of the rationals $\mathbf{Q}$ is completely solved. [Devlin] Let $\phi(n)$ be given by $\phi(1) = 1$ and

$$\phi(n) = \sum_{l=1}^{n-1} \binom{2n-2}{2l-1} \phi(l)\phi(n-l).$$

We write

$$\mathbf{Q} \to (\mathbf{Q})^n_{<\omega/\phi(n)}$$

to mean that for every partition of the n-element subsets of $\mathbf{Q}$ into $k \in \omega$ parts there is an order embedding from $\mathbf{Q}$ into $\mathbf{Q}$ which contains n-element subsets of only $\phi(n)$ parts of the partition. Then [Devlin]

$$\mathbf{Q} \to (\mathbf{Q})^n_{<\omega/\phi(n)} \quad \text{and} \quad \mathbf{Q} \not\to (\mathbf{Q})^n_{<\omega/(\phi(n)-1)}.$$

This theorem followed by finding a canonical partition of the n-element subsets of $\mathbf{Q}$ into $\phi(n)$ parts.

The function $\phi(n)$ is quite interesting. Let $E(n)$ be the sequence of Entringer or Euler numbers, not Eulerian numbers. Their exponential generating function is

$$\mathcal{E}(x) = \sum_{n=0}^{\infty} E_n \frac{x^n}{n!} = \frac{1}{\cos(x)} + \tan(x).$$

The number of the increasing complete binary trees on $2n+1$ elements is E_{2n+1}.

References

[Woodrow] A.H. Lachlan and R.E. Woodrow, *Countable Ultrahomogeneous Undirected Graphs*, Trans. Amer. Math. Soc. **262** (1980), 51-94.

[Lachlan2] A.H. Lachlan *On countable stable structures which are homogeneous for a finite relational language*, Israel J. Math. **49** (1984), no. 1-3, 69-153.

[Lachlan1] A.H. Lachlan, *Countable homogeneous tournaments*, Trans. Amer. Math. Soc. 284 (1984), 431-461.

[Cherlin] G. Cherlin, *The classification of countable homogeneous directed graphs and countable homogeneous n-tournaments*, Mem. Amer. Math. Soc. **131** (1998), no.621. xiv+161pp.

[Schmerl] J. Schmerl, *Countable homogeneous partially ordered sets*, Alg. Universalis 9 (1979), 317-321.

[Henson] C.W. Henson, *Countable homogeneous relational structures and $\aleph_0$-categorical theories*, J. Symbolic Logic, **37**, (1972), 494-500.

[Pouzet1] M.Pouzet, *Application de la notion de relation presque-enchainable au dénombrement des restrictions finies d'une relation*, Zeitschr. f. mathem. Logic und Grundlagen der Mathematik, **Bd 27**, S 289-332 (1981).

[ElSauer1] M. El-Zahar, N.W. Sauer, *Ramsey-type properties of relational structures*, Discrete Mathematics **94** (1991) 1-10.

[Zhu] K. Böröcky, Jr., N.W. Sauer and X. Zhu, *Inexhaustible Homogeneous Structures.* Discrete Mathematics 115(1993) 57-63, North-Holland

[ElSauer2] M. EL-Zahar, N.W. Sauer, *On the Divisibility of Homogeneous Hypergraphs.* Combinatorica **14** (2) (1994) 1-7

[Rodl] P. Komjáth and V. Rödl, Coloring of Universal Graphs, *Graphs and Combinatorics* **2**, (1986), pp.55-60.

[ElSauer3] M. El-Zahar, N.W. Sauer, *On the Divisibility of Homogeneous Directed Graphs.* Can. J. Math. Vol.**45**(2), 1993 pp. 284-294

[ElSauer4] M. El-Zahar, N.W. Sauer, *The Indivisibility of the Homogeneous K_n-free graphs.* Journal of Combinatorial Theory, Series B, Vol. 47, No. 2, October 1989.

[Posa] P. Erdős, A. Hajnal, L. Pósa, *Strong embeddings of graphs into colored graphs*, Colloquia Mathematica Societatis János Bolyai, **10, Vol. I**, Infinite and finite sets, page 585-595, Edited by A. Hajnal, R. Rado, Vera T. Sós.

[Sauer1] N.Sauer, *Coloring finite substructures of countable structures*, Bolyai Society Mathematical Studies, **11**.

[PouSauer] M. Pouzet, N. Sauer *Edge partitions of the Rado graph*, Combinatorica, **16**, (4), (1996) 1-16

[Sauer2] N. Sauer *Edge partitions of the countable triangle free homogeneous graph*, Discrete Mathematics 185 (1998) 137-181

[Devlin] D. Devlin, *Some partition theorems and ultrafilters on ω*, Ph. D. Thesis, Dartmouth College, 1979.

[Cameron1] P.J. Cameron, *Transitivity of permutation groups on unordered sets*, Math. Z. 148 (1976), 127-139.

[Cameron2] P.J. Cameron *Orbits of permutation groups on unordered sets, II*, J. London Math. Soc. (2), 23 (1981), 249-264.

[Cameron3] P.J. Cameron *Some treelike objects*, Quart. J. Math. Oxford (2) 38 (1987), 155-183.

[Cameron4] P.J. Cameron, *Oligomorphic permutation groups*, LMS Lecture Notes, CUP, Cambridge, 1990.

[Gardiner] A. Gardiner, *Homogeneous graphs*, J. Comb. Theory B 20 (1976), 94-102.

[Macpherson] H.D. Macpherson, *Orbits of infinite permutation groups*, Proc. London Math. Soc. (3) 51 (1985), 246-284.