Canonical vertex partitions

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Abstract

Let σ be a finite relational signature and $\mathcal{T}$ a set of finite complete relational structures of signature σ and $\mathrm{H}_{\mathcal{T}}$ the countable homogeneous relational structure of signature σ which does not embed any of the structures in $\mathcal{T}$.

In the case that σ consists of at most binary relations and $\mathcal{T}$ is finite the vertex partition behaviour of $\mathrm{H}_{\mathcal{T}}$ is completely analysed; in the sense that it is shown that a canonical partition exists and the size of this partition in terms of the structures in $\mathcal{T}$ is determined. If $\mathcal{T}$ is infinite some results are obtained but a complete analysis is still missing.

Some general results are presented which are intended to be used in further investigations in case that σ contains relational symbols of arity larger than two or that the set of bounds $\mathcal{T}$ is infinite.

1 Introduction

A *relational signature* σ consists of a finite set of relation symbols together with a finite arity for each of the relation symbols. The relational signature σ is *binary* if every relational symbol of σ has arity at most two. A *relational structure* A *of signature* σ consists of a set $\langle \mathrm{A} \rangle$ together with a subset $R^{\mathrm{A}} \subseteq \langle \mathrm{A} \rangle^n$ for every $n \in \omega$ and n-ary relation symbol R of σ. If $(x_0, x_1, \dots, x_{n-1}) \in R^{\mathrm{A}}$ we write $R^{\mathrm{A}}(x_0, x_1, \dots, x_{n-1})$ or $R(x_0, x_1, \dots, x_{n-1})$ if the structure A is understood. All relational structures A under consideration will be countable, that is the set $\langle \mathrm{A} \rangle$ is finite or countably infinite.

Let A and B be two relational structures of the same signature σ. The injection $f : \langle \mathrm{A} \rangle \mapsto \langle \mathrm{B} \rangle$ is an *embedding* of A into B if for all $n \in \omega$ and n-ary relation symbols R of σ

$$R^{\mathrm{A}}(x_0, x_1, \dots, x_{n-1}) \text{ if and only if } R^{\mathrm{B}}(f(x_0), f(x_1), \dots, f(x_{n-1})).$$

If the embedding f of A to B is surjective then f is an *isomorphism*.

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The structure A is an *induced substructure* of B if $\langle \mathrm{A} \rangle \subseteq \langle \mathrm{B} \rangle$ and the identity map on $\langle \mathrm{A} \rangle$ is an embedding. If $S \subseteq \langle \mathrm{B} \rangle$ then $\mathrm{B}|S$ denotes the induced substructure of B with $\langle \mathrm{B}|S \rangle = S$.

If S is a subset of the domain of the function f then $f[S] := \{f(s) \mid s \in S\}$. The cardinality of a set S is denoted by $|S|$.

Let A be a relational structure. The expression

$$\mathrm{A} \to (\mathrm{A})_n$$

for $n \in \omega$ stands for: For every function $\gamma : \langle \mathrm{A} \rangle \mapsto n$ there is an embedding f of A into A so that γ is constant on $f[\langle \mathrm{A} \rangle]$. Note that $\mathrm{A} \to (\mathrm{A})_2$ implies $\mathrm{A} \to (\mathrm{A})_n$ for every $n \in \omega$. A relational structure A is *indivisible* if $\mathrm{A} \to (\mathrm{A})_2$. $\mathrm{A} \not\to (\mathrm{A})_n$ is the negation of $\mathrm{A} \to (\mathrm{A})_n$.

The expression

$$\mathrm{A} \to (\mathrm{A})_{n/k}$$

for $n, k \in \omega$ stands for: For every function $\gamma : \langle \mathrm{A} \rangle \mapsto n$ there is an embedding f of A into A so that $|\gamma[f[\langle \mathrm{A} \rangle]]| \leq k$. It follows that $\mathrm{A} \to (\mathrm{A})_n$ is equivalent to $\mathrm{A} \to (\mathrm{A})_{n/1}$. $\mathrm{A} \not\to (\mathrm{A})_{n/k}$ is the negation of $\mathrm{A} \to (\mathrm{A})_{n/k}$.

The partition $P := (P_0, P_1, P_2, \dots, P_{k-1})$ of $\langle \mathrm{A} \rangle$ into $k \in \omega$ classes is a *canonical partition* of A if:

1. $P_i \cap f[\langle \mathrm{A} \rangle] \neq \emptyset$ for all embeddings f of A into A and for all $i \in k$.
2. For every function $\gamma : \langle \mathrm{A} \rangle \mapsto n \in \omega$ there is an embedding f of A into A so that for every $i \in k$:
   (a) $f[P_i] \subseteq P_i$.
   (b) The function γ is constant on $f[P_i]$.

It follows that a relational structure A is indivisible if and only if $(\langle \mathrm{A} \rangle)$ is a canonical partition of A. If A has a canonical partition into k classes then, according to Lemma 2.3, $\mathrm{A} \to (\mathrm{A})_{n/k}$ and $\mathrm{A} \not\to (\mathrm{A})_{n/l}$ if $l < k$. Hence if A has a canonical partition into k classes it does not have a canonical partition into $n \neq k$ classes. (Actually, a canonical partition is in a certain sense unique, but this will not be needed here.)

The results of this paper deal with canonical partitions of $\mathcal{T}$-free homogeneous structures $\mathrm{H}_{\mathcal{T}}$. (See section 4 for a definition.) It turns out that if $\mathcal{T}$ is a finite set of relational structures and the signature of the structures in $\mathcal{T}$ is binary then $\mathrm{H}_{\mathcal{T}}$ has a canonical partition of size k, where k is the size of a largest anti-chain of the partial order $(\mathbf{orb}(\mathcal{T}); \subseteq)$; see Theorem 9.2. The partial order $(\mathbf{orb}(\mathcal{T}); \subseteq)$ can be effectively determined from the structures in $\mathcal{T}$ given that $\mathcal{T}$ is finite. There is a finite algorithm which places every element of $\mathrm{H}_{\mathcal{T}}$ into one of the parts of the canonical partition. The partial order $(\mathbf{orb}(\mathcal{T}); \subseteq)$ is finite if $\mathcal{T}$ is finite.

If the set of relational structures $\mathcal{T}$ is infinite and the sizes of the anti-chains of the partial order $(\mathbf{orb}(\mathcal{T}); \subseteq)$ are unbounded then $\mathrm{H}_{\mathcal{T}}$ does not have a canonical partition; see Theorem 5.1.

If the signature of the structures in $\mathcal{T}$ is binary and the set $\mathcal{T}$ is infinite and the width of the partial order $(\mathbf{orb}(\mathcal{T}); \subseteq)$ is a finite number k, partial results are obtained; see Theorem 8.1. One of the assumptions of Theorem 8.1 is, that the chains into which $(\mathbf{orb}(\mathcal{T}); \subseteq)$ is partitioned, are finitary. It is not known if this assumption is really necessary and if so under which circumstances it holds. The obvious instances in which the chains are finitary are covered by Corollary 8.1 and Lemma 8.2. Example 10.2 presents an instance in which Theorem 8.1 can be applied and which goes beyond the situations covered by Corollary 8.1 and Lemma 8.2.

A homogeneous structure $\mathrm{H}_{\mathcal{T}}$ with binary signature is indivisible if and only if $(\mathbf{orb}(\mathcal{T}); \subseteq)$ is a chain; see Theorem 9.4. For general signature, if $\mathrm{H}_{\mathcal{T}}$ is indivisible then $(\mathbf{orb}(\mathrm{H}_{\mathcal{T}}); \subseteq)$ is a chain. (Follows from Theorem 5.1.)

Chapters one to six do not restrict the arity of the signature. Some of the results apply to general relational structures, not just homogeneous ones, and will be used as basic observations for further investigations into general vertex partition problems. Chapters seven to ten of this paper deal with free homogeneous structures $\mathrm{H}_{\mathcal{T}}$ with binary signature and a possibly infinite set $\mathcal{T}$ of relational structures.

The case of homogeneous structures $\mathrm{H}_{\mathcal{T}}$ with non binary signature will be investigated in a subsequent paper which will make use of the first six sections of this paper and restrict itself to finite sets $\mathcal{T}$ of relational structures. The situation in the non binary case is quite different and requires different arguments and a different notational setup. There seems to be no hope at present to also deal with an infinite boundary set $\mathcal{T}$.

Komjath and Rödl [2] proved that the triangle free homogeneous graph is indivisible. In [3] it is proven that in general the K_n-free homogeneous graph is indivisible. The paper [6] contains a related vertex partition result. In [4] it is proven that the oriented graphs $\mathrm{H}_{\mathcal{T}}$, with $\mathcal{T}$ a finite set of tournaments, are indivisible if and only if the ages of the orbits of $\mathrm{H}_{\mathcal{T}}$ form a total chain. In [5] this result is generalized to infinite sets of tournaments $\mathcal{T}$.

Section 6 of this paper follows an argument in [5] closely. This argument in [5] would not have to be repeated here if it would just generalize from oriented graphs to more general binary relational structures. Unfortunately we need a, what seems to be only slightly stronger version, which resisted all attempts to prove using the result in [5]. In order to avoid such a problem in the future the result here is cast in as general a form as possible. Co-ideals are extensively used in [8] and it is quite likely that the co-ideal Theorem 6.1 will also be needed for partition results of substructures other than vertices's.

In [7] the partial orders $(\mathbf{orb}(\mathcal{T}); \subseteq)$ are characterized in the case that $\mathcal{T}$ is a set of finite tournaments. It is easy to see that essentially the same characterization holds in the general case of binary relational structures. It is even simpler in this case as structures like $\mathrm{W}_n \oplus \mathrm{W}_m$ as in Example 10.2 can be used. In [7] it is also proven that $\mathrm{H}_{\mathcal{T}} \to (\mathrm{H}_{\mathcal{T}})_{n/k}$ for $\mathcal{T}$ a finite set of tournaments, where k is the size of a largest anti-chain of $(\mathbf{orb}(\mathcal{T}); \subseteq)$.

2 Canonical partition Theorem

Lemma 2.1. *Let* A *be a relational structure and* $P := (P_i; i \in k)$ *a partition of* $\langle \mathrm{A} \rangle$ *into* $k \in \omega$ *classes so that:*

a. $P_i \cap f[P_i] \neq \emptyset$ *for all embeddings* f *of* A *into* A *and for all* $i \in k$.

b. *For every function* $\gamma : \langle \mathrm{A} \rangle \mapsto n \in \omega$ *there is an embedding* f *of* A *into* A *so that the function* γ *is constant on* $f[P_i]$ *for every* $i \in k$.

Let f *be an embedding of* A *into* A *with image* A^*.

Then there exists an embedding g *of* A^* *into* A^* *so that* $g \circ f[P_i] \subseteq P_i \cap f[P_i]]$ *for every* $i \in k$.

Proof. The relational structure A^* is an isomorphic copy of A which implies that $(f[P_i]; i \in k)$ is a partition of $\langle \mathrm{A}^* \rangle$ satisfying items a. and b. Let $\gamma : \langle \mathrm{A}^* \rangle \mapsto k \times k$ be the function so that $\gamma(x) = (i, j)$ if $x \in P_i \cap f[P_j]$.

There is an embedding g of A^* into A^* so that γ is constant on $g[f[P_i]]$ for every $i \in k$. It follows from item a. applied to the structure A that $g \circ f[P_i] \cap P_i \neq \emptyset$ and applied to A^* that $g \circ f[P_i] \cap f[P_i] = g[f[P_i]] \cap f[P_i] \neq \emptyset$. Hence γ maps the elements of $g \circ f[P_i]$ to (i, i) for all $i \in k$.

□

Lemma 2.2. *Let* A *be a relational structure and* $P := (P_i; i \in k)$ *a partition of* $\langle \mathrm{A} \rangle$ *into* $k \in \omega$ *classes so that:*

a. $f(P_i) \not\subseteq P_j$ *for all embeddings* f *of* A *into* A *and for all* $i, j \in k$ *with* $i \neq j$.

b. *For every function* $\gamma : \langle \mathrm{A} \rangle \mapsto n \in \omega$ *there is an embedding* f *of* A *into* A *so that the function* γ *is constant on* $f[P_i]$ *for every* $i \in k$.

Then P *is a canonical partition with* $P_i \cap f[P_i] \neq \emptyset$ *for every* $i \in k$.

Proof. Assume for a contradiction that the embedding f of A into A is such that, say, $P_{k-1} \cap f[P_{k-1}] = \emptyset$. Let A^* be the isomorphic copy of A which is the image of f. The partition $(f[P_i]; i \in k)$ is a partition of $\langle \mathrm{A}^* \rangle$ satisfying items a. and b.

Let $\gamma : \langle \mathrm{A}^* \rangle \mapsto k$ be the function with $\gamma(x) = i$ if $x \in P_i$. There exists an embedding g of A^* into A^* so that γ is constant to some $i < k-1$, on $g[f[P_{k-1}]]$. This contradicts item a. because then $g \circ f[P_{k-1}] \subseteq P_i$ with $i \neq k-1$.

Hence $P_i \cap f[P_i] \neq \emptyset$ for every embedding f of A into A and every $i \in k$. Item 1. of the definition of canonical partition follows.

In order to establish item 2. of the definition of canonical partition let $\gamma : \langle \mathrm{A} \rangle \mapsto n \in \omega$. There exists an embedding f of A into A so that γ is constant on $f[P_i]$ for every $i \in k$. It follows from Lemma 2.1 that there is an embedding g of the image A^* of f into A^* so that $g \circ f[P_i] \subseteq P_i \cap f[P_i]$ for all $i \in k$.

□

Lemma 2.3. *Let* $P := (P_0, P_1, P_2, \ldots, P_{k-1})$ *be a canonical partition of the relational structure* A. *Then:*

1. *$P_i \cap f[P_i] \neq \emptyset$ for every embedding f of* A *into* A *and every $i \in k$.*

2. A $\to$ (A)$_{n/k}$ *for every $n \in \omega$ and k is the smallest such number.*

Proof. Item 1. follows from Lemma 2.2.

The relation A $\to$ (A)$_{n/k}$ follows from item 2.(b) of the definition of canonical partition. If A $\to$ (A)$_{n/l}$ with $l < k$ let $\gamma : \langle \mathrm{A} \rangle \mapsto k$ be the function with $\gamma(x) = i$ if $x \in P_i$. There is an embedding f of A into A so that $|\gamma[f[\langle \mathrm{A} \rangle]]| \leq l < k$ in contradiction to item 1. of the definition of canonical partition. □

Let $\langle \mathrm{A} \rangle$ be a relational structure.

The set $Q := \{Q_0, Q_1, Q_2, Q_3, \dots\}$ of pairwise disjoint subsets of $\langle \mathrm{A} \rangle$ is a *reducing set of subsets* of $\langle \mathrm{A} \rangle$ if:

For every function $\gamma : \langle \mathrm{A} \rangle \mapsto n \in \omega$ there is an embedding f of A into A so that γ is constant on $f[Q_i]$ for every i. If Q is reducing and a partition of $\langle \mathrm{A} \rangle$ then it is a *reducing partition* of A. If the set Q consists of the single term Q_0 then Q_0 is a *reducing set of elements of* A. Note that if S is a reducing set of elements of A then so is every subset of S.

The set of subsets $\mathbb{C}$ of a set S is a *co-ideal* of S if:

1. $S \in \mathbb{C}$ and $\emptyset \notin \mathbb{C}$.

2. $L \in \mathbb{C}$ and $L \subseteq H \subseteq S$ implies $H \in \mathbb{C}$.

3. $L, H \subseteq S$ and $L \cup H \in \mathbb{C}$ implies $L \in \mathbb{C}$ or $H \in \mathbb{C}$.

The elements of $\mathbb{C}$ will often be called *large* subsets of S and those subsets of S not in $\mathbb{C}$ the *small* subsets of S.

The subset P of $\langle \mathrm{A} \rangle$ is *persistent* if there is a coideal $\mathbb{C}$ of large subsets of P so that $f[S] \cap P$ is large for every embedding f of A into A and every large subset S of P.

The partition $P := (P_0, P_1, P_2, \dots)$ of the subset V of $\langle \mathrm{A} \rangle$ is a *persistent* partition of V if the set P_i is persistent for every i. If P is a persistent partition of $\langle \mathrm{A} \rangle$ then we say that it is a persistent partition of A. Note that if $P := (P_0, P_1, P_2, \dots)$ is a *persistent* partition of $V \subseteq \langle \mathrm{A} \rangle$ then $(P_0 \cup (\langle \mathrm{A} \rangle - V), P_1, P_2, \dots)$ is a persistent partition of A. (Take a subset S of $P_0 \cup (\langle \mathrm{A} \rangle - V)$ to be large if $S \cap P_0$ is large.)

Lemma 2.4. *Let $P := (P_0, P_1, P_2, \dots)$ be a persistent partition of the relational structure* A *with $\mathbb{C}_i$ the set of large subsets of P_i. Let f be an embedding of* A *into* A*, then:*

1. *If $S \in \mathbb{C}_i$ is a large subset of P_i then*

$$f[S] \not\subseteq \bigcup_{j \neq i} P_j.$$

2. *If $S \in \mathbb{C}_i$ is a large subset of P_i then $S_i := \{x \in S \mid f(x) \in P_i\}$ is a large subset of P_i.*

3. $f[\langle \mathrm{A}\rangle] \cap P_i \neq \emptyset$ *for every* $i \in k$.

Proof. Item 1. follows trivially from the definition of persistent set because the empty set is not large. If S_i is a small subset of P_i then $S - S_i = \{x \in S \mid f(x) \notin P_i\}$ is large in contradiction to item 1. Item 3. follows because P_i is large and hence $f[P_i] \cap P_i \neq \emptyset$ for every $i \in k$, according to item 1. of this Lemma.

□

Theorem 2.1. *Let* $\langle \mathrm{A}\rangle$ *be a relational structure. If* A *has a persistent partition* $P := (P_i; i \in k)$ *then every reducing partition* $Q := (Q_i; i \in k)$ *of* A *is a canonical partition of* A*.*

Proof. Let $P := (P_i; i \in k)$ be a persistent partition of A with a set $\mathbb{C}_i$ of large subsets of P_i for every $i \in k$ and let $Q := (Q_i; i \in k)$ be a reducing partition of A. We have to establish item a. of Lemma 2.2 for the reducing partition Q.

Let $\gamma : \langle \mathrm{A}\rangle \mapsto k$ be the function with $\gamma(x) = i$ for $x \in P_i$.

Because Q is a reducing partition of A there is an embedding f of A into A so that γ is constant on $f[Q_i]$ for every $i \in k$. Let $\gamma[f[Q_i]] := \{\pi(i)\}$. The function π is a permutation of the elements of k for otherwise $f[\langle \mathrm{A}\rangle] \cap P_i = \emptyset$ for some $i \in k$. Contradicting item 3. of Lemma 2.4. We assume without loss that the ordering of Q is such that $\gamma[f[Q_i]] = \{i\}$ for all $i \in k$. That is $f[Q_i] \subseteq P_i$.

Let $P_{i,j} = P_i \cap Q_j$. Then $f[P_{i,j}] \subseteq f[Q_j] \subseteq P_j$. It follows from the definition of persistent partition and Lemma 2.4 that $P_{i,i}$ is a large subset of P_i while $P_{i,j}$ is a small subset of P_i for all $j \in k$ not equal to i. Because $P_{i,j} \subseteq Q_j$, every set Q_j of the partition Q is partitioned into the sets $(P_{i,j}; i \in k)$ where $P_{i,j}$ is a large subset of P_i if and only of $i = j$.

Let g be an embedding of A into A and assume for a contradiction to item a. of Lemma 2.2 that $g[Q_0] \subseteq Q_1$. Then $g[P_{0,0}] \subseteq Q_1$. The set Q_1 is partitioned into the sets $(P_{0,1}, P_{1,1}, P_{2,1}, \dots, P_{k-1,1})$. Let R_i be the set of elements in $P_{0,0}$ which are mapped by g into $P_{i,1}$.

Then one of the sets R_i, say R_t is a large subset of P_0. If $t \neq 0$ then g would map a large subset of P_0 into one of the sets P_i for $0 < i \in k$ in contradiction to item 1. of Lemma 2.4. If $t = 0$ then g would map a large subset of P_0 into a small subset of P_0 in contradiction to the definition of persistent set.

□

3 Relational structures

Let A be a relational structure with signature σ. The expression $\mathrm{A} - S$ for $S \subseteq \langle \mathrm{A}\rangle$ stands for $\mathrm{A}|(\langle \mathrm{A}\rangle - S)$ where the operator $-$ means set difference. Also $\mathrm{A} - a$ stands for $\mathrm{A} - \{a\}$. We will write $a \in U$ if $a \in \langle \mathrm{A}\rangle$ and U a unary relational symbol in σ and $U(a)$. The subset $U \subseteq \langle \mathrm{A}\rangle$ is a *unary subset* of A if for all $a, b \in U$ the set of unary relations which hold at a is equal to the set of unary relations which hold at b. Note that the unary subsets of A form a partition of $\langle \mathrm{A}\rangle$.

The set rel(A) is the set of all finite subsets $S = \{x_i \mid i \in n \in \omega\}$ of $\langle \mathrm{A} \rangle$ so that there is an n-ary relational symbol R of σ with $R^{\mathrm{A}}(x_0, x_1, \dots, x_{n-1})$. The elements in the set $\{a_i \mid i \in n \in \omega\} \subseteq V \subseteq \langle \mathrm{A} \rangle$ are *adjacent within* V if there is $S \in \mathrm{rel}(\mathrm{A})$ with $\{a_i \mid i \in n\} \subseteq S$. The structure A is *complete* if any two elements of A are adjacent within $\langle \mathrm{A} \rangle$.

The *skeleton* of A, skel(A), is the set of finite induced substructures of A and the *age* of A, age(A), is the set of relational structures isomorphic to one of the structures in skel(A). If B is another σ-structure then an isomorphism of an element in skel(A) to an element in skel(B) is a *local isomorphism* of A to B. A local isomorphism of A is a local isomorphism of A to A. The set Bound(A) is the set of all finite σ-structures not in the age of A and bound(A) is the set of minimal, under embeddings, elements in Bound(A).

The set $\mathcal{A}$ of finite relational structures with signature σ is an *age* if it is closed under isomorphism, induced substructures and if for any two elements A and B of $\mathcal{A}$ there is a $\mathrm{D} \in \mathcal{A}$ so that both A and B have an embedding into D. Note that if A is a relational structure then age(A) is an age. Every age has a representative; see **10.2.1** of [1]. Then $\mathrm{bound}(\mathcal{A}) := \mathrm{bound}(\mathrm{A})$ for any relational structure A with $\mathrm{age}(\mathrm{A}) = \mathcal{A}$.

The structure A is *weakly indivisible* if for every partition (P_0, P_1) of $\langle \mathrm{A} \rangle$ with $\mathrm{age}(\mathrm{A}|P_0) \neq \mathrm{age}(\mathrm{A})$ there is an embedding of A into $\mathrm{A}|P_1$. The structure A is *age indivisible* if for every partition (P_0, P_1) of $\langle \mathrm{A} \rangle$ there is $i \in 2$ so that $\mathrm{age}(\mathrm{A}|P_i) = \mathrm{age}(\mathrm{A})$.

Theorem 3.1. *Let* A *and* B *be two relational structures in signature* σ *with* $\mathrm{age}(\mathrm{A}) = \mathrm{age}(\mathrm{B})$*. Then* A *is age indivisible if and only if* B *is age indivisible.*

Proof. The Theorem follows from [9] **A. 4.1** as well as from [10] Theorem 1. □

On account of Theorem 3.1 we can speak of an indivisible age. Note that if A is weakly indivisible then its age is indivisible.

The relational signature σ' is an *expansion* of the relational signature σ if every relational symbol of σ is a relational symbol in σ' and has in σ the same arity as in σ'. If σ' is an expansion of σ then σ is a *reduction* of σ'.

Let σ' be an expansion of σ. The relational σ'-structure B is an *expansion* of the σ-structure A if $\langle \mathrm{B} \rangle = \langle \mathrm{A} \rangle$ and if for all $n \in \omega$ and n-ary relations R in σ and n-tuples of elements of $\langle \mathrm{B} \rangle$:

$$R^{\mathrm{A}}(x_0, x_1, \dots, x_{n-1}) \text{ if and only if } R^{\mathrm{B}}(x_0, x_1, \dots, x_{n-1}).$$

If B is an expansion of A then A is a reduction of B.

Lemma 3.1. *Let the* σ'*-structure* B *be an expansion of the* σ*-structure* A*. If* B *is age indivisible then* A *is age indivisible.*

Proof. Let (P_0, P_1) be a partition of $\langle \mathrm{A} \rangle$ and assume that $\mathrm{age}(\mathrm{B}|P_0) = \mathrm{age}(\mathrm{B})$. Let $\mathrm{R} \in \mathrm{age}(\mathrm{A})$. Then there is $\mathrm{S} \in \mathrm{skel}(\mathrm{A})$ which is isomorphic to R and $\mathrm{S}' \in \mathrm{skel}(\mathrm{B})$ which is an expansion of S to a σ'-structure. Let $\mathrm{R}' \in \mathrm{skel}(\mathrm{B}|\mathrm{P}_0)$ isomorphic to S and $\bar{\mathrm{R}}$ the reduction of R' to a σ-structure. Then $\bar{\mathrm{R}} \in \mathrm{skel}(\mathrm{A}|\mathrm{P}_0)$ and $\bar{\mathrm{R}}$ is isomorphic to R. □

The relational structures A and B in signature σ are *compatible* if $\mathrm{A}|(\langle\mathrm{A}\rangle \cap \langle\mathrm{B}\rangle) = \mathrm{B}|(\langle\mathrm{A}\rangle \cap \langle\mathrm{B}\rangle)$. If A and B are compatible then the *free amalgam of* A *and* B, $\mathrm{A} \amalg \mathrm{B}$, is the σ-structure C with $\langle\mathrm{C}\rangle = \langle\mathrm{A}\rangle \cup \langle\mathrm{B}\rangle$ and $\mathrm{C}|\langle\mathrm{A}\rangle = \mathrm{A}$ and $\mathrm{C}|\langle\mathrm{B}\rangle = \mathrm{B}$ and $\mathrm{rel}(\mathrm{C}) = \mathrm{rel}(\mathrm{A}) \cup \mathrm{rel}(\mathrm{B})$. A set $\mathcal{A}$ of relational structures is *freely amalgamable* if for every two compatible elements of $\mathcal{A}$ their free amalgam is again an element of $\mathcal{A}$.

Theorem 3.2. *An age $\mathcal{A}$ is freely amalgamable if and only if* $\mathrm{bound}(\mathcal{A})$ *is a set of finite complete relational structures which can pairwise not be embedded into each other.*

Proof. See **A.2.5** of [9]. □

More generally, an *amalgam* of two compatible relational structures A and B is any relational structure D so that there is a function f which maps $\langle\mathrm{A}\rangle \cup \langle\mathrm{B}\rangle$ into $\langle\mathrm{D}\rangle$ so that f restricted to $\langle\mathrm{A}\rangle$ is an isomorphism and f restricted to $\langle\mathrm{B}\rangle$ is an isomorphism. A set $\mathcal{A}$ of relational structures is *amalgamable* if every two compatible elements of $\mathcal{A}$ have an amalgam which is again an element of $\mathcal{A}$.

4 Homogeneous structures

The relational structure A has the *mapping extension property* if for every structure $\mathrm{B} \in \mathrm{age}(\mathrm{A})$ with $x \in \langle\mathrm{B}\rangle$ and embedding f of $\mathrm{B} - x$ into A there is an extension of f to an embedding f' of B into A. A countable structure is *homogeneous* if it has the mapping extension property.

Theorem 4.1. *The age of every homogeneous structure is amalgamable and if $\mathcal{A}$ is an amalgamable age then there is a homogeneous structure* H *with* $\mathrm{age}(\mathrm{H}) = \mathcal{A}$. *If* G *and* H *are homogeneous structures with* $\mathrm{age}(\mathrm{G}) = \mathrm{age}(\mathrm{H})$ *then* G *and* H *are isomorphic. The following are equivalent:*

1. *The countable relational structure* H *is homogeneous.*
2. *Every local isomorphism f of* H *has an extension to an automorphism of* H.
3. *If* A *is a countable relational structure with* $\mathrm{age}(\mathrm{A}) \subseteq \mathrm{age}(\mathrm{H})$ *and* $\mathrm{B} \in \mathrm{skel}(\mathrm{A})$ *and f an embedding of* B *into* H *then there is an extension f' of f to an embedding of* A *into* H.

Proof. See **A.1.1** to **A.1.3** of [9]. □

The set $\mathcal{T}$ of σ-structures is a *free boundary set* if $\mathcal{T}$ is a set of finite complete σ-structures which can pairwise not be embedded into each other.

Let $\mathcal{T}$ be a free boundary set and $\mathrm{Forb}(\mathcal{T})$ denote the set of finite σ-structures into which none of the elements of $\mathcal{T}$ can be embedded. It follows that $\mathrm{Forb}(\mathcal{T})$ is an age with $\mathrm{bound}(\mathrm{Forb}(\mathcal{T})) = \mathcal{T}$. The age of $\mathrm{Forb}(\mathcal{T})$ is freely amalgamable according to Theorem 3.2. We denote by $\mathrm{H}_\mathcal{T}$ the unique homogeneous structure with $\text{age}(\mathrm{H}_\mathcal{T}) = \text{Forb}(\mathcal{T})$. The homogeneous structure $\mathrm{H}_\mathcal{T}$ is the *free homogeneous structure with boundary* $\mathcal{T}$ or the $\mathcal{T}$*-free homogeneous structure*. A homogeneous structure H is *free* if every structure in bound(H) is complete. It follows that if H is free then it is the bound(H)-free homogeneous structure $\mathrm{H}_{\text{bound(H)}}$.

Theorem 4.2. *Let $\mathcal{T}$ be a free boundary set. The homogeneous structure $\mathrm{H}_\mathcal{T}$ is weakly indivisible if every two elements of $\langle \mathrm{H}_\mathcal{T} \rangle$ satisfy the same set of unary relations. (That is if $\langle \mathrm{H}_\mathcal{T} \rangle$ is a unary set.)*

Proof. See Theorem **A.4.3 of [9].** □

Let A be a relational structure with signature σ. The elements $x, y \in \langle \mathrm{A} \rangle - F$ are of the same *type* with respect to the finite subset F of $\langle \mathrm{A} \rangle$ if there is a local isomorphism f of A which is the identity map on F and maps x to y.

An *orbit* $\mathrm{X} = (X, \mathbb{F}(\mathrm{X}))$ of a homogeneous structure H is a pair consisting of a nonempty subset $X := \mathbb{I}(\mathrm{X})$ of $\langle \mathrm{H} \rangle$ and a finite subset $\mathbb{F}(\mathrm{X})$, the *base* of X so that:

1. $\mathbb{F}(\mathrm{X}) \cap X = \emptyset$.
2. Any two elements x and y in X are of the same type over $\mathbb{F}(\mathrm{X})$.
3. If $x \in X$ and $y \in \langle \mathrm{H} \rangle$ are of the same type over $\mathbb{F}(\mathrm{X})$ then $y \in X$.

Let X be an orbit of $\mathrm{H}_\mathcal{T}$. Then $\mathbb{F}_*(\mathrm{X})$ is the set of elements $b \in \mathbb{F}(\mathrm{X})$ so that if $a \in X$ then a and b are adjacent within $\mathrm{H}_\mathcal{T} | (\mathbb{F}(\mathrm{X}) \cup \{a\})$. Note that the set $\mathbb{F}_*(\mathrm{X})$ does not depend on the particular choice of $a \in X$. Let $\mathbb{F}_0(\mathrm{X}) := \mathbb{F}(\mathrm{X}) - \mathbb{F}_*(\mathrm{X})$. Let J be a finite subset of $\langle \mathrm{H}_\mathcal{T} \rangle$. Then $\mathbb{F}^{-1}(J)$ is the set of all orbits X of $\mathrm{H}_\mathcal{T}$ so that $\mathbb{F}(\mathrm{X}) = J$. The set $\mathbb{F}^{-1}(J)$ is finite because the signature σ is assumed to be finite.

It is notationally convenient to identify X with the structure $\mathrm{H}|X$. Hence we write skel(X) for skel(H|X) and age(X) for age($\mathrm{H}|X$) and similarly for bound(X) and Bound(X). We say that the structure A has an embedding into X if it has an embedding into $\mathrm{H}|X$. The orbit X has the mapping extension property if $\mathrm{H}|X$ has the mapping extension property and it is age-indivisible if $\mathrm{H}|X$ is age-indivisible.

Let X and Y be two orbits of $\mathrm{H}_\mathcal{T}$ and $S \subseteq \mathbb{F}(\mathrm{X})$. Then X/S is the orbit with $X \subseteq \mathbb{I}(\mathrm{X}/S)$ and $\mathbb{F}(\mathrm{X}/S) = S$. The orbits X and Y are *compatible* if $\mathrm{X}/(\mathbb{F}(\mathrm{X}) \cap \mathbb{F}(\mathrm{Y})) = \mathrm{Y}/(\mathbb{F}(\mathrm{X}) \cap \mathbb{F}(\mathrm{Y}))$. Note that if $X \cap Y \neq \emptyset$ then X and Y are compatible but it might be the case that X and Y are compatible and $X \cap Y = \emptyset$.

The orbit Y is a *continuation* of the orbit X if $\mathrm{X} = \mathrm{Y}/\mathbb{F}(\mathrm{X})$. Hence if Y is a continuation of X then $\text{age}(\mathrm{Y}) \subseteq \text{age}(\mathrm{X})$ and X and Y are compatible. If $S \subseteq \mathbb{F}(\mathrm{X})$ then X is a continuation of X/S. The orbit Y is a *refinement* of the orbit X if Y is a continuation of X and $\text{age}(\mathrm{Y}) = \text{age}(\mathrm{X})$. Note that a continuation of a continuation is a continuation and that a refinement of a refinement is a refinement.

Let $\mathbf{orb}(\mathrm{H}) := \{\mathrm{age}(\mathrm{X}) \mid \mathrm{X} \text{ is an orbit of } \mathrm{H}\}$ for every homogeneous structure H and more generally let $\mathbf{orb}(\mathrm{V}) := \{\mathrm{age}(\mathrm{X}) \mid \mathrm{X} \text{ is a continuation of } \mathrm{V}\}$.

The ages in $\mathbf{orb}(\mathrm{H})$ are partially ordered under $\subseteq$. Then $(\mathbf{orb}(\mathrm{H}); \subseteq)$ is the partial order of the ages of the orbits of the homogeneous structure H. Let $\mathbf{c} \in \mathbf{orb}(\mathrm{H})$ and X an orbit of H with $\mathbf{c} \subseteq \mathrm{age}(\mathrm{X})$. A continuation Y of X is a $\mathbf{c}$-*restriction* of X if $\mathrm{age}(\mathrm{Y}) = \mathbf{c}$.

Let U be a unary subset of the homogeneous structure H. Then the pair $(U, \emptyset)$ is an orbit of H, called a *unary orbit*. If σ does not contain a unary relation symbol then $(\langle \mathrm{H} \rangle, \emptyset)$ is an orbit of H. Note that the structure $\mathrm{H}|U$ has the mapping extension property and hence it is a homogeneous structure. Observe that if the age of H is freely amalgamable then the age of $\mathrm{H}|U$ is freely amalgamable and if X is an orbit of H then there is a unary subset U of H so that $X \subseteq U$. Hence every orbit of H is a subset of a unary orbit which in turn implies that the age of every orbit is a subset of the age of some unary orbit.

Lemma 4.1. *Let $\mathcal{T}$ be a free boundary set and $\mathrm{H}_\mathcal{T}$ the corresponding $\mathcal{T}$-free homogeneous structure. Then every orbit* X *of* $\mathrm{H}_\mathcal{T}$ *is age indivisible.*

Proof. According to [9] Theorem **A.4.6** there is an expansion σ' of σ and an expansion of the σ-structure $\mathrm{H}_\mathcal{T}|X$ to a homogeneous σ'-structure G whose age is freely amalgamable. Because every two elements of X are in the same set of unary relations it follows from Theorem 4.2 that G is age indivisible and hence from Lemma 3.1 that $\mathrm{H}|X$ is age indivisible.

□

Corollary 4.1. *Let $\mathcal{T}$ be a free boundary set and $\mathrm{H}_\mathcal{T}$ the corresponding $\mathcal{T}$-free homogeneous structure. Let* X *be an orbit of* $\mathrm{H}_\mathcal{T}$*. Then X is infinite.*

Lemma 4.2. *Let $\mathcal{T}$ be a free boundary set and $\mathrm{H}_\mathcal{T}$ the corresponding $\mathcal{T}$-free homogeneous structure. Let* X *and* Y *be two orbits of* $\mathrm{H}_\mathcal{T}$ *so that* $\mathbb{F}(\mathrm{X}) \cap \mathbb{F}(\mathrm{Y}) = \emptyset$ *and so that no element of* $\mathbb{F}(\mathrm{X})$ *is adjacent to an element of* $\mathbb{F}(\mathrm{Y})$ *within* $\mathbb{F}(\mathrm{X}) \cup \mathbb{F}(\mathrm{Y})$*. Let Z be the set of all elements $z \in X \cap Y$ and so, that if $x \in \mathbb{F}(\mathrm{X})$ and $y \in \mathbb{F}(\mathrm{Y})$, then x, y and z are not adjacent within $\{z\} \cup \mathbb{F}(\mathrm{X}) \cup \mathbb{F}(\mathrm{Y})$.*

Then there exists an orbit Z *with* $\mathbb{F}(\mathrm{Z}) = \mathbb{F}(\mathrm{X}) \cup \mathbb{F}(\mathrm{Y})$ *and* $\mathbb{I}(\mathrm{Z}) = Z$ *and* $\mathrm{age}(\mathrm{Z}) = \mathrm{age}(\mathrm{X}) \cap \mathrm{age}(\mathrm{Y})$.

Proof. Let $\mathrm{A} \in \mathrm{skel}(\mathrm{X})$ and $\mathrm{B} \in \mathrm{skel}(\mathrm{Y})$ and f a local isomorphism of A to B. Let $\bar{\mathrm{A}} := \mathrm{H}_\mathcal{T}|(\langle \mathrm{A} \rangle \cup \mathbb{F}(\mathrm{X}))$. Let A′ be the structure with $\langle \mathrm{A}' \rangle = \langle \mathrm{B} \rangle \cup \mathbb{F}(\mathrm{X})$ and so that the function which is the identity on $\mathbb{F}(\mathrm{X})$ and f when restricted to $\langle \mathrm{A} \rangle$ is an isomorphism of $\bar{\mathrm{A}}$ to A′. Let $\mathrm{B}' := \mathrm{H}_\mathcal{T}|(\langle \mathrm{B} \rangle) \cup \mathbb{F}(\mathrm{Y}))$.

Then $\mathrm{A}' \amalg \mathrm{B}'$ is an element in $\mathrm{age}(\mathrm{H}_\mathcal{T})$ and hence there is an extension h of the identity on $\mathbb{F}(\mathrm{X}) \cup \mathbb{F}(\mathrm{Y})$ to an embedding of $\mathrm{A}' \amalg \mathrm{B}'$. The function h maps $\langle \mathrm{B} \rangle$ into Z.

Hence Z contains the intersection of the ages of X and Y and is therefore not empty because both orbits are not empty; containing singleton substructures. It follows that Z exists and that $\mathrm{age}(\mathrm{Z}) = \mathrm{age}(\mathrm{X}) \cap \mathrm{age}(\mathrm{Y})$. □

Lemma 4.3. *Let $\mathcal{T}$ be a free boundary set and $\mathrm{H}_\mathcal{T}$ the corresponding $\mathcal{T}$-free homogeneous structure. Let* X *be an orbit of* $\mathrm{H}_\mathcal{T}$*. Then* $\mathrm{age}(\mathrm{X}) = \mathrm{age}(\mathrm{X}|\mathbb{F}_*(\mathrm{X}))$.

Proof. Let $\mathrm{Y} = \mathrm{X}|\mathbb{F}_*(\mathrm{X})$. We have to prove that every element in $\mathrm{skel}(\mathrm{Y})$ is an element in $\mathrm{age}(\mathrm{X})$. Let B be a finite subset of Y. The structures $\mathrm{H}_\mathcal{T}|(B \cup \mathbb{F}_*(\mathrm{X}))$ and $\mathrm{H}_\mathcal{T}|\mathbb{F}(\mathrm{X})$ are compatible and hence can be freely amalgamated to the structure $\mathrm{C} := \big(\mathrm{H}_\mathcal{T}|(B \cup \mathbb{F}(\mathrm{Y}))\big) \amalg \big(\mathrm{H}_\mathcal{T}|\mathbb{F}(\mathrm{X})\big)$. It follows from the mapping extension property of $\mathrm{H}_\mathcal{T}$ that the identity map on $\mathbb{F}(\mathrm{X})$ has an extension to an embedding f of C into $\mathrm{H}_\mathcal{T}$. The function f maps the structure $\mathrm{H}_\mathcal{T}|B$ into X.

□

Let X be an orbit of $\mathrm{H}_\mathcal{T}$ and F a finite subset of $\langle \mathrm{H}_\mathcal{T} \rangle$ with $\mathbb{F}(\mathrm{X}) \cap F = \emptyset$ then the orbit Y with $\mathbb{F}(\mathrm{Y}) = \mathbb{F}(\mathrm{X}) \cup F$ and $F \subseteq \mathbb{F}_0(\mathrm{X})$ is called the *free F-continuation* of X.

Corollary 4.2. *Let $\mathcal{T}$ be a free boundary set and $\mathrm{H}_\mathcal{T}$ the corresponding $\mathcal{T}$-free homogeneous structure. If* X *and* Y *are two compatible orbits of* $\mathrm{H}_\mathcal{T}$ *with* $\mathbb{F}_*(\mathrm{X}) = \mathbb{F}_*(\mathrm{Y})$ *then* $\mathrm{age}(\mathrm{X}) = \mathrm{age}(\mathrm{Y})$. *In particular if* Y *is the free F-continuation of* X *for some set F then* $\mathrm{age}(\mathrm{X}) = \mathrm{age}(\mathrm{Y})$.

Lemma 4.4. *Let $\mathcal{T}$ be a free boundary set and* $\mathrm{H}_\mathcal{T}$ *the corresponding $\mathcal{T}$-free homogeneous structure. Let* X *be an orbit of* $\mathrm{H}_\mathcal{T}$ *and* $\mathbf{c} \in \mathbf{orb}(\mathrm{H}_\mathcal{T})$ *with* $\mathrm{age}(\mathrm{X}) \supseteq \mathbf{c}$ *and F a finite subset of* $\langle \mathrm{H}_\mathcal{T} \rangle$.

Then there is a continuation Z *of* X *with* $\mathrm{age}(Z) = \mathbf{c}$ *and* $\mathbb{F}(\mathrm{Z}) \cap F = \mathbb{F}(\mathrm{X}) \cap F$ *and no element in* $\mathbb{F}(\mathrm{Z}) - \mathbb{F}(\mathrm{X})$ *is adjacent to an element in* $\mathbb{F}(\mathrm{X}) \cup F$ *within* $\mathbb{F}(\mathrm{X}) \cup \mathbb{F}(\mathrm{Z}) \cup F$.

Proof. There is an orbit Y with $\mathrm{age}(\mathrm{Y}) = \mathbf{c}$. Let A be a σ-structure with $\langle \mathrm{A} \rangle \cap \langle \mathrm{H}_\mathcal{T} \rangle = \emptyset$ so that there is an isomorphism f of $\mathrm{H}_\mathcal{T}|\mathbb{F}(\mathrm{Y})$ to A. Let B be the structure with $\langle \mathrm{B} \rangle = \mathbb{F}(\mathrm{X}) \cup \langle \mathrm{A} \rangle \cup F$ and $\mathrm{B}|(\mathbb{F}(\mathrm{X}) \cup F) = \mathrm{H}_\mathcal{T}|(\mathbb{F}(X) \cup F)$ and $\mathrm{B}|\langle \mathrm{A} \rangle = \mathrm{A}$ and no vertex in $\langle \mathrm{A} \rangle$ is adjacent to a vertex in $\mathbb{F}(\mathrm{X}) \cup F$ within $\langle \mathrm{A} \rangle \cup \mathbb{F}(\mathrm{X}) \cup F$. The structure B is in the age of $\mathrm{H}_\mathcal{T}$ and hence there is an extension g of the identity map on $\mathbb{F}(\mathrm{X}) \cup F$ to an embedding of B into $\mathrm{H}_\mathcal{T}$.

Let h be an expansion of the local automorphism $g \circ f$ to an automorphism of $\mathrm{H}_\mathcal{T}$. Let Y' be the orbit with $\mathbb{F}(\mathrm{Y}') = (g \circ f)[\mathbb{F}(\mathrm{Y})]$ and so that if $y \in Y$ then $h(y) \in Y'$. It follows that $\mathrm{age}(\mathrm{Y}') = \mathbf{c}$ and that $\mathbb{F}(\mathrm{X}) \cap \mathbb{F}(\mathrm{Y}) = \emptyset$ and that no element in $\mathbb{F}(\mathrm{X})$ is adjacent to an element in $\mathbb{F}(\mathrm{Y}')$ within $\mathbb{F}(\mathrm{X}) \cup \mathbb{F}(\mathrm{Y}')$.

Hence Lemma 4.2 applies. There is an orbit Z with $\mathbb{F}(\mathrm{Z}) = \mathbb{F}(\mathrm{X}) \cup \mathbb{F}(\mathrm{Y}')$ and $\mathbb{I}(\mathrm{Z}) = X \cap Y'$ and $\mathrm{age}(\mathrm{Z}) = \mathrm{age}(X) \cap \mathrm{age}(\mathrm{Y}) = \mathrm{age}(\mathrm{X}) \cap \mathbf{c} = \mathbf{c}$.

□

5 Persistent partitions

Let H be a countable homogeneous structure in signature σ and $\langle \mathrm{H} \rangle := \{b_i \mid i \in \omega\}$ an enumeration of H into an ω-sequence. We will write $b_i \leq b_j$ if $i \leq j$ and $b_i < b_j$ if $i < j$. (Assuming that $\leq$ is not one of the relation symbols in σ.)

Let R and S be subsets of $\langle \mathrm{H} \rangle$. Then $R < S$ stands for: If $r \in R$ and $s \in S$ then $r < s$. Also $R < t$ for t an element in $\langle \mathrm{H} \rangle$ if $R < \{t\}$. If R and S are finite then R is *lexicographically smaller* then S if the largest element of the symmetric difference of R and S is an element of S.

Let X be an orbit of H and f an embedding of H into H. Let $f^*(\mathrm{X})$ be the orbit $\mathrm{Y} = (Y, f[\mathbb{F}(\mathrm{X})])$ of H so that $f[X] \subseteq Y$. It follows that $\mathrm{age}(\mathrm{X}) = \mathrm{age}(f^*(\mathrm{X}))$. Denote by $\mathbf{R}(\mathrm{X})$ the set of orbits Y of H with $\mathrm{age}(\mathrm{Y}) \not\supseteq \mathrm{age}(\mathrm{X})$ and $\mathbb{F}(\mathrm{Y})$ lexicographically smaller than $\mathbb{F}(\mathrm{X})$. Then

$$\Lambda(\mathrm{X}) := X - \bigcup_{\mathrm{Y} \in \mathbf{R}(\mathrm{X})} Y.$$

Let $\bar{\Lambda}(\mathrm{X}) := f[X] - \Lambda(\mathrm{X})$.

Lemma 5.1. *Let* X *be an age indivisible orbit of the homogeneous structure* H. *Let* $S \subseteq X$ *with* $\mathrm{age}(\mathrm{H}|S) = \mathrm{age}(\mathrm{X})$ *and* f *an embedding of* H *into* H. *Then*

$$\mathrm{age}\Big(\mathrm{H}|\big(f[S] \cap \Lambda(f^*(\mathrm{X}))\big)\Big) = \mathrm{age}(\mathrm{X}) \tag{1}$$

and

$$\mathrm{age}\Big(\mathrm{H}|\big(f[S] \cap \bar{\Lambda}(f^*(\mathrm{X}))\big)\Big) \neq \mathrm{age}(\mathrm{X}). \tag{2}$$

Proof. The set $\mathbf{R}(f^*(\mathrm{X}))$ is finite and $\mathrm{age}(\mathrm{H}|(Y \cap f[S])) \subseteq \mathrm{age}(\mathrm{Y})$ and hence $\mathrm{age}(\mathrm{H}|(Y \cap f[S])) \neq \mathrm{age}(f^*(\mathrm{X})) = \mathrm{age}(\mathrm{X})$ for every $\mathrm{Y} \in \mathbf{R}(f^*(\mathrm{X}))$. The age of $\mathrm{H}|f[S]$ is equal to $\mathrm{age}(\mathrm{H}|S) = \mathrm{age}(\mathrm{X})$ and hence $\mathrm{H}|f[S]$ is age indivisible according to Theorem 3.1. It follows that

$$\mathrm{age}\Big(\mathrm{H}\Big|\Big(\bigcup_{\mathrm{Y} \in \mathbf{R}(f^*(\mathrm{X}))} (Y \cap f[S])\Big)\Big) \neq \mathrm{age}(f[S]) = \mathrm{age}(X)$$

which implies formula (2). Formula (1) follows because

$$\Big(\bigcup_{\mathrm{Y} \in \mathbf{R}(f^*(\mathrm{X}))} (Y \cap f[S])\Big) \cup \Big(f[S] \cap \Lambda(f^*(\mathrm{X}))\Big) = f[S]$$

and $\mathrm{age}(f[S])$ is indivisible.

□

Let $\mathbf{c} \in (\mathbf{orb}(\mathrm{H}); \subseteq)$. Then

$$\Lambda(\mathbf{c}) := \bigcup_{\mathrm{age}(\mathrm{X}) = \mathbf{c}} \Lambda(\mathrm{X}).$$

Lemma 5.2. *If* $\mathbf{c}$ *and* $\mathbf{d}$ *are two non comparable elements of* $(\mathbf{orb}(\mathrm{H}); \subseteq)$ *then the sets* $\Lambda(\mathbf{c})$ *and* $\Lambda(\mathbf{d})$ *are disjoint.*

Proof. Let $x \in \Lambda(\mathbf{c}) \cap \Lambda(\mathbf{d})$. Then there is an orbit X with $\mathrm{age}(\mathrm{X}) = \mathbf{c}$ and an orbit Y with $\mathrm{age}(\mathrm{Y}) = \mathbf{d}$ so that $x \in \Lambda(\mathrm{X}) \cap \Lambda(\mathrm{Y})$. Because one of $\mathbb{F}(\mathrm{X})$ and $\mathbb{F}(\mathrm{Y})$ is lexicographically smaller than the other this is in contradiction to the definition of $\Lambda(\mathrm{X})$ or $\Lambda(\mathrm{Y})$. □

A subset $S \in \Lambda(\mathbf{c})$ is *large* if there is an orbit X with $\mathrm{age}(\mathrm{X}) = \mathbf{c}$ so that $\mathrm{age}(\mathrm{H}|(S \cap X)) = \mathrm{age}(\mathrm{X}) = \mathbf{c}$.

Lemma 5.3. *Let* $\mathbf{c} \in \mathbf{orb}(\mathrm{H})$ *with* $\mathbf{c}$ *indivisible and let* f *be an embedding of* H *into* H *and* S *a large subset of* $\Lambda(\mathbf{c})$*. Then:*

a. The large subsets of $\Lambda(\mathbf{c})$ *form a co-ideal of* $\Lambda(\mathbf{c})$*.*

b. The set $f[S] \cap \Lambda(\mathbf{c})$ *is large.*

Proof. Let S be a large subset of $\Lambda(\mathbf{c})$ and X an orbit so that $\mathrm{age}(\mathrm{H}|(S \cap X)) = \mathrm{age}(\mathrm{X})$.

Proof of item a.: Item 3. of the definition of co-ideal follows because age(X) is indivisible and item 2. is trivially satisfied. The empty set is not large because an orbit is by definition not empty. The set $\Lambda(\mathbf{c})$ is large according to Lemma 5.1 item 1. with f the identity embedding and $S = \mathrm{X} \in \mathbf{c}$.

Item b. follows from Lemma 5.1 item 1. with S replaced by $S \cap X$ because $\Lambda(f^*(\mathrm{X})) \subseteq \Lambda(\mathbf{c})$.

□

Theorem 5.1. *Let* H *be a homogeneous structure so that every orbit of* H *is age indivisible and let* $(\mathbf{c}_0, \mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_{n-1})$ *be an anti-chain of* $\mathbf{orb}(\mathrm{H})$*.*

Then the sets $\big(\Lambda(\mathbf{c}_0), \Lambda(\mathbf{c}_1), \Lambda(\mathbf{c}_2), \dots, \Lambda(\mathbf{c}_{n-1})\big)$ *form a persistent partition of their union and*

$$\Big(\Lambda(\mathbf{c}_0), \Lambda(\mathbf{c}_1), \Lambda(\mathbf{c}_2), \dots, \Lambda(\mathbf{c}_{n-1}) \cup \big(\langle \mathrm{H} \rangle - \bigcup_{i \in n-1} \Lambda(\mathbf{c}_i)\big)\Big)$$

is a persistent partition of H*.*

If the sizes of the anti-chains in $(\mathbf{orb}(\mathrm{H}_{\mathcal{T}}); \subseteq)$ *are unbounded then*

$$\mathrm{H}_{\mathcal{T}} \not\rightarrow (\mathrm{H}_{\mathcal{T}})_{n/n-1}$$

for every $n \in \omega$*. In particular* $\mathrm{H}_{\mathcal{T}}$ *does not have a canonical partition.*

Proof. The sets $\big(\Lambda(\mathbf{c}_0), \Lambda(\mathbf{c}_1), \Lambda(\mathbf{c}_2), \dots, \Lambda(\mathbf{c}_{k-1})\big)$ are pairwise disjoint according to Lemma 5.2 and each of them is persistent according to Lemma 5.3.

Let $n \in \omega$. If the sizes of the anti-chains in $(\mathbf{orb}(\mathrm{H}_{\mathcal{T}}); \subseteq)$ are unbounded then there exists an anti-chain $(\mathbf{c}_0, \mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_{n-1})$ of n elements in $(\mathbf{orb}(\mathrm{H}_{\mathcal{T}}); \subseteq)$. Color $x \in \langle \mathrm{H}_{\mathcal{T}} \rangle$ with color i if $x \in \Lambda(\mathbf{c}_i)$. Color the remaining elements with color 0.

□

6 An important co-ideal

The co-ideal $\mathbb{C}$ on the set H, the partial order $(\mathbf{P}, \subseteq)$ with maximum $\mathbf{m}$, the partial order $(\mathfrak{J}, \preceq)$ with maximum M, the function $\mathbb{I}$ of $\mathfrak{J}$ into $\mathbb{C}$, and the function *age* of $\mathfrak{J}$ into $\mathbf{P}$ have property

$$\mho(\mathbb{C}, H, (\mathbf{P}, \subseteq), \mathbf{m}, (\mathfrak{J}, \preceq), M, \mathbb{I}, age)$$

if for all $X, Y \in \mathfrak{J}$:

1. $\mathbb{I}(M) = H$ and $age(M) = \mathbf{m}$.
2. $X \preceq Y$ implies $\mathbb{I}(X) \subseteq \mathbb{I}(Y)$.
3. $X \preceq Y$ implies $age(X) \subseteq age(Y)$.
4. If $\mathbf{b} \in \mathbf{P}$ with $\mathbf{b} \subseteq age(X)$ then there is $Y \in \mathfrak{J}$ with $Y \preceq X$ and $age(Y) = \mathbf{b}$.

Let $\mho(\mathbb{C}, H, (\mathbf{P}, \subseteq), \mathbf{m}, (\mathfrak{J}, \preceq), M, \mathbb{I}, age)$ and $X, Y \in \mathbb{C}$. If $Y \preceq X$ and $age(Y) = age(X)$ then Y is a *ref* of X. If $Y \preceq X$ and $age(X) \subseteq \mathbf{b} = age(Y)$ then Y is a $\mathbf{b}$-*rest* of X.

Let $\mathbf{C}$ be a chain of $\mathbf{P}$ which contains $\mathbf{m}$. A subset S of H is $\mathbf{C}$-*large* if there is a unary relation Φ on $\{X \in \mathfrak{J} \mid age(X) \in \mathbf{C}\}$ so that $\Phi(M)$ and so that $\Phi(X)$ implies $S \cap \mathbb{I}(X) \in \mathbb{C}$ and so that the following formula $\phi(X)$ holds:

$$\phi(X) := \begin{array}{ll} \text{For all } \mathbf{c} \in \mathbf{C} \text{ with } age(X) \supseteq \mathbf{c} & (1) \\ \text{there exists a ref } Y \text{ of } X & (2) \\ \text{so that for all refs } Z \text{ of } Y & (3) \\ \text{there exists a } \mathbf{c}\text{-rest } R \text{ of } Z \text{ with } \Phi(R). & (4) \end{array}$$

The unary relation Φ is the *witness* of S being $\mathbf{C}$-large.

Theorem 6.1. *Let* $\mho(\mathbb{C}, H, (\mathbf{P}, \subseteq), \mathbf{m}, (\mathfrak{J}, \preceq), M, \mathbb{I}, age)$ *and* $\mathbf{C}$ *a chain of* $(\mathbf{P}, \subseteq)$ *containing the element* $\mathbf{m}$.

Then the set of $\mathbf{C}$*-large subsets of* H *is a co-ideal on* H.

Proof. In order to see that H is $\mathbf{C}$-large let $\Phi(X)$ for every $X \in \mathfrak{J}$ with $age(X) \in \mathbf{C}$. Then $\Phi(M)$ because $age(H) = \mathbf{m} \in \mathbf{C}$. If $\phi(X)$ with $age(X) \in \mathbf{C}$ then $\mathbb{I}(X) \in \mathbb{C}$ and hence $H \cap \mathbb{I}(X) = \mathbb{I}(X) \in \mathbb{C}$. It remains to verify $\phi(X)$ for $X \in \mathfrak{J}$ with $age(X) \in \mathbf{C}$.

For $\mathbf{c} \subseteq age(X) \in \mathbf{C}$ and $\mathbf{c} \in \mathbf{C}$ let Y in line (2) of formula $\phi(X)$ be equal to X. Given a ref Z of $Y = X$ there is a $\mathbf{c}$-rest R of Z according to item 4. of the definition of property $\mho$.

Let S with witness Φ be $\mathbf{C}$-large and $S \subseteq T \subseteq H$. Then T is $\mathbf{C}$-large with the same witness Φ.

Let S with witness Φ be $\mathbf{C}$-large and $S_1 \cup S_2 = S$. We will show that one of the sets S_i is $\mathbf{C}$-large. We first determine unary relations Φ_1 and Φ_2 on the set $\{X \in \mathfrak{J} \mid age(X) \in \mathbf{C} \text{ and } \Phi(X)\} := \mathbf{C}_\Phi$ using the following two-player game Γ. The unary relations Φ_1 and Φ_2 are to be used as witnesses for S_1 or S_2 respectively to be large.

Let $\mathrm{X} \in \mathbf{C}_\Phi$. The game $\Gamma(\mathrm{X})$ starts in state $(\mathrm{X}, 0)$ with player **I** to move.

0. If the game is in state $(U, 0)$ for some element $\mathrm{U} \in \mathbf{C}_\Phi$ then it is the turn of player **I** to move. Player **I** selects $\mathbf{c} \in \mathbf{C}$ with $age(U) \supseteq \mathbf{c}$ and the game moves into state $(U, \mathbf{c}, 1)$.

1. If the game is in state $(U, \mathbf{c}, 1)$ then it is the turn of player **II** to move. Player **II** selects a ref V of U and the game moves to state $(V, \mathbf{c}, 2)$.

2. If the game is in state $(V, \mathbf{c}, 2)$ then it is the turn of player **I** to move. Player **I** selects a ref W of V. The game moves to state $(W, \mathbf{c}, 3)$.

3. If the game is in state $(W, \mathbf{c}, 3)$ then it is the turn of player **II** to move. Player **II** selects a **c**-rest R of W and the game moves to state $(R, 0)$. Then it is again the turn of player **I** to move.

The game ends with a win of player **I** if it is in a state $(R, 0)$ for which $\mathbb{I}(R) \cap S_2 \notin \mathbb{C}$ or for which $\neg\Phi(\mathrm{R})$. We let $\Phi_2(\mathrm{X})$ if player **I** does not have a winning strategy in the game $\Gamma(X)$. (Player **I** has a winning strategy if no matter how player **II** plays the game will always end after finitely many moves in a state $(R, 0)$ for which $\mathbb{I}(R) \cap S_2 \notin \mathbb{C}$ or for which $\neg\Phi(\mathrm{R})$.) It follows that if player **I** does not have a winning strategy in the game $\Gamma(\mathrm{X})$ then $\mathbb{I}(X) \cap S_2 \in \mathbb{C}$. That is $\Phi_2(\mathrm{X})$ implies that $\mathbb{I}(X) \cap S_2 \in \mathbb{C}$.

Let $\phi_2(X)$ be formula $\phi(X)$ with Φ_2 replacing Φ in line (4). It follows from the definition of the game Γ and the definition of $\Phi_2(X)$ that if $\Phi_2(X)$ then $\phi_2(X)$. Hence if $\Phi_2(H)$ then S_2 is **C**-large with witness Φ_2.

We will say player **I** *has a win* at a state of the game if player **I** has a winning strategy when the game starts at this state.

If $X \in \mathbf{C}_\Phi$ and $\mathbb{I}(X) \cap S_1 \notin \mathbb{C}$ then player **I** does not have a win at X. Because player **II** chooses for V in state $(X, \mathbf{c}, 1)$ the element Y given by line (2) of formula $\phi(X)$. Then no matter what ref W player **I** selects we get from formula $\phi(X)$ that $\Phi(R)$, hence $S \cap \mathbb{I}(R) \in \mathbb{C}$, for every **c**-rest R of W.

We obtain from $\mathbb{I}(X) \cap S_1 \notin \mathbb{C}$ and $\mathbb{I}(R) \subseteq \mathbb{I}(X)$ that $\mathbb{I}(R) \cap S_1 \notin \mathbb{C}$ and because $S \cap \mathbb{I}(R) \in \mathbb{C}$ and $S = S_1 \cup S_2$ that $\mathbb{I}(R) \cap S_2 \in \mathbb{C}$. Hence player **II** can never force the game into a state (R,0) for which $\mathbb{I}(R) \cap S_2 \notin \mathbb{C}$ or for which $\neg\Phi(\mathrm{R})$.

Let $\Phi_1(X)$ if $X \in \mathbf{C}_\Phi$ and player **I** has a win at X. Then $\Phi_1(X)$ implies that $\mathbb{I}(X) \cap S_1 \in \mathbb{C}$. Let $\phi_1(X)$ be formula $\phi(X)$ with Φ_1 replacing Φ in line (4). It follows from the following Lemma 6.1 that $\Phi_1(X)$ implies $\phi_1(\mathrm{X})$. Hence if $\Phi_1(H)$ then S_1 is **C**-large with witness Φ_1.

The element H is an element of $\mathbf{C}_\Phi$. If player **I** has a win at state $(H, 0)$ then $\Phi_1(H)$. If player **I** does not have a win at state $(H, 0)$ then $\Phi_2(H)$. Hence S_1 or S_2 is **C**-large.

□

Lemma 6.1. *Let $X \in \mathbf{C}_\Phi$ with $\Phi_1(X)$. Then $\phi_1(X)$.*

Proof. Because $\Phi_1(X)$ player **I** has a winning strategy in the game $\Gamma(X)$. For line (1) of formula ϕ_1 let $\mathbf{c} \in \mathbf{C}$ with $age(\mathrm{X}) \subseteq \mathbf{c}$ be given. Let $\mathbf{c}'$ be the element of **C** chosen by player **I**. The game moves to state $(X, \mathbf{c}', 1)$ with a win for player **I**.

Case 1.: $\mathbf{c} \supseteq \mathbf{c}'$.

We will prove that the formula consisting of lines (2) to (4) of formula $\phi_1(X)$ is satisfied and that there is a $\mathbf{c}$-rest R of X so that $\Phi(R)$ and player $\mathbf{I}$ has a win at state $(R, 0)$ of the game $\Gamma(X)$.

From $X \in \mathbf{C}_\Phi$ we get $\Phi(X)$ and hence $\phi(X)$. Then for $\mathbf{c}$ in line (1) of $\phi(X)$ let Y' be an element given by line (2) of formula $\phi(X)$. It follows that Y' is a ref of X so that if Z is a ref of Y' the set of all $\mathbf{c}$-rests R of Z with $\Phi(R)$ is not empty.

Let Y be the ref of Y' chosen by player $\mathbf{I}$ when in state $(Y', \mathbf{c}', 2)$ moving the game to state $(Y, \mathbf{c}', 3)$ with a win for player $\mathbf{I}$. Because player $\mathbf{I}$ has made a winning move, player $\mathbf{I}$ has a win at all states of the form $(R', 0)$ in which R' is a $\mathbf{c}'$-rest of Y. We will prove that this element Y can be used as a "Y" in line (2) of formula $\phi_1(X)$.

In order to validate line (3) of formula $\phi_1(X)$ let a ref Z of Y be given. We have to prove that there exists a $\mathbf{c}$-rest R of Z so that $\Phi(R)$ and so that $(R, 0)$ is a winning position for player $\mathbf{I}$.

Let R be a $\mathbf{c}$-rest of Z with $\Phi(R)$. Assume for a contradiction that $(R, 0)$ is not a winning state for player $\mathbf{I}$. Let player $\mathbf{I}$ choose $\mathbf{c}'$ when starting the game $\Gamma(R)$ in state $(R, 0)$. Let V be the ref chosen by player $\mathbf{II}$ and let player $\mathbf{I}$ select V in state $(V, \mathbf{c}', 2)$. Because $(R, 0)$ is not a winning state for player $\mathbf{I}$ there exists a $\mathbf{c}'$ rest R' of R so that player $\mathbf{I}$ does not have a win at state $(R', 0)$. This is a contradiction because R' is a $\mathbf{c}'$ rest of Y.

Case 2.: $\mathbf{c}' \supset \mathbf{c}$.

We will prove that the formula consisting of lines (2),(3) and (4) of formula $\phi_1(X)$ is satisfied. From $X \in \mathbf{C}_\Phi$ we get $\Phi(X)$ and hence $\phi(X)$. Then for $\mathbf{c}'$ in line (1) of $\phi(X)$ let Y' be an element given by line (2) of formula $\phi(X)$. It follows that Y' is a ref of X so that if Z is a ref of Y' the set of all $\mathbf{c}'$-rests R_0 of Z with $\Phi(R_0)$ is not empty.

Let Y be the ref of Y' chosen by player $\mathbf{I}$ when in state $(Y', \mathbf{c}', 2)$ moving the game to state $(Y, \mathbf{c}', 3)$ with a win for player $\mathbf{I}$. Because player $\mathbf{I}$ has made a winning move, player $\mathbf{I}$ has a win at all states of the form $(R', 0)$ in which R' is a $\mathbf{c}'$-rest of Y. We will prove that this element Y can be used as a "Y" in line (2) of formula $\phi_1(X)$.

In order to validate line (3) of formula $\phi_1(X)$ let a ref Z of Y be given. We have to prove that there exists a $\mathbf{c}$-rest R of Z so that $\Phi(R)$ and $(R, 0)$ is a winning position for player $\mathbf{I}$. Let R_0 be a $\mathbf{c}'$-rest of Z with $\Phi(R_0)$. Then player $\mathbf{I}$ has a win at state $(R_0, 0)$. We continue to play the game following a winning strategy of player $\mathbf{I}$. the game will progress through states:

$(\mathrm{R}_0, 0)$, $(\mathrm{R}_0, \mathbf{c}_0, 1)$, $(\mathrm{V}_0, \mathbf{c}_0, 2)$, $(\mathrm{W}_0, \mathbf{c}_0, 3)$,
$(\mathrm{R}_1, 0)$, $(\mathrm{R}_1, \mathbf{c}_1, 1)$, $(\mathrm{V}_1, \mathbf{c}_1, 2)$, $(\mathrm{W}_1, \mathbf{c}_1, 3)$,
$(\mathrm{R}_2, 0)$, $(\mathrm{R}_2, \mathbf{c}_2, 1)$, $(\mathrm{V}_2, \mathbf{c}_2, 2)$, $(\mathrm{W}_2, \mathbf{c}_2, 3)$,
$(\mathrm{R}_3, 0)$, $(\mathrm{R}_3, \mathbf{c}_3, 1)$, $(\mathrm{V}_3, \mathbf{c}_3, 2)$, $(\mathrm{W}_3, \mathbf{c}_3, 3)$,
. . . .
$(\mathrm{R}_i, 0)$, $(\mathrm{R}_i, \mathbf{c}_i, 1)$, $(\mathrm{V}_i, \mathbf{c}_i, 2)$, $(\mathrm{W}_i, \mathbf{c}_i, 3)$,
$(\mathrm{R}_{i+1}, 0)$, $(\mathrm{R}_{i+1}, \mathbf{c}_{i+1}, 1)$, $(\mathrm{V}_{i+1}, \mathbf{c}_{i+1}, 2)$, $(\mathrm{W}_{i+1}, \mathbf{c}_{i+1}, 3)$,
. . . .

The game will move through winning states of player **I** so that $\Phi(R_i)$ for all i. Player **I** chooses $\mathbf{c}_i$ if in state $(R_i, 0)$ and we, as player **II** select V_i so that it satisfies line (2) of formula $\phi(R_i)$. Then player **I** selects W_i and player **II** R_{i+1} with $\Phi(R_{i+1})$. This is always possible because of the choice of V_i. Note that $age(R_{i+1}) = \mathbf{c_i}$ for all i.

Because $\mathbf{c}_0 \supseteq \mathbf{c}_1 \supseteq \mathbf{c}_2 \dots$ there is either a number i so that $\mathbf{c}_i \supset \mathbf{c} \supseteq \mathbf{c}_{i+1}$ or, because player **I** has a winning strategy at state $(R_0, 0)$, the game ends after finitely many rounds, with a win of player **I** in some state $(\mathrm{R}_n, 0)$ with $age(\mathrm{R}_n) = \mathbf{c}_{n-1} \supset \mathbf{c}$ and $S_2 \cap \mathbb{I}(R_n) \notin \mathbb{C}$.

If $\mathbf{c}_i \supset \mathbf{c} \supseteq \mathbf{c}_{i+1}$ for some $i \in \omega$ we are in the situation of case 1. There is a **c**-rest R of R_i so that $\Phi(\mathrm{R})$ and player **I** has a win at state $(\mathrm{R}, 0)$ of the game $\Gamma(\mathrm{X})$. The element R is the desired **c**-rest of the element Z.

If the game ends after finitely many rounds with a win of player **I** in a state $(R_n, 0)$ and $\mathbf{c}_{n-1} \supset \mathbf{c}$ then $\mathbb{I}(R_n) \cap S_2 \notin \mathbb{C}$. Let R be a **c**-rest of R_n. Such an element R exists because $\Phi(R_n)$ and we can use formula ϕ to obtain R. Because $\mathbb{I}(R_n) \cap S_2 \notin \mathbb{C}$ and $\mathbb{I}(R) \subseteq R_n$ we get $\mathbb{I}(R) \cap S_2 \notin \mathbb{C}$. It follows that R is a **c**-rest of Z and $(R, 0)$ is a winning position of player **I**.

□

Let $\mho(\mathbb{C}, H, (\mathbf{P}, \subseteq), \mathbf{m}, (\mathfrak{J}, \preceq), M, \mathbb{I}, age)$ and $\mathbf{C}$ a chain of $(\mathbf{P}, \subseteq)$ containing the element $\mathbf{m}$. Let S be a **C**-large subset of H with witness Φ. Let $\mathbf{C}_\Phi := \{X \in \mathfrak{J} \mid age(X) \in \mathbf{C} \text{ and } \phi(X)\}$.

For $X \in \mathfrak{J}$ with $age(X) \in \mathbf{C}$ let $\psi(X)$ if $\phi(Y)$ for every ref Y of X. It follows that if $\psi(X)$ then $\psi(Y)$ for every ref Y of X and $\psi(X)$ implies $\phi(X)$.

Lemma 6.2. *Every element $X \in \mathbf{C}_\Phi$ has a ref Y with $\psi(Y)$.*

Proof. Let $X \in \mathbf{C}_\Phi$. We use formula $\phi(X)$ in the instance $age(X)$ for **c** in line (1). Formula $\phi(X)$ returns a ref Y of X. We will prove that $\psi(Y)$.

Let V be a ref of Y. In order to prove $\phi(V)$ let $\mathbf{c} \in \mathbf{C}$ with $\mathbf{c} \subseteq age(V)$. Because of the choice of Y there is an $age(X)$-rest, that is a ref W of V, so that $\Phi(W)$ hence $\phi(W)$. Formula $\phi(W)$ returns in line (2) an element U if prompted in line (1) with **c**. We use this element U to establish line (2) of formula $\phi(V)$. Let Z be a ref of U. Then there is, according to formula $\phi(W)$, a **c**-rest R of Z with $\Phi(R)$.

□

Lemma 6.3. *Let $\psi(X)$. For every $\mathbf{c} \in \mathbf{C}$ with $age(X) \supseteq \mathbf{c}$ there is a **c**-rest R of X with $\psi(R)$ and $\mathbb{I}(X) \cap S \in \mathbb{C}$ and $\psi(Y)$ for every ref Y of X.*

Proof. If $\psi(X)$ then $\phi(X)$ which implies that there is a **c**-rest R' of X with $\Phi(R')$. It follows from Lemma 6.2 that R' has a ref R with $\psi(R)$.

There is a refinement R' of X with $\Phi(R')$ and hence $\mathbb{I}(R') \cap S \in \mathbb{C}$ which implies $\mathbb{I}(X) \cap S \in \mathbb{C}$ because $R' \subseteq X$.

□

Let H be a homogeneous σ-structure. Note that if U is a unary orbit of H and **C** with $\mathrm{age(U)} \in \mathbf{C}$ is a chain of $(\mathbf{orb}(\mathrm{H}); \subseteq)$ then age(U) is the maximum of the chain $\mathbf{C}$ and if X is an orbit of H with age(X) $\in \mathbf{C}$ then X is a continuation of U.

Let $\mathbf{C}$ be a chain of $\mathbf{orb}(\mathrm{H})$ and $\mathrm{U} \in \mathbf{C}$ a unary orbit and T be a finite subset of $\langle \mathrm{H} \rangle$ then:

$$\mathfrak{J}(\mathbf{C}, \mathrm{U}, T) := \{(\mathrm{X}, F) \mid \mathrm{X} \text{ is an orbit of } \mathrm{H}_{\mathcal{T}} \text{ with } \mathrm{age}(\mathrm{X}) \in \mathbf{C} \text{ and } T \cap \mathbb{F}(\mathrm{X}) \subseteq \mathbb{F}_0(\mathrm{X}) \text{ and } \mathbb{F}(\mathrm{X}) \subseteq F \subseteq \langle \mathrm{H} \rangle \text{ and } F \text{ is finite}\}.$$

The partial order relation $\preceq$ on $\mathfrak{J}(\mathbf{C}, \mathrm{U}, T)$ is given by $(\mathrm{X}, A) \preceq (\mathrm{Y}, B)$ if:

1. X is a continuation of Y.
2. $A \supseteq B$.
3. $(\mathbb{F}(\mathrm{X}) - \mathbb{F}(\mathrm{Y})) \cap B \subseteq \mathbb{F}_0(\mathrm{X})$.

Let $(\mathrm{X}, A) \preceq (\mathrm{Y}, B)$ be elements in $\mathfrak{J}(\mathrm{U}, T)$. Then $\mathrm{age}(\mathrm{X}, F) := \mathrm{age}(\mathrm{X})$ and $\mathbb{I}(\mathrm{X}, F) := \mathbb{I}(\mathrm{X})$ and (X, A) is a *refinement* of (Y, B) if $\mathrm{age}(\mathrm{X}) = \mathrm{age}(\mathrm{X}, A) = \mathrm{age}(\mathrm{Y}, B) = \mathrm{age}(\mathrm{Y})$ and if $\mathrm{age}(\mathrm{X}) = \mathbf{c} \in \mathbf{C}$ then (X, A) is a $\mathbf{c}$-*restriction* of (Y, B).

Lemma 6.4. *Let $\mathcal{T}$ be a free boundary set of σ-structures and $\mathrm{H}_{\mathcal{T}}$ the corresponding homogeneous $\mathcal{T}$-free structure. Let U be a unary orbit of $\mathrm{H}_{\mathcal{T}}$ and $\mathbf{C}$ a chain of $\mathbf{orb}(\mathrm{U})$ and T a finite subset of $\langle \mathrm{H}_{\mathcal{T}} \rangle$. Let $\mathbb{C}$ be the co-ideal of infinite subsets of U. Then:*

1. $$\mho\big(\mathbb{C}, U, (\mathbf{C}, \subseteq), \mathrm{age}(\mathrm{U}), (\mathfrak{J}(\mathbf{C}, \mathrm{U}, T), \preceq), (\mathrm{U}, T), \mathbb{I}, \mathrm{age}\big).$$

   The interpretation of ref is refinement and the interpretation of $\mathbf{c}$-res is $\mathbf{c}$-restriction.

2. *If $\{S_i \mid i \in n \in \omega\}$ is a set of subsets of U with $\bigcup_{i \in n} S_i = U$ then S_i is $\mathbf{C}$-large for some $i \in n$.*

3. *If $S \subseteq U$ is $\mathbf{C}$-large then there is a unary relation $\psi_{\mathbf{C}}$ on the elements of $\mathfrak{J}(\mathbf{C}, \mathrm{U}, T)$ so that for all elements $(\mathrm{X}, F) \in \mathfrak{J}(\mathbf{C}, \mathrm{U}, T)$ with $\psi_{\mathbf{C}}(\mathrm{X}, F)$:*

   (a) *There is a refinement $(\mathrm{R}, B) \in \mathfrak{J}(\mathbf{C}, U, T)$ of the element (U, T) with $\psi_{\mathbf{C}}(\mathrm{R}, B)$.*

   (b) *$X \cap S$ is infinite.*

   (c) *Let $\mathbf{c} \in \mathbf{C}$ with $\mathbf{c} \subseteq \mathrm{age}(\mathrm{X})$. Then there exists a $\mathbf{c}$-restriction (R, G) of (X, F) with $\psi_{\mathbf{C}}(\mathrm{R}, G)$.*

   (d) *$\psi_{\mathbf{C}}(\mathrm{Y}, G)$ for every refinement (Y, G) of (X, F).*

Proof. Item 1.: That $\mathbb{I}$ is a function of $\mathfrak{J}(\mathbf{C}, \mathrm{U}, T)$ into the set of infinite subsets of U follows from Corollary 4.1. Items 1., 2. and 3. of the definition of $\mho$ are easily checked Item 4. follows from Lemma 4.4.

Item 2.: Follows from Theorem 6.1

Item 3.: Item 3. (a) follows from Lemma 6.2 because $\phi(\mathrm{U}, T)$ according to the definition of **C**-large set. Items 3. (b), (c) and (d) follow from Lemma 6.3.

□

Theorem 6.2. *Let $\mathcal{T}$ be a free boundary set of σ-structures and $\mathrm{H}_{\mathcal{T}}$ the corresponding homogeneous $\mathcal{T}$-free structure. Let* U *be a unary orbit of $\mathrm{H}_{\mathcal{T}}$ and* **C** *a chain of* $\mathbf{orb}(\mathrm{U})$.

Then there exists, for every finite subset T of $\langle \mathrm{H}_{\mathcal{T}} \rangle$, a unary relation **C***-large on the subsets of U, so that if S is* **C***-large, then there exists unary relation $\psi_{\mathbf{C}}$ on the set of orbits* X *of $\mathrm{H}_{\mathcal{T}}$ with $\mathrm{age}(\mathrm{X}) \in \mathbf{C}$ and $T \cap \mathbb{F}(\mathrm{X}) \subseteq \mathbb{F}_0(\mathrm{X})$, so that if $\psi(X)$ then there is a finite subset $\mathbb{E}_{\mathbf{C}}(\mathrm{X}) \supseteq \mathbb{F}(\mathrm{X})$ of $\langle \mathrm{H}_{\mathcal{T}} \rangle$ so that:*

1. *If $\{S_i \mid i \in n \in \omega\}$ is a set of subsets of U with $\bigcup_{i \in n} S_i = U$ then S_i is* **C***-large for some $i \in n$. (Implying* U *is $\mathbf{C}-large$).*
2. *There is a refinement* R *of the orbit* U *with $\psi_{\mathbf{C}}(\mathrm{R})$ and $T \cap \mathbb{F}(\mathrm{R}) \subseteq \mathbb{F}_0(\mathrm{R})$.*
3. *If $\psi_{\mathbf{C}}(\mathrm{X})$ then $X \cap S$ is infinite.*
4. *Let $\psi_{\mathbf{C}}(\mathrm{X})$ and $\mathbf{c} \in \mathbf{C}$ with $\mathbf{c} \subseteq \mathrm{age}(\mathrm{X})$ and $E \supseteq \mathbb{E}_{\mathbf{C}}(\mathrm{X})$. Then there is a* **c***-restriction* R *of* X *with $\psi_{\mathbf{C}}(\mathrm{R})$ and $(\mathbb{F}(\mathrm{R}) - \mathbb{F}(\mathrm{X})) \cap E \subseteq \mathbb{F}_0(\mathrm{R})$.*
5. *Let $\psi_{\mathbf{C}}(\mathrm{R})$ and* X *a refinement of* R *with $\big(\mathbb{F}(\mathrm{X}) - \mathbb{F}(\mathrm{R}) \cap \mathbb{E}_{\mathbf{C}}(\mathrm{R})\big) \subseteq \mathbb{F}_0(\mathrm{X})$. Then $\psi_{\mathbf{C}}(\mathrm{X})$.*
6. *$\psi_{\mathbf{C}}(\mathrm{X})$ implies $T \cap \mathbb{F}(\mathrm{X}) \subseteq \mathbb{F}_0(\mathrm{X})$.*

Proof. It follows from Lemma 6.4 that

$$\mho\big(\mathbb{C}, U, (\mathbf{C}, \subseteq), \mathrm{age}(\mathrm{U}), (\mathfrak{J}(\mathbf{C}, \mathrm{U}, T), \preceq), (\mathrm{U}, T), \mathbb{I}, \mathrm{age}\big).$$

Hence item 1. follows from item 2. of Lemma 6.4.

Given a **C**-large subset S of U let $\psi_{\mathbf{C}}(\mathrm{X})$ if there is a finite subset F of $\langle \mathrm{H}_{\mathcal{T}} \rangle$ with $\psi_{\mathbf{C}}(\mathrm{X}, F)$. Let $\mathbb{E}(\mathrm{X}) = F$ for some finite set with $\psi_{\mathbf{C}}(\mathrm{X}, F)$.

Item 2.: It follows from item 3.(a) of Lemma 6.4 that there is a refinement $(\mathrm{R}, B) \in \mathfrak{J}(\mathbf{C}, U, T)$ of the element (U, T) with $\psi_{\mathbf{C}}(\mathrm{R}, B)$. Hence R is a refinement of U and $\psi_{\mathbf{C}}(\mathrm{R})$.

Item 3.: If $\psi_{\mathbf{C}}(\mathrm{X})$ then there is $(\mathrm{X}, F) \in \mathfrak{J}(\mathbf{C}, U, T)$ with $\psi_{\mathbf{C}}(\mathrm{X}, F)$. Then $X \cap S$ is infinite from Lemma 6.4 item 3.(b)

Item 4.: Let $F = \mathbb{E}_{\mathbf{C}}(\mathrm{X})$. Then $\psi_{\mathbf{C}}(\mathrm{X}, F)$ and $(\mathrm{X}, E) \in \mathfrak{J}(\mathbf{C}, U, T)$ and (X, E) is a refinement of (X, F). According to Lemma 6.4 item 3.(c) there is a **c**-restriction (R, G) of (X, E) with $\psi_{\mathbf{C}}(\mathrm{R}, G)$. This implies, according to the definition of **c**-restriction, that $(\mathbb{F}(\mathrm{R}) - \mathbb{F}(\mathrm{X})) \cap E \subseteq \mathbb{F}_0(\mathrm{X})$.

Item 5.: Then $\psi_{\mathbf{C}}(\mathrm{R}, \mathbb{E}_{\mathbf{C}}(\mathrm{R}))$ and $(\mathrm{X}, \mathbb{E}_{\mathbf{C}}(\mathrm{X}))$ is a refinement of $(\mathrm{R}, \mathbb{E}_{\mathbf{C}}(\mathrm{R}))$. Hence $\psi_{\mathbf{C}}((\mathrm{X}, \mathbb{E}_{\mathbf{C}}(\mathrm{X})))$, according to Lemma 6.4 item 3.(d), which in turn implies $\psi_{\mathbf{C}}(\mathrm{X})$.

□

7 Free binary homogeneous structures

Let σ be a binary relational signature and A a relational structure with signature σ. It follows that if two elements a and b of A are adjacent within some subset of $\langle \mathrm{A} \rangle$ then they are adjacent within all subsets of $\langle \mathrm{A} \rangle$ which contain a and b. Hence it is sufficient to say that a and b are adjacent without referring to a particular set. The relational structure A is complete if any two elements a and b of A are adjacent whenever $a \neq b$.

Theorem 7.1. *Let $\mathcal{T}$ be a free boundary set and $\mathrm{H}_{\mathcal{T}}$ the $\mathcal{T}$-free homogeneous structure in the binary signature σ. Let X be an orbit of $\mathrm{H}_{\mathcal{T}}$.*

Then $\mathrm{H}_{\mathcal{T}}|X$ is a free homogeneous structure. The free homogeneous structure $\mathrm{H}_{\mathcal{T}}|X$ is weakly indivisible.

Proof. Let $\mathrm{B} \in \mathrm{age}(\mathrm{H}_{\mathcal{T}}|X)$ and $b \in \langle \mathrm{B} \rangle$ and $\mathrm{A} = \mathrm{B} - b$ and $\mathrm{A} \in \mathrm{skel}(\mathrm{H}_{\mathcal{T}}|\mathrm{X})$. Let $\mathrm{B}' \in \mathrm{skel}(\mathrm{H}_{\mathcal{T}}|\mathrm{X})$ and so that there is an isomorphism f of B to B′. Let A′ be the image of A under f. There exists a local isomorphism h' and hence an automorphism h of $\mathrm{H}_{\mathcal{T}}$ which maps A′ to A and is the identity on $\mathbb{F}(\mathrm{X})$. (This is not necessarily the case if σ contains relational symbols of arity larger than two.) It follows that the image of B′ under h is an element of skel(X). The function $h \circ f$ is the identity on A. This implies that the function which is the identity on A and maps b to $h \circ f(b)$ is an embedding of B into $\mathrm{H}_{\mathcal{T}}|X$. It follows that the structure $\mathrm{H}_{\mathcal{T}}|X$ has the mapping extension property.

Let A and B be two compatible elements of $\mathrm{age}(\mathrm{H}_{\mathcal{T}}|X)$. Let A′ be the structure with $\langle \mathrm{A}' \rangle = \langle \mathrm{A} \rangle \cup \mathbb{F}(\mathrm{X})$ and so that $\mathrm{A}'|\langle \mathrm{A} \rangle = \mathrm{A}$ and there is an embedding of A′ into $\mathrm{H}_{\mathcal{T}}$ which is the identity on $\mathbb{F}(\mathrm{X})$ and maps A into X. Such a structure and embedding exist because of the mapping extension property of $\mathrm{H}_{\mathcal{T}}$ and because $\mathrm{A} \in age(\mathrm{X})$. Let B′ the structure with $\langle \mathrm{B}' \rangle = \langle \mathrm{B} \rangle \cup \mathbb{F}(\mathrm{X})$ and so that $\mathrm{B}'|\langle \mathrm{B} \rangle = \mathrm{B}$ and there is an embedding of B′ into $\mathrm{H}_{\mathcal{T}}$ which is the identity on $\mathbb{F}(\mathrm{X})$ and maps B into X.

The structures A′ and B′ are compatible. The restriction of $\mathrm{A}' \amalg \mathrm{B}'$ to $\langle \mathrm{A} \rangle \cup \langle \mathrm{B} \rangle$ is the free amalgam $\mathrm{A} \amalg \mathrm{B}$ of A and B. It follows that the age of X is freely amalgamable and hence according to Theorem 3.2 that bound(X) is a free boundary set.

Hence $\mathrm{H}_{\mathcal{T}}|X$ is a free homogeneous structure. We observed earlier that every two elements of X satisfy the same unary relations. The free homogeneous structure $\mathrm{H}_{\mathcal{T}}|X$ is weakly indivisible according to Theorem 4.1 item 5.

□

Let X and Y be two orbits with $X \cap Y \neq \emptyset$. Then $\mathrm{X} \cap \mathrm{Y}$ is the orbit with $\mathbb{F}(\mathrm{X} \cap \mathrm{Y}) = \mathbb{F}(\mathrm{X}) \cap \mathbb{F}(\mathrm{Y})$ and $\mathbb{I}(\mathrm{X} \cap \mathrm{Y}) = X \cap Y$. (This definition requires binary signature.)

Lemma 7.1. *Let $\mathcal{T}$ be a free boundary set and $\mathrm{H}_{\mathcal{T}}$ the $\mathcal{T}$-free homogeneous structure in the binary signature σ. Let* R *and* S *be compatible orbits of $\mathrm{H}_{\mathcal{T}}$.*

If no vertex in $\mathbb{F}(\mathrm{R}) - \mathbb{F}(\mathrm{S})$ is adjacent to a vertex in $\mathbb{F}(\mathrm{S}) - \mathbb{F}(\mathrm{R})$ then $R \cap S \neq \emptyset$.

Proof. Let $a \in R$ and $\mathrm{A} = \mathrm{H}_{\mathcal{T}}|(\mathbb{F}(\mathrm{R}) \cup \{a\})$. Let $b \in S$. Let B be the structure so that $\langle \mathrm{B} \rangle = \mathbb{F}(\mathrm{S}) \cup \{a\}$ and so that the function which is the identity on $\mathbb{F}(\mathrm{S})$ and maps b to a is an isomorphism of $\mathrm{H}_{\mathcal{T}}|(\mathbb{F}(\mathrm{S}) \cup \{b\})$ to B. The structures A and B are compatible because the orbits R and S are compatible.

The structure $\mathrm{A} \amalg \mathrm{B}$ is an element of $\mathrm{age}(\mathrm{H}_{\mathcal{T}})$ and hence, because of the mapping extension property of $\mathrm{H}_{\mathcal{T}}$, has an embedding h into $\mathrm{H}_{\mathcal{T}}$ so that h restricted to $\mathbb{F}(\mathrm{R}) \cup \mathbb{F}(\mathrm{S})$ is the identity map. Then $h(a) \in R \cap S$.

□

Lemma 7.2. *Let $\mathcal{T}$ be a free boundary set and $\mathrm{H}_{\mathcal{T}}$ the $\mathcal{T}$-free homogeneous structure in the binary signature σ. Let* R *and* S *be compatible orbits of $\mathrm{H}_{\mathcal{T}}$.*

If no vertex in $\mathbb{F}(\mathrm{R}) - \mathbb{F}(\mathrm{S})$ is adjacent to a vertex in $\mathbb{F}(\mathrm{S}) - \mathbb{F}(\mathrm{R})$ then the orbit $\mathrm{R} \cap \mathrm{S}$ exists and $\mathrm{age}(\mathrm{R} \cap \mathrm{S}) = \mathrm{age}(\mathrm{R}) \cap \mathrm{age}(\mathrm{S})$.

Proof. The orbit $\mathrm{R} \cap \mathrm{S}$ exists according to Lemma 7.1. Clearly $\mathrm{age}(\mathrm{R} \cap \mathrm{S}) \subseteq \mathrm{age}(\mathrm{R}) \cap \mathrm{age}(\mathrm{S})$.

Conversely, let A in $\mathrm{age}(\mathrm{R}) \cap \mathrm{age}(\mathrm{S})$. Let $\mathrm{A}_{\mathrm{R}} \in \mathrm{skel}(\mathrm{R})$ be isomorphic to A and $\mathrm{A}_{\mathrm{S}} \in \mathrm{skel}(\mathrm{S})$ isomorphic to A. Let $\mathrm{A}'_{\mathrm{R}} = \mathrm{H}_{\mathcal{T}}|(\langle \mathrm{A}_{\mathrm{R}} \rangle \cup \mathbb{F}(\mathrm{R}))$ and $\mathrm{A}'_{S} = \mathrm{H}_{\mathcal{T}}|(\langle \mathrm{A}_{\mathrm{S}} \rangle \cup \mathbb{F}(\mathrm{S}))$. Let f be an isomorphism of A_{R} to A_{S}.

Let A''_{R} be the structure with $\langle \mathrm{A}''_{R} \rangle = \mathrm{A}_{S} \cup \mathbb{F}(\mathrm{R})$ and so that the function which is the identity on $\mathbb{F}_{\mathrm{R}}$ and agrees with f on A_{R} is an isomorphism of A'_{R} to $\mathrm{A}''_{\mathrm{R}}$. The structures A''_{R} and A'_{S} are compatible because the orbits R and S are compatible. Because of the mapping extension property of $\mathrm{H}_{\mathcal{T}}$ there is an embedding of $\mathrm{A}''_{R} \amalg \mathrm{A}'_{\mathrm{S}}$ into $\mathrm{H}_{\mathcal{T}}$ which is the identity on $\mathbb{F}(\mathrm{R}) \cup \mathbb{F}(\mathrm{S})$ and hence maps A_{S} into $R \cap S$.

□

Lemma 7.3. *Let $\mathcal{T}$ be a free boundary set and $\mathrm{H}_{\mathcal{T}}$ the $\mathcal{T}$-free homogeneous structure in the binary signature σ. Let* X *and* Q *be orbits of $\mathrm{H}_{\mathcal{T}}$ so that* Q *is a refinement of $\mathrm{X}/(\mathbb{F}(\mathrm{X}) \cap \mathbb{F}(\mathrm{Q}))$ and every element of $\mathbb{F}(\mathrm{X}) - \mathbb{F}(\mathrm{Q})$ is an element of Q.*

Then the orbit $\mathrm{X} \cap \mathrm{Q}$ exists and is a refinement of X.

Proof. Let $\mathrm{R} = \mathrm{X}/(\mathbb{F}(\mathrm{X}) \cap \mathbb{F}(\mathrm{Q}))$ and $D = \mathbb{F}(\mathrm{X}) - \mathbb{F}(\mathrm{Q})$. Let $\mathrm{A} \in \mathrm{skel}(\mathrm{X})$ and $\mathrm{B} := \mathrm{H}_{\mathcal{T}}|(\langle \mathrm{A} \rangle \cup D)$.

Then $\mathrm{A} \in \mathrm{skel}(\mathrm{R})$. Also $D \subseteq R$ because $D \subseteq Q$ and Q is a continuation of R. It follows that $\mathrm{B} \in \mathrm{skel}(\mathrm{R})$

Hence $\mathrm{B} \in \mathrm{age}(\mathrm{Q})$ because Q is a refinement of R. The structure $\mathrm{H}_{\mathcal{T}}/Q$ is homogeneous according to Theorem 7.1. It follows from Theorem 4.1 item 3. that the identity map on D has an extension f to an embedding of B into Q. The embedding f maps A into $X \cap Q$ due to the definition of B and the definition of D. (Using for A a structure induced by a singleton element of X shows that $X \cap Q$ is not empty.)

Hence $\text{age}(\mathrm{X}) = \text{age}(\mathrm{H}_{\mathcal{T}}|(X \cap Q))$. ☐

Lemma 7.4. *Let $\mathcal{T}$ be a free boundary set and $\mathrm{H}_{\mathcal{T}}$ the $\mathcal{T}$-free homogeneous structure in the binary signature σ. Let X and Q be two orbits of $\mathrm{H}_{\mathcal{T}}$ so that the orbit Q is a refinement of $\mathrm{X}/(\mathbb{F}(\mathrm{X}) \cap \mathbb{F}(\mathrm{Q}))$ and every element $x \in \mathbb{F}(\mathrm{X}) - \mathbb{F}(\mathrm{Q})$ which is adjacent to an element in $\mathbb{F}(\mathrm{Q}) - \mathbb{F}(\mathrm{X})$ is an element of Q.*

Then the orbit $\mathrm{X} \cap \mathrm{Q}$ exists and is a refinement of X.

Proof. Let S be the set of elements in $\mathbb{F}(\mathrm{X}) - \mathbb{F}(\mathrm{Q})$ which are in Q and let T be the set of elements in $\mathbb{F}(\mathrm{X}) - \mathbb{F}(\mathrm{Q})$ which are not in Q and hence not adjacent to any element in $\mathbb{F}(\mathrm{Q}) - \mathbb{F}(\mathrm{X})$. Let $R = \mathbb{F}(\mathrm{X}) \cap \mathbb{F}(\mathrm{Q})$. Then (R, S, T) is a partition of $\mathbb{F}(\mathrm{X})$.

Let $\mathrm{X}' = \mathrm{X}/(R \cup S)$. The orbits X' for X and Q for Q satisfy the conditions of Lemma 7.3. Hence $\mathrm{X}' \cap \mathrm{Q}$ exists and is a refinement of X'.

The orbits X for R and $\mathrm{X}' \cap \mathrm{Q}$ for S satisfy the conditions of Lemma 7.2 because $\mathbb{F}(\mathrm{X}) - \mathbb{F}(\mathrm{X}' \cap \mathrm{Q}) = T$ and $\mathbb{F}(\mathrm{X}' \cap \mathrm{Q}) - \mathbb{F}(\mathrm{X}) = \mathbb{F}(\mathrm{Q}) - \mathbb{F}(\mathrm{X})$. Hence $\mathrm{X} \cap (\mathrm{X}' \cap \mathrm{Q}) = \mathrm{X} \cap \mathrm{Q}$ exists and $\text{age}(\mathrm{X} \cap \mathrm{Q}) = \text{age}(\mathrm{X} \cap (\mathrm{X}' \cap \mathrm{Q})) = \text{age}(\mathrm{X}) \cap \text{age}(\mathrm{X}' \cap \mathrm{Q}) = \text{age}(\mathrm{X}) \cap \text{age}(\mathrm{X}') = \text{age}(\mathrm{X})$.

☐

Lemma 7.5. *Let $\mathcal{T}$ be a free boundary set and $\mathrm{H}_{\mathcal{T}}$ the $\mathcal{T}$-free homogeneous structure in the binary signature σ. Let X, Q and P be orbits of $\mathrm{H}_{\mathcal{T}}$ so that:*

1. *$\mathrm{Q} \cap \mathrm{X}$ exists and is a refinement of X.*
2. *P is a continuation of Q.*
3. *If $x \in \mathbb{F}(\mathrm{X}) - \mathbb{F}(\mathrm{P})$ is adjacent to an element in $\mathbb{F}(\mathrm{P}) - \mathbb{F}(\mathrm{Q}) - \mathbb{F}(\mathrm{X})$ then $x \in P$.*
4. *P is a refinement of $\mathrm{X}/(\mathbb{F}(\mathrm{X}) \cap \mathbb{F}(\mathrm{P}))$.*

Then $\mathrm{X} \cap \mathrm{P}$ exists and is a refinement of X.

Proof.
Claim 1: *The orbit P is a refinement of $(\mathrm{Q} \cap \mathrm{X})/(\mathbb{F}(\mathrm{Q} \cap \mathrm{X}) \cap \mathbb{F}(\mathrm{P}))$.*

$\text{age}(\mathrm{P}) = \text{age}(\mathrm{X}/(\mathbb{F}(\mathrm{X})) \cap \mathbb{F}(\mathrm{P})) \subseteq \text{age}((\mathrm{Q} \cap \mathrm{X})/(\mathbb{F}(\mathrm{Q} \cap \mathrm{X}) \cap \mathbb{F}(\mathrm{P}))) \subseteq \text{age}(\mathrm{P})$.

Claim 2: *If $x \in \mathbb{F}(\mathrm{Q} \cap \mathrm{X}) - \mathbb{F}(\mathrm{P})$ is adjacent to an element in $\mathbb{F}(\mathrm{P}) - \mathbb{F}(\mathrm{Q} \cap \mathrm{X})$ then $x \in P$.*

We have $\mathbb{F}(\mathrm{Q} \cap \mathrm{X}) - \mathbb{F}(\mathrm{P}) = \mathbb{F}(\mathrm{X}) - \mathbb{F}(\mathrm{P})$ and $\mathbb{F}(\mathrm{P}) - \mathbb{F}(\mathrm{Q} \cap \mathrm{X}) = \mathbb{F}(\mathrm{P}) - \mathbb{F}(\mathrm{Q}) - \mathbb{F}(\mathrm{X})$.

It follows from Claim 1 and Claim 2 that $\mathrm{Q} \cap \mathrm{X}$ for X and P for Q satisfy the conditions of Lemma 7.4. Hence the orbit $(\mathrm{Q} \cap \mathrm{X}) \cap \mathrm{P} = \mathrm{X} \cap \mathrm{P}$ exists and is a refinement of $\mathrm{Q} \cap \mathrm{X}$ which in turn is a refinement of X.

☐

The sequence $\mathcal{Q} = (\mathrm{Q}_i; i \in n \in \omega)$ of orbits of $\mathrm{H}_\mathcal{T}$ is *decreasing* if Q_i is a continuation of Q_j for all $j \in i \in n$. If $\mathcal{Q} = (\mathrm{Q}_i; i \in n \in \omega)$ then $\mathbb{F}(\mathrm{Q}_{-1}) := \emptyset$.

Let X be an orbit of $\mathrm{H}_\mathcal{T}$. The sequence $\mathcal{Q} = (\mathrm{Q}_i; i \in n+1)$ is a *closed refinement sequence* of X if for every $i \in n+1$ and

$$D_i := \mathbb{F}(\mathrm{Q}_i) - \mathbb{F}(\mathrm{X}) - \mathbb{F}(\mathrm{Q}_{i-1}) :$$

1. The sequence $\mathcal{Q}$ is decreasing.
2. If $x \in \mathbb{F}(\mathrm{X}) - \mathbb{F}(\mathrm{Q}_i)$ is adjacent to an element in D_i then $x \in Q_i$.
3. Q_i is a refinement of $\mathrm{X}/(\mathbb{F}(\mathrm{X}) \cap \mathbb{F}(\mathrm{Q}_i))$.

The sequence $\mathcal{Q} = (\mathrm{Q}_i; i \in n+1)$ is an *open refinement sequence* of X if for every $i \in n+1$ and

$$D_i := \mathbb{F}(\mathrm{Q}_i) - \mathbb{F}(\mathrm{X}) - \mathbb{F}(\mathrm{Q}_{i-1}) :$$

1. The sequence $\mathcal{Q}$ is decreasing.
2. If $x \in \mathbb{F}(\mathrm{X}) - \mathbb{F}(\mathrm{Q}_i)$ is adjacent to an element in D_i then $x \in Q_i$.
3. If $i \neq n$ then Q_i is a refinement of $\mathrm{X}/(\mathbb{F}(\mathrm{X}) \cap \mathbb{F}(\mathrm{Q}_i))$.
4. $\big(\mathbb{F}(\mathrm{X}) - \mathbb{F}(\mathrm{Q}_{n-1})\big) = \big(\mathbb{F}(\mathrm{X}) - \mathbb{F}(\mathrm{Q}_n)\big)$.
5. No element in $\mathbb{F}(\mathrm{X}) - \mathbb{F}(\mathrm{Q}_{n-1})$ is adjacent to an element in $(\mathbb{F}(\mathrm{Q}_n) - \mathbb{F}(\mathrm{Q}_{n-1}))$.

The sequence $\mathcal{Q} = (\mathrm{Q}_i; i \in n+1)$ is a refinement sequence of X if it is a closed refinement sequence or an open refinement sequence. A refinement sequence $\mathcal{Q} = (\mathrm{Q}_i; i \in n+1)$ is *full* if the orbit Q_n is a continuation of the orbit X. The sequence $(\mathrm{Q}_i; i \in n)$ is an *initial segment* of the sequence $(Q_i; i \in m)$ if $n < m$. Note that every initial segment of a refinement sequence of an orbit X is a closed refinement sequence of X.

Let $\mathcal{Q} = (\mathrm{Q}_i; i \in n+1)$ be a refinement sequence of the orbit X. The *age* of $\mathcal{Q}$, $\mathrm{age}(\mathcal{Q})$, is the age of the orbit Q_n and $\mathbb{F}(\mathcal{Q}) := \mathbb{F}(Q_n)$ and $\mathbb{I}(\mathcal{Q}) := \mathbb{I}(\mathrm{Q}_n)$ and $\mathbb{J}(\mathcal{Q}) := \mathrm{X}$. The *top* of the refinement sequence $\mathcal{Q} = (\mathrm{Q}_i; i \in n+1)$ is the orbit Q_n.

Lemma 7.6. *Let* X *be an orbit of* $\mathrm{H}_\mathcal{T}$ *and* $\mathcal{Q} = (\mathrm{Q}_i; i \in n \in \omega)$ *a closed refinement sequence of* X.

Then $\mathrm{Q}_i \cap \mathrm{X}$ *exists and* $\mathrm{age}(\mathrm{Q}_i \cap \mathrm{X}) = \mathrm{age}(\mathrm{X})$ *for all* $i \in n$.

Proof. By induction on n using Lemma 7.5. If $n = 0$ the Lemma follows from Lemma 7.4.

□

Lemma 7.7. *Let* $\mathcal{Q} = (\mathrm{Q}_i; i \in n+1 \in \omega)$ *be an open refinement sequence of* X *with* $\mathrm{age}(\mathrm{X}) = \mathrm{age}(\mathrm{Q}_n)$. *Let* $\mathrm{P}_i = \mathrm{Q}_i$ *for all* $i \in n$ *and* $\mathrm{P}_n = \mathrm{Q}_n \cap \mathrm{X}$. *Then* $\mathcal{P} = (\mathrm{P}_i; i \in n+1)$ *is a full and closed refinement sequence of* X.

Proof. Let $\mathrm{Q}_{n-1} \cap \mathrm{X} := \mathrm{R}$. It follows from Lemma 7.6 that $\mathrm{age}(\mathrm{R}) = \mathrm{age}(\mathrm{X})$. The orbits R for R and Q_n for S satisfy the conditions of Lemma 7.2. It follows that the orbit $\mathrm{R} \cap \mathrm{Q}_n$ exists and $\mathrm{age}(\mathrm{R} \cap \mathrm{Q}_n) = \mathrm{age}(\mathrm{R}) \cap \mathrm{age}(\mathrm{Q}_n) = \mathrm{age}(\mathrm{X}) \cap \mathrm{age}(\mathrm{Q}_n) = \mathrm{age}(\mathrm{X})$.

The set $\mathbb{F}\big((\mathrm{Q}_{n-1} \cap \mathrm{X}) \cap \mathrm{Q}_n\big) = \mathbb{F}(\mathrm{X} \cap \mathrm{Q}_n)$ because of item 4. of the definition of open refinement sequence and because Q_n is a continuation of Q_{n-1}. This in turn, together with the fact that Q_n is a continuation of Q_{n-1}, implies that $(\mathrm{Q}_{n-1} \cap \mathrm{X}) \cap \mathrm{Q}_n = \mathrm{X} \cap \mathrm{Q}_n$.

Hence $\mathrm{P}_n = (\mathrm{X} \cap \mathrm{Q}_n)$ exists and $\mathrm{age}(\mathrm{P}_n) = \mathrm{age}(\mathrm{X} \cap \mathrm{Q}_n) = \mathrm{age}((\mathrm{Q}_{n-1} \cap \mathrm{X}) \cap \mathrm{Q}_n) = \mathrm{age}(\mathrm{R} \cap \mathrm{Q}_n) = \mathrm{age}(\mathrm{X})$. Hence P_n is a refinement of $\mathrm{X}/(\mathbb{F}(\mathrm{X}) \cap \mathbb{F}(\mathrm{P}_n)) = \mathrm{X}$ satisfying item 3. of the definition of closed refinement sequence in the case $i = n$. The other conditions for $\mathcal{P}$ to be a closed refinement sequence are easily checked.

□

The full and closed refinement sequence $\mathcal{P}$ constructed in Lemma 7.7 is the *closure* of the refinement sequence $\mathcal{Q}$ of X.

Corollary 7.1. *Let $\mathcal{Q}$ be an open refinement sequence for the orbit* X *with* $\mathrm{age}(\mathcal{Q}) = \mathrm{age}(\mathrm{X})$*. Then the closure of $\mathcal{Q}$ is a full and closed refinement sequence for* X*.*

Lemma 7.8. *Let $\mathcal{Q} = (Q_i; i \in n)$ be a full refinement sequence of the orbit* X*. Let Q_n be a continuation of* Q*. Then $\mathcal{Q}' = (Q_i; i \in n+1)$ is an open refinement sequence of* X*.*

Proof. The conditions for an open refinement sequence are easily checked. □

Lemma 7.9. *Let $\mathcal{Q} = (\mathrm{Q}_i; i \in n)$ be a refinement sequence of* X *and $k \in n$. Let $a \in Q_k - \mathbb{F}(\mathrm{Q}_{n-1}) - \mathbb{F}(\mathrm{X})$ and so that a is not adjacent to any vertex in $\mathbb{F}(\mathrm{Q}_{n-1}) - \mathbb{F}(\mathrm{Q}_k) - \mathbb{F}(\mathrm{X})$. Let* Y *be a continuation of* X *with $\mathbb{F}(\mathrm{Y}) = \mathbb{F}(\mathrm{X}) \cup \{a\}$.*

Then $\mathcal{Q}$ is a refinement sequence of Y*. The sequence $\mathcal{Q}$ is closed as a refinement sequence of* Y *if it is closed as a refinement sequence of* X *and open as a refinement sequence of* Y *if it is open as a refinement sequence of* X*.*

Proof. The conditions for $\mathcal{Q}$ to be a refinement sequence of Y are easily checked.
□

Let J be a finite subset of $\langle \mathrm{H}_{\mathcal{T}} \rangle$ and $\mathcal{Q} := (\mathrm{Q}_i; \in n)$ and $\mathcal{P} := (\mathrm{P}; i \in m)$ be decreasing sequences of orbits. The sequences $\mathcal{Q}$ and $\mathcal{P}$ are *branched over* J, if there is a number β so that $\mathrm{Q}_j = \mathrm{P}_j$ for all $j \in \beta$ and so that

$$\big(\mathbb{F}_*(\mathrm{Q}_{n-1}) - \mathbb{F}(\mathrm{Q}_{\beta-1}) - J\big) \cap \big(\mathbb{F}_*(\mathrm{P}_{m-1}) - \mathbb{F}(\mathrm{Q}_{\beta-1}) - J\big) = \emptyset.$$

The number β is the *branching number* of the branched pair $(\mathrm{Q}_i; i \in n)$ and $(\mathrm{P}_i; i \in n)$. It follows that the sequences $(\mathrm{Q}_i; i \in n)$ and $(\mathrm{Q}_i; i \in m)$ with $n \leq m$ are branched with branching number n.

The refinement sequence $\mathcal{Q} = (Q_i; i \in n)$ of X is *branched* with the refinement sequence $\mathcal{P} = (\mathrm{P}_i; i \in m))$ of Y if $\mathbb{F}(\mathrm{X}) = \mathbb{F}(\mathrm{Y})$ and the sequences $\mathcal{Q}$ and $\mathcal{P}$ are branched over $\mathbb{F}(\mathrm{X})$.

A set $\mathfrak{Q}$ of refinement sequences is a *branched set* of refinement sequences for the finite subset J of $\langle \mathrm{H}_\mathcal{T} \rangle$ if for all refinement sequences $\mathcal{P}$ and $\mathcal{Q}$ in $\mathfrak{Q}$:

1. $\mathcal{P}$ and $\mathcal{Q}$ are branched over J.
2. Every initial segment of $\mathcal{Q}$ is an element of $\mathfrak{Q}$.

Let $\mathfrak{Q}$ be a branched set of refinement sequences for J. Then $\mathbb{F}(\mathfrak{Q}) := \bigcup_{\mathcal{Q} \in \mathfrak{Q}} \mathbb{F}(\mathcal{Q})$.

Lemma 7.10. *Let $\mathfrak{Q}$ be a branched set of refinement sequences for J. Let $\mathcal{Q} \in \mathfrak{Q}$ and either $a \in \mathbb{I}(\mathcal{Q}) - \mathbb{F}(\mathfrak{Q}) - J$ and so that a is not adjacent to any element in $\mathbb{F}(\mathfrak{Q}) - \mathbb{F}(\mathcal{Q})$, or $a \in \langle \mathrm{H}_\mathcal{T} \rangle - \mathbb{F}(\mathfrak{Q}) - J$ and so that a is not adjacent to any element in $\mathbb{F}(\mathfrak{Q})$.*

Then $\mathfrak{Q}$ is a branched set of refinement sequences for $J \cup \{a\}$. The closed refinement sequences in $\mathfrak{Q}$ remain closed and the open refinement sequences remain open.

Proof. The Lemma follows in the first case from Lemma 7.9 and in the second case trivially. □

Lemma 7.11. *Let $\mathfrak{Q}$ be a branched set of refinement sequences over J and $\mathcal{Q} \in \mathfrak{Q}$ so that $\mathcal{Q}$ is a refinement sequence for the orbit* X *with $\mathbb{F}(\mathrm{X}) = J$ and so that* $\mathrm{age}(\mathrm{X}) = \mathrm{age}(\mathcal{Q})$. *Let $\mathcal{Q}'$ be the closure of the refinement sequence $\mathcal{Q}$ of* X.

Then $\mathfrak{Q} - \{\mathcal{Q}\} \cup \{\mathcal{Q}'\}$ is a branched set of refinement sequences for J.

Proof. Follows from Corollary 7.1. □

Lemma 7.12. *Let $\mathfrak{Q}$ be a branched set of refinement sequences over J and $\mathcal{Q} = (Q_i; i \in n) \in \mathfrak{Q}$ a full refinement sequence for the orbit* X *with $\mathbb{F}(\mathrm{X}) = J$. Let Q_n be a continuation of Q_{n-1} so that $\big(\mathbb{F}_*(\mathrm{Q}_n) - \mathbb{F}(Q_{n-1})\big) \cap \mathbb{F}(\mathfrak{Q}) = \emptyset$. Let $\mathcal{Q}' = (\mathrm{Q}_i; i \in n+1)$.*

Then $\mathcal{Q}'$ is an open refinement sequence and $\mathfrak{Q} \cup \mathcal{Q}'$ is a branching set of refinement sequences over J.

Proof. Follows from Lemma 7.11. □

8 Reducing sets and partitions

For this section σ is a binary relational language and $\mathcal{T}$ a free boundary set in signature σ with $\mathrm{H}_\mathcal{T}$ the corresponding $\mathcal{T}$-free homogeneous structure.

Let $(b_i; i \in \omega)$ be an enumeration of $\langle \mathrm{H}_\mathcal{T} \rangle$ into an ω sequence. For $a \in \langle \mathrm{H} \rangle$ let $\emptyset_a$ be the orbit with $\mathbb{F}(\emptyset_a) = \{y \in \langle \mathrm{H} \rangle \mid y < a\}$ and $a \in \mathbb{I}(\emptyset_a)$. Note that $\emptyset(a)$ depends on the enumeration of $\langle \mathrm{H}_\mathcal{T} \rangle$. An orbit X of $\mathrm{H}_\mathcal{T}$ is *n-initial* if $\mathbb{F}(\mathrm{X}) = \{b_i \mid i \in n\}$ and it is *initial* if it is n-initial for some $n \in \omega$.

Let $\mathbf{C}$ be a chain in $\mathbf{orb}(\mathrm{H}_\mathcal{T})$ and X an initial orbit with $\mathrm{age}(\mathrm{X}) \in \mathbf{C}$. The initial continuation Y of X with $\mathrm{age}(\mathrm{Y}) \in \mathbf{C}$ is $(\mathbf{C}, \mathrm{X})$-*maximal* if $\mathrm{X} \neq \mathrm{Y}$ and there is no initial continuation Z of X with $\mathrm{age}(\mathrm{Z}) \in \mathbf{C}$ and $\mathrm{X} \neq \mathrm{Z} \neq \mathrm{Y}$ so that Y is a continuation of Z. Let $N(\mathbf{C}, \mathrm{X})$ be the set of $(\mathbf{C}, \mathrm{X})$-maximal initial orbits. Let

$$M(\mathbf{C}, \mathrm{X}) := \bigcup_{\mathrm{X} \in \mathrm{cont}(\mathrm{Y})} N(\mathbf{C}, \mathrm{Y}).$$

The chain $\mathbf{C}$ is X-*finitary* if the set $\{\mathrm{age}(\mathrm{Y}) \mid \mathrm{Y} \in N(\mathbf{C}, \mathrm{X})\}$ is finite. The chain $\mathbf{C}$ is finitary if it is X-finitary for every initial orbit X with $\mathrm{age}(\mathrm{X}) \in \mathbf{C}$.

Lemma 8.1. *Let $\mathcal{T}$ be a free boundary set and $\mathrm{H}_\mathcal{T}$ the $\mathcal{T}$-free homogeneous structure in the binary signature σ. Let $\mathbf{C}$ be a chain of $(\mathbf{orb}(\mathrm{H}_\mathcal{T}); \subseteq)$ so that whenever $\mathbf{d} \in \mathbf{orb}(\mathrm{H}_\mathcal{T})$ and $\mathbf{c} \in \mathbf{C}$ with $\mathbf{c} \supseteq \mathbf{d}$ then $\mathbf{d} \in \mathbf{C}$.*

Then $\mathbf{C}$ is finitary.

Proof. It follows that if X is n-initial and Y is $(\mathbf{C}, \mathrm{X})$-maximal then Y is $n+1$-initial. Because σ is finite there are only finitely many continuations of X which are $n+1$-initial.

□

Corollary 8.1. *Let $\mathcal{T}$ be a free boundary set and $\mathrm{H}_\mathcal{T}$ the $\mathcal{T}$-free homogeneous structure in the binary signature σ. Let U be a unitary orbit of $\mathrm{H}_\mathcal{T}$ so that the partial order $(\mathbf{orb}(\mathrm{U}), \subseteq)$ is a chain $\mathbf{C}$.*

Then $\mathbf{C}$ is finitary. In particular if $(\mathbf{orb}(\mathrm{H}_\mathcal{T}); \subseteq)$ is a chain $\mathbf{C}$ then $\mathbf{C}$ is finitary.

Lemma 8.2. *Let $\mathcal{T}$ be a free boundary set and $\mathrm{H}_\mathcal{T}$ the $\mathcal{T}$-free homogeneous structure in the binary signature σ. If $\mathbf{orb}(\mathrm{H}_\mathcal{T})$ is finite then every chain of the partial order $(\mathbf{orb}(\mathrm{H}_\mathcal{T}); \subseteq)$ is finitary.*

Proof. Obvious. □

If $\mathbf{C}$ is a chain of the partial order $(\mathbf{orb}(\mathrm{H}_\mathcal{T}); \subseteq)$ then $\mathbf{C}(\mathrm{H}_\mathcal{T})$ is the set of elements $a \in \langle \mathrm{H}_\mathcal{T} \rangle$ so that $\mathrm{age}(\emptyset(a)) \in \mathbf{C}$.

Theorem 8.1. *Let $\mathcal{T}$ be a free boundary set and $\mathrm{H}_\mathcal{T}$ the $\mathcal{T}$-free homogeneous structure in the binary signature σ and let $\mathfrak{C}$ be a finite set of finitary chains of $(\mathbf{orb}(\mathrm{H}_\mathcal{T}); \subseteq)$.*

For every $\mathbf{C} \in \mathfrak{C}$ there is a unary orbit $\mathrm{U}_\mathbf{C}$ so that $\mathrm{age}(\mathrm{U}_\mathbf{C}) \in \mathbf{C}$ and a subset $S_\mathbf{C}$ of $\mathbf{C}(\mathrm{H}_\mathcal{T})$ so that if $\mathbf{C}, \mathbf{D} \in \mathfrak{C}$ with $\mathbf{C} \neq \mathbf{D}$ then $S_\mathbf{C} \cap S_\mathbf{D} = \emptyset$.

For every $\mathbf{C} \in \mathfrak{C}$ let $(S_{\mathbf{C},i}; i \in r \in \omega)$ be a partition of S_C.

Then there exists an embedding f of $\mathrm{H}_\mathcal{T}$ into $\mathrm{H}_\mathcal{T}$ and for every $\mathbf{C} \in \mathfrak{C}$ an $i_\mathbf{C} \in r$ so that $f[S_\mathbf{C}] \subseteq S_{\mathbf{C}, i_\mathbf{C}}$. That is, the set of subsets $\{S_\mathbf{C} \mid \mathbf{C} \in \mathfrak{C}\}$ is a reducing set of subsets of $\langle \mathrm{H}_\mathcal{T} \rangle$.

Proof. Let $\{b_i \mid i \in \omega\}$ an enumeration of $\mathrm{H}_\mathcal{T}$ into an ω sequence.

Let $\mathbf{C} \in \mathfrak{C}$ and $T_\mathbf{C}$ a finite subset of $U_\mathbf{C}$. Then according to Theorem 6.2 one of the sets $S_{\mathbf{C},i}$, say $S_{\mathbf{C},i_\mathbf{C}}$, is $\mathbf{C}$-large. We will prove that there exists an embedding f of $\mathrm{H}_\mathcal{T}$ into $\mathrm{H}_\mathcal{T}$ so that $f[S_\mathbf{C}] \subseteq S_{\mathbf{C}, i_\mathbf{C}}$.

We obtain, using item 2. of Theorem 6.2, for every $\mathbf{C} \in \mathfrak{C}$, a refinement $\mathrm{Q}_0^{\mathbf{C}}$ of $\mathrm{U}_{\mathbf{C}}$ with $\psi_{\mathbf{C}}(\mathrm{Q}_0^{\mathbf{C}})$ and so that $T_{\mathbf{C}} \cap \mathbb{F}(\mathrm{Q}_0^{\mathbf{C}}) \subseteq \mathbb{F}_0(\mathrm{Q}_0^{\mathbf{C}})$. We can ensure, by choosing the sets $T_{\mathbf{C}}$ appropriately along some ordering of $\mathfrak{C}$, that $\mathbb{F}(\mathrm{Q}_0^{\mathbf{C}}) \cap \mathbb{F}(\mathrm{Q}_0^{\mathbf{D}}) \subseteq \mathbb{F}_0(\mathrm{Q}_0^{\mathbf{C}})$ or $\mathbb{F}(\mathrm{Q}_0^{\mathbf{C}}) \cap \mathbb{F}(\mathrm{Q}_0^{\mathbf{D}}) \subseteq \mathbb{F}_0(\mathrm{Q}_0^{\mathbf{D}})$ if $\mathbf{C} \neq \mathbf{D}$. It follows that the set $\mathfrak{Q}_0 := \{(Q_0^{\mathbf{C}}) \mid \mathbf{C} \in \mathfrak{C}_0\}$ is a branched set of refinement sequences for $\emptyset$.

Note that the branched set $\mathfrak{Q}_0$ has the following property: If $\mathbf{C} \in \mathfrak{C}$ and X is an orbit with $\mathbb{F}(\mathrm{X}) = \emptyset$ and $\mathrm{age}(\mathrm{X}) \in \mathbf{C}$ there exists a full refinement sequence $\mathcal{Q} \in \mathfrak{Q}_0$ of X with top Q so that $\psi_{\mathbf{C}}(\mathrm{Q})$. (The orbits X with $\mathbb{F}(\mathrm{X}) = \emptyset$ being the unitary orbits or $\mathrm{H}_{\mathcal{T}}$.)

Let $J = \{a_i \mid i \in n \in \omega\}$ be a subset of $\langle \mathrm{H}_{\mathcal{T}} \rangle$ so that the function g with $g(a_i) = b_i$ for all $i \in n$ is a local isomorphism. If X is an orbit with $\mathbb{F}(\mathrm{X}) = J$ then $g(\mathrm{X})$ is the orbit with $\mathbb{F}(g(\mathrm{X})) = \{b_i \mid i \in n\}$ and $\mathbb{I}(g(\mathrm{X})) = g'[X]$ where g' is an extension of g to an automorphism of $\mathrm{H}_{\mathcal{T}}$. It follows that $\mathrm{age}(\mathrm{X}) = \mathrm{age}(g(\mathrm{X}))$.

An initial configuration of length $n \in \omega$ is a branched set $\mathfrak{Q}$ of refinement sequences over a set $J = \{a_i \mid i \in n\}$ so that:

1. The function f_n so that $f_n(b_i) = a_i$ for all $i \in n$ is a local isomorphism of $\mathrm{H}_{\mathcal{T}}$ with f_m the restriction of f_n to $\{a_i \mid i \in m\}$.
2. If $i \in n$ and $b_i \in S_{\mathbf{C}}$ then $f_n(b_i) \in S_{\mathbf{C}, i_{\mathbf{C}}}$.
3. For every orbit X and every chain $\mathbf{C} \in \mathfrak{C}$ with $\mathbb{F}(\mathrm{X}) = J$ and $\mathrm{age}(\mathrm{X}) \in \mathbf{C}$ there exists a full and closed refinement sequence $\mathcal{Q} \in \mathfrak{Q}$ of X with top Q so that $\psi_{\mathbf{C}}(\mathrm{Q})$.
4. The set $\mathfrak{Q}$ is $n-1$ complete.

Where $\mathfrak{Q}$ is *m-complete* for $m \in n$ if for every $\mathbf{C} \in \mathfrak{C}$ and every orbit X with $\mathbb{F}(\mathrm{X}) = \{a_i \mid i \in m\}$ and $\mathrm{Z} \in N(\mathbf{C}, f_m(\mathrm{X}))$ there is an open refinement sequence $\mathcal{Q}_{\mathbf{C},\mathrm{Z}} \in \mathfrak{Q}$ of X with top $\mathrm{Q}_{\mathbf{C},\mathrm{Z}}$ so that:

1. $\mathrm{age}(\mathrm{Z}) = \mathrm{age}(\mathcal{Q}_{\mathbf{C},Z}) = \mathrm{age}(\mathrm{Q}_{\mathbf{C},Z})$.
2. $\psi_{\mathbf{C}}(\mathrm{Q}_{\mathbf{C},\mathrm{Z}})$.
3. If $a \in J - \mathbb{F}(\mathrm{Q}_{\mathbf{C},\mathrm{Z}})$ then $a \notin \mathbb{E}_{\mathbf{C}}(\mathrm{Q}_{\mathbf{C},\mathrm{Z}})$.

The set $\mathfrak{Q}$ is *complete* if it is n-complete. Let $\mathbf{C} \in \mathfrak{C}$ and let X be an orbit with $\mathbb{F}(\mathrm{X}) = \{a_i \mid i \in m\}$ and $\mathrm{Z} \in N(\mathbf{C}, f_m(\mathrm{X}))$. The orbit Z is $(\mathbf{C}, \mathrm{X})$-*complete* if there is an open refinement sequence $\mathcal{Q}_{\mathbf{C},\mathrm{Z}} \in \mathfrak{Q}$ of X with top $\mathrm{Q}_{\mathbf{C},\mathrm{Z}}$ satisfying items 1. 2. and 3. above. If every $\mathrm{Z} \in N(\mathbf{C}, f_m(\mathrm{X}))$ is complete then the pair $(\mathrm{X}, \mathbf{C})$ is complete. It follows that if, for every $m \leq n$, every pair$(\mathrm{X}, \mathbf{C})$, with $\mathbb{F}(\mathrm{X}) = \{a_i \mid i \in m\}$ and $\mathrm{age}(\mathrm{X}) \in \mathbf{C}$ and $\mathbf{C} \in \mathfrak{C}$, is complete, then $\mathfrak{Q}$ is complete.

Let $\mathfrak{Q}$ be an initial configuration over a set J. Then $\mathbb{E}(\mathfrak{Q})$ is the union of $\mathbb{F}(\mathfrak{Q})$ with the union of all $\mathbb{E}_{\mathbf{C}}(\mathrm{Q})$ where $\mathbf{C} \in \mathfrak{C}$ and Q is a top of one of the refinement sequences in $\mathfrak{Q}$ so that $\psi_{\mathbf{C}}(\mathrm{Q})$.

Note that the branched set $\mathfrak{Q}_0$ is an initial configuration of length 0.

Claim 1: *Every initial configuration* $\mathfrak{Q}$ *over* J *of length* n *can be extended to a complete initial configuration* $\mathfrak{Q}'$ *over* J *of length* n.

Because $\mathfrak{Q}$ is $n-1$-complete we have only to complete orbits X with $\mathbb{F}(\mathrm{X}) = \{b_i \mid i \in n\} = J$. Let X be such an orbit and $\mathbf{C} \in \mathfrak{C}$ with $\mathrm{age}(\mathrm{X}) \in \mathbf{C}$. Let $\mathrm{Z} \in N(\mathbf{C}, f_n(\mathrm{X}))$ so that Z is not complete.

Let $\mathcal{Q}$ with top Q be the full refinement sequence of X given by item 3. of the definition of initial configuration when prompted with X for X and $\mathbf{C}$ for $\mathbf{C}$. Let $E = \mathbb{F}(\mathfrak{Q}) \cup \mathbb{E}_{\mathbf{C}}(\mathrm{Q})$ and $\mathbf{c} = \mathrm{age}(\mathrm{Z})$.

Then there exists, according to Lemma 6.4, a $\mathbf{c}$-restriction $\mathrm{Q}_{\mathbf{C},\mathrm{Z}}$ of Q with $\big(\mathbb{F}_*(\mathrm{Q}_{\mathbf{C},\mathrm{Z}}) - \mathbb{F}(Q)\big) \cap E = \emptyset$ and $\psi_{\mathbf{C}}(\mathrm{Q}_{\mathbf{C},\mathrm{Z}})$. It follows from Lemma 7.12 that extending $\mathcal{Q}$ by $\mathrm{Q}_{\mathbf{C},\mathrm{Z}}$ to $\mathcal{Q}'$ yields a branching set $\mathfrak{Q}'$. The orbit Z is $(\mathbf{C}, \mathrm{X})$-complete. The set $\mathfrak{Q}'$ is an initial configuration over J.

Completing the initial configuration $\mathfrak{Q}$ is a finite process because $\mathfrak{C}$ is finite and the set of orbits X with $\mathbb{F}(\mathrm{X}) = J$ is finite and the sets $N(\mathbf{C}, f_n(\mathrm{X}))$ are finite.

Claim 2: *Every complete initial configuration* $\mathfrak{Q}$ *over* J *of length* n *can be extended to an initial configuration* $\mathfrak{Q}'$ *over* $J \cup \{a_n\}$ *of length* $n+1$ *and suitable element* $a_n \in \langle \mathrm{H}_{\mathcal{T}} \rangle - J$.

Let V be the orbit with $\mathbb{F}(\mathrm{V}) = J$ and $b_n \in \mathbb{I}(f_n(\mathrm{V}))$. Then $\emptyset(b_n) = f_n(\mathrm{V})$. If there is a chain $\mathbf{D} \in \mathfrak{C}$ so that $b_n \in S_{\mathbf{D}}$ then this chain $\mathbf{D}$ is unique. In this case $\mathrm{age}(\mathrm{V}) = \mathrm{age}(f_n(\mathrm{V})) = \mathrm{age}(\emptyset(b_n)) \in \mathbf{C}$. Item 3. of the definition of initial configuration with V for X and $\mathbf{D}$ for $\mathbf{C}$ produces a full refinement sequence $\mathbf{Q} \in \mathfrak{Q}$ of V with top Q for which $\psi_{\mathbf{D}}(\mathrm{Q})$.

Let R be the continuation of Q with $\mathbb{F}(\mathrm{R}) = \mathbb{F}(\mathfrak{Q})$ and $\mathbb{F}(\mathrm{R}) - \mathbb{F}(\mathrm{Q}) \subseteq \mathbb{F}_0(\mathrm{R})$. (Note $J \subseteq \mathbb{F}(\mathrm{R})$.) It follows from Theorem 6.2 item 5. that $\psi_{\mathbf{D}}(\mathrm{R})$. According to Theorem 6.2 item 3. there are infinitely many elements of $S_{\mathbf{D},i_{\mathbf{D}}}$ in R.

Let $a_n \in R - \mathbb{E}(\mathfrak{Q})$. If there is no chain $\mathbf{D} \in \mathfrak{C}$ so that $b_n \in S_{\mathbf{D}}$ let a_n be an element in $V - \mathbb{E}(\mathfrak{Q})$ and so that it is not adjacent to any element in $\mathbb{E}(\mathfrak{Q}) - J$. In either case it follows from Lemma 7.10 that $\mathfrak{Q}$ is a branched set over the set $J \cup \{a_n\}$.

Item 1. of the definition of initial configuration follows because $a_n \in V$ and item 2. because of the choice of a_n in the first case. In order to satisfy item 3. we will close the appropriate refinement sequences in $\mathfrak{Q}$ using Lemma 7.7 and Lemma 7.11. This of course will not change the validity of items 1. and 2.

Let f_{n+1} be the extension of f_n to $\{a_i \mid i \in n+1\}$ so that $f_{n+1}(a_n) = b_n$.

Let X be an orbit with $\mathbb{F}(\mathrm{X}) = J \cup \{a_n\}$ and $\mathbf{C} \in \mathfrak{C}$ with $\mathrm{age}(\mathrm{X}) \in \mathbf{C}$. There is a largest number $m \in n+1$ so that $\mathrm{age}(\mathrm{X}/\{a_i \mid i \in m\}) \in \mathbf{C}$. The orbit $f_{n+1}(\mathrm{X})$ is an initial continuation of the initial orbit $f_{n+1}(\mathrm{X})/\{b_i \mid i \in m\}$. The orbit $f_{n+1}(\mathrm{X})$ is $(\mathbf{C}, f_{n+1}(\mathrm{X})/\{b_i \mid i \in m\})$-maximal because of the maximality of m. Let $\mathrm{Z} = f_{n+1}(\mathrm{X})$ then $\mathrm{Z} \in N(\mathbf{C}, f_{n+1}(\mathrm{X})/\{b_i \mid i \in m\})$.

Because $\mathfrak{Q}$ is complete there is an open refinement sequence $\mathcal{Q}_{\mathbf{C},\mathrm{Z}} \in \mathfrak{Q}$ of X with top $\mathrm{Q}_{\mathbf{C},\mathrm{Z}}$ so that $\mathrm{age}(\mathrm{Z}) = \mathrm{age}(\mathrm{Q}_{\mathbf{C},\mathrm{Z}})$ and $\psi_{\mathbf{C}}(Q_{\mathbf{C},\mathrm{Z}})$ and if $a \in J - \mathbb{F}(\mathrm{Q}_{\mathbf{C},\mathrm{Z}})$ then $a \notin \mathbb{E}_{\mathbf{C}}(\mathrm{Q}_{\mathbf{C},\mathrm{Z}})$. It follows from Lemma 7.7 that the closure of the refinement sequence exists and is a full and closed refinement sequence of X. It follows from

Lemma 7.11 that replacing in $\mathfrak{Q}$ the refinement sequence $\mathbf{Q}$ with the closure of $\mathbf{Q}$ leads again to a branched set of refinement sequences for $J \cup \{a\}$.

The top of the closure of $\mathcal{Q}$ is the orbit $\mathrm{P} = \mathrm{X} \cap Q_{\mathbf{C},\mathrm{Z}}$. It follows from Lemma 7.7 and the definition of full refinement sequence that P is a refinement of X. Hence P is a refinement of $\mathrm{Q}_{\mathbf{C},\mathrm{Z}}$ because $\mathrm{age}(\mathrm{X}) = \mathrm{age}(\mathrm{Z}) = \mathrm{age}(\mathrm{Q}_{\mathbf{C},\mathrm{Z}})$.

In order to satisfy item 3. of the definition of initial configuration we have to prove that $\psi_{\mathbf{C}}(\mathrm{P})$. Using Theorem 6.2 item 5. with P for X and $\mathrm{Q}_{\mathbf{C},\mathrm{Z}}$ for R we obtain $\psi_{\mathbf{C}}(\mathrm{P})$. The condition $\big(\mathbb{F}(\mathrm{P}) - \mathbb{F}(\mathrm{Q}_{\mathbf{C},\mathrm{Z}})\big) \cap \mathbb{E}_{\mathbf{C}}(\mathrm{P}) \supseteq \mathbb{F}_0(\mathrm{P})$ is satisfied because if $a \in J - \mathbb{F}(\mathrm{Q}_{\mathbf{C},\mathrm{Z}})$ then $a \notin \mathbb{E}_{\mathbf{C}}(\mathrm{Q}_{\mathbf{C},\mathrm{Z}})$.

We close, step by step, all of the appropriate open refinement sequences and obtain the initial configuration $\mathfrak{Q}'$ over $J \cup \{a_n\}$. The configuration $\mathfrak{Q}'$ is $n+1-1$-complete because the configuration $\mathfrak{Q}$ is n-complete.

Finally we obtain the required embedding f as the union of the f_n with $n \in \omega$.

□

Theorem 8.2. *Let $\mathcal{T}$ be a free boundary set and $\mathrm{H}_{\mathcal{T}}$ the $\mathcal{T}$-free homogeneous structure in the binary signature σ so that $\mathbf{orb}(\mathrm{H}_{\mathcal{T}})$ is finite. Let $\mathfrak{C}$ be a set of maximal chains of $(\mathbf{orb}(\mathrm{H}_{\mathcal{T}}); \subseteq)$ with $\bigcup \mathfrak{C} = \mathbf{orb}(\mathrm{H}_{\mathcal{T}})$ and so that $|\mathfrak{C}|$ is minimal under those conditions.*

Let $\mathcal{Q} := \{S_{\mathbf{C}} \mid \mathbf{C} \in \mathfrak{C}\}$ be a partition of $\langle \mathrm{H}_{\mathcal{T}} \rangle$ into finitely many classes so that $S_{\mathbf{C}} \subseteq \mathbf{C}(\mathrm{H}_{\mathcal{T}})$ for every $\mathbf{C} \in \mathfrak{C}$.

Then $\mathcal{Q}$ is a canonical partition of $\mathrm{H}_{\mathcal{T}}$.

Proof. It follows from Lemma 8.2 and Theorem 8.1 that $\mathcal{Q}$ is a reducing partition of $\langle \mathrm{H}_{\mathcal{T}} \rangle$. If $\mathbf{c}_0, \mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_{n-1}$ is a largest anti-chain of $(\mathbf{orb}(\mathrm{H}_{\mathcal{T}}); \subseteq)$ then $|\mathfrak{C}| = n$ according to Dilworth Theorem, see [11]. Because every orbit of $\mathrm{H}_{\mathcal{T}}$ is age-indivisible according to Theorem 4.1, Theorem 5.1 applies saying that there exists a persistent partition of of $\mathrm{H}_{\mathcal{T}}$ into n classes.

Hence, the partition $\mathcal{Q}$ of $\langle \mathrm{H}_{\mathcal{T}} \rangle$ is a canonical partition of $\mathrm{H}_{\mathcal{T}}$ according to Theorem 2.1.

□

Theorem 8.3. *Let $\mathcal{T}$ be a free boundary set and $\mathrm{H}_{\mathcal{T}}$ the $\mathcal{T}$-free homogeneous structure in the binary signature σ so that the partial order $(\mathbf{orb}\mathrm{U}, \subseteq)$ is a chain for every unary orbit U of $\mathrm{H}_{\mathcal{T}}$.*

Then the partition of $\mathrm{H}_{\mathcal{T}}$ into unary sets is a canonical partition.

Proof. If the partial order $(\mathbf{orb}\mathrm{U}, \subseteq)$ is a chain for every unary orbit U of $\mathrm{H}_{\mathcal{T}}$ then $(\mathbf{orb}(\mathrm{H}_{\mathcal{T}}); \subseteq)$ consists of disjoint chains, one for every unary set. Those chains are finitary according to Corollary 8.1. If $\mathbf{C}$ is the chain containing the unary orbit U then $\mathbf{C}(\mathrm{H}_{\mathcal{T}}) = U$. Hence, the partition of $\langle \mathrm{H}_{\mathcal{T}} \rangle$ into unary sets is a reducing partition of $\langle \mathrm{H}_{\mathcal{T}} \rangle$, according to Theorem 8.1.

The partial order $\mathbf{orb}(\mathrm{H}_{\mathcal{T}})$ has an anti-chain the size of the number, say k, of unary subsets of $\langle \mathrm{H}_{\mathcal{T}} \rangle$. Hence $\mathrm{H}_{\mathcal{T}}$ has a persistent partition into k classes according to Theorem 5.1.

The partition into unary sets is a canonical partition according to Theorem 2.1.

□

Corollary 8.2. *Let $\mathcal{T}$ be a free boundary set and $\mathrm{H}_{\mathcal{T}}$ the $\mathcal{T}$-free homogeneous structure in the binary signature σ so that the partial order $(\mathbf{orb}\mathrm{H}_{\mathcal{T}}, \subseteq)$ is a chain.*

Then $\mathrm{H}_{\mathcal{T}}$ is indivisible.

If $\mathrm{H}_{\mathcal{T}}$ is indivisible then the partial order $(\mathbf{orb}(\mathrm{H}_{\mathcal{T}}); \subseteq)$ is a chain.

Proof. If the partial order $(\mathbf{orb}(\mathrm{H}_{\mathcal{T}}); \subseteq)$ is a chain the $\langle \mathrm{H}_{\mathcal{T}} \rangle$ is a unary set. Hence $\mathrm{H}_{\mathcal{T}}$ has a canonical partition with a single class according to Theorem 8.3 and hence is indivisible.

If $(\mathbf{orb}(\mathrm{H}_{\mathcal{T}}); \subseteq)$ has an anti-chain with two elements then it has a persistent sequence of two elements according to Theorem 5.1 which implies that $\mathrm{H}_{\mathcal{T}}$ is not indivisible.

□

9 Bounds

Let σ be a relational signature which does not contain the symbols $\leq$ or $<$. Let A be a relational structure in signature σ and $\leq$ a total order on A. We denote by $\mathrm{A}_{\leq}$ the expansion of A by the additional relation $\leq$. The signature of $\mathrm{A}_{\leq}$ is then $\sigma \cup \{\leq\}$.

Let T be a finite relational structure with n elements and signature σ. Then ord(T) denotes the set of all expansions of T to a structure of signature $\sigma \cup \{\leq\}$ in which $\leq$ is a total order. Hence $|\mathrm{ord}(\mathrm{T})| = n!$. If $\mathcal{T}$ is a set of structures with signature σ then $\mathrm{ord}(\mathcal{T})$ is the union of ord(T) over all $\mathrm{T} \in \mathcal{T}$.

Let $\mathrm{T}_{\leq} \in \mathrm{ord}(\mathrm{T})$. For t an element of $\mathrm{T}_{\leq}$ denote by $< t$ the set of elements of $\mathrm{T}_{\leq}$ which are strictly below t in the total order $\leq$, by $t \leq$ the set of elements larger than or equal to t and so on. We will use induced substructures of T of the form $\mathrm{T}|t \leq$ and $\mathrm{T}| \leq t$ and so on.

The element t of $\mathrm{T}_{\leq}$ is a *break point* of $\mathrm{T}_{\leq}$ if:

1. The set $t \leq$ contains at least two elements.

2. If $x, y \in t \leq$ then x and y are of the same type with respect to the set $< t$.

Note that if σ does not contain unary relations and $\mathrm{T}_{\leq}$ contains at least two elements then the smallest element of $\mathrm{T}_{\leq}$ is always a break point. The smallest element of $\mathrm{T}_{\leq}$ is a *trivial break point*.

Let $\mathcal{T}$ be a set of finite structures with signature σ and let U be a general unary relation of σ. Let $\mathrm{cut}_U(\mathcal{T})$ be the set of functions β so that:

1. The domain of β contains all elements $\mathrm{T}_{\leq} \in \mathrm{ord}(\mathcal{T})$ for which every $x \in \langle \mathrm{T}_{\leq} \rangle$ is an element of U and is contained in all $\mathrm{T}_{\leq} \in \mathrm{ord}(\mathcal{T})$ which contain a break point which is in the relation U.

2. If $T_\leq$ is an element in the domain of β then $\beta(T_\leq)$ is a break point t of $T_\leq$ with $t \in U$.

Let $\mathrm{cut}(\mathcal{T})$ be the union of the sets $\mathrm{cut}_U(\mathcal{T})$ over all general unary relations U of σ. For $\beta \in \mathrm{cut}(\mathcal{T})$ let $\mathrm{broken}_u(\beta)$ be the set of all σ-structures R for which there exist

1. a structure $T_\leq \in \mathrm{ord}(\mathcal{T})$,
2. a break point t of $T_\leq$ with $t \leq \beta(T_\leq)$,
3. a structure $S_\leq \in \mathrm{ord}(\mathcal{T})$ and a break point s of $S_\leq$,
4. an order preserving embedding f of $S| \leq s$ into $T_\leq$ so that $f(s) = t$,

so that $R = S|s \leq$. Note that the elements of $\mathrm{broken}_u(\beta)$ have signature σ and not $\sigma \cup \{\leq\}$.

For $\beta \in \mathrm{cut}(\mathcal{T})$ let $\mathrm{broken}_d(\beta)$ be the set of all σ-structures R for which there are

1. a structure $T \in \mathcal{T}$,
2. a total order $\leq$ of $\langle T \rangle$ to an expansion $T_\leq \in \mathrm{domain}\beta$,
3. a break point t of $T_\leq$ with $t \leq \beta(T_\leq)$

so that $R = T| \leq t$. If $R = T| \leq t$ is an element of $\mathrm{broken}_d\beta$ then t is the *root* of R.

Let $\mathcal{T}$ be a set of finite complete structures in signature σ and $H_\mathcal{T}$ the corresponding $\mathcal{T}$-free homogeneous structure. Let X be an orbit of $H_\mathcal{T}$ which is a continuation of the unary orbit U. We will associate with X a function $\beta_X \in \mathrm{cut}_U(\mathcal{T})$.

Let $T_\leq \in \mathrm{ord}(\mathcal{T})$ and $x \in X$. If there is a break point t of $T_\leq$ and an embedding f of $T|(\leq t)$ into $H_\mathcal{T}|(\mathbb{F}(X) \cup \{x\})$ with $f(t) = x$ then $T_\leq \in \mathrm{domain}(\beta_X)$. In this case $\beta_X(T_\leq)$ is the largest break point t of $T_\leq$ so that there is an embedding f of $T|(\leq t)$ into $H|(\mathbb{F}(X) \cup \{x\})$ with $f(t) = x$. Note that β_X does not depend on the particular element $x \in X$ and that $\beta_X \in \mathrm{cut}_U(\mathcal{T})$.

Lemma 9.1. *Let $\mathcal{T}$ be a set of finite complete σ-structures and $H_\mathcal{T}$ the corresponding $\mathcal{T}$-free homogeneous structure. Let X be an orbit of $H_\mathcal{T}$. Then*

$$\mathrm{age}(X) \supseteq \mathrm{Forb}(\mathrm{broken}_u(\beta_X)).$$

Proof. Let $A \in \mathrm{Forb}(\mathrm{broken}_u(\beta_X))$ and assume without loss, that $\langle A \rangle \cap \langle H_\mathcal{T} \rangle = \emptyset$. Let B be the σ-structure with $\langle B \rangle = \mathbb{F}(X) \cup \langle A \rangle$ and so that B restricted to $\langle A \rangle$ = A and B restricted to $\mathbb{F}(X)$ equals $H_\mathcal{T}$ restricted to $\mathbb{F}(X)$ and if $a \in \langle A \rangle$ then the function which is the identity on $\mathbb{F}(X)$ and maps a into X is an embedding of B restricted to $\{a\} \cup \mathbb{F}(X)$ into $H_\mathcal{T}$.

If $A \notin \mathrm{age}(X)$ then the identity map on $\mathbb{F}(X)$ does not have an extension to an embedding of B into $H_\mathcal{T}$. Hence $B \notin \mathrm{age}(H_\mathcal{T})$ which implies that there is

$\mathrm{T} \in \mathcal{T}$ and an embedding f of T into B. Let $\leq$ be a total order on T to the ordered structure $\mathrm{T}_{\leq}$ so that the set of elements T_u of T which are mapped by f into A is above the set of elements T_d of T which are mapped by f into $\mathbb{F}(\mathrm{X})$. Let t be the smallest element in T_u. Then t is a break point of $\mathrm{T}_{\leq}$.

It follows that $\mathrm{T}|T_u \in \mathrm{broken}_u(\beta_{\mathrm{X}})$ contradicting $\mathrm{A} \in \mathrm{Forb}(\mathrm{broken}_u(\beta_{\mathrm{X}}))$ because f embeds $\mathrm{T}|T_u$ into A. □

Lemma 9.2. *Let σ be a binary relational signature and $\mathcal{T}$ a set of finite complete σ-structures and $\mathrm{H}_{\mathcal{T}}$ the corresponding $\mathcal{T}$-free homogeneous structure. Let* X *be an orbit of* $\mathrm{H}_{\mathcal{T}}$. *Then*

$$\mathrm{age}(\mathrm{X}) = \mathrm{Forb}(\mathrm{broken}_u(\beta_{\mathrm{X}})).$$

Proof. Using Lemma 9.1 we have to proof

$$\mathrm{age}(\mathrm{X}) \subseteq \mathrm{Forb}(\mathrm{broken}_u(\beta_{\mathrm{X}})).$$

Let $\mathrm{A} \in \mathrm{skel}(\mathrm{X})$ and assume for a contradiction that $\mathrm{A} \notin \mathrm{Forb}(\mathrm{broken}_u(\beta_{\mathrm{X}}))$. Then there exists $\mathrm{T}_{\leq} \in \mathrm{domain}(\beta_{\mathrm{X}})$ and a break point $t \leq \beta_{\mathrm{X}}(\mathrm{T}_{\leq})$ and an embedding g of $\mathrm{T}|t \leq$ into A and for $x \in X$ an embedding f of $\mathrm{T}| \leq t$ into $\mathrm{H}_{\mathcal{T}}|(\mathbb{F}(\mathrm{X}) \cup \{x\})$ with $f(t) = x$. Let f' be the restriction of f to $< t$. We arrived at a contradiction because the function $g \cup f'$ is an embedding of T into $\mathrm{H}_{\mathcal{T}}$. □

The element $\beta \in \mathrm{cut}(\mathcal{T})$ is *finite* if there are only finitely many isomorphism types in the set $\mathrm{broken}_d(\beta)$, that is $\mathrm{broken}_d(\beta)$ is finite.

Lemma 9.3. *Let $\mathcal{T}$ be a free boundary set in signature σ and let* X *be an orbit of the homogeneous structure $\mathrm{H}_{\mathcal{T}}$. Then $\beta_{\mathrm{X}} \in \mathrm{ord}(\mathcal{T})$ is finite and there exists a general unary relation U on σ so that all of the roots of the elements in* $\mathrm{broken}_d(\beta)$ *are in the relation* U*; that is $\beta_{\mathrm{X}} \in \mathrm{cut}_U(\mathcal{T})$ and β_{X} is finite.*

Proof. The structure $\mathrm{T}| < t$ has an embedding into the finite structure $\mathrm{H}_{\mathcal{T}}|\mathbb{F}(\mathrm{X})$ for every break point $t \leq \beta_{\mathrm{X}}(\mathrm{T}_{\leq})$ of $\mathrm{T}_{\leq}$. (Let $\mathrm{A} = \mathrm{T}| < t$. There are only finitely many structures of the form $\mathrm{T}'| \leq t'$ so that $\mathrm{T}'| < t$ is isomorphic to A.)

If X is an orbit of $\mathrm{H}_{\mathcal{T}}$ then there is a unary subset U of $\langle \mathrm{H}_{\mathcal{T}} \rangle$ so that $X \subseteq U$. □

Lemma 9.4. *Let $\mathcal{T}$ be a free boundary set and $\beta \in \mathrm{cut}(\mathcal{T})$. Then no element of $\mathcal{T}$ can be embedded into an element of* $\mathrm{broken}_d(\beta)$.

Proof. The Lemma will follow if no element $\mathrm{T} \in \mathcal{T}$ is equal to $\mathrm{T}| \leq t$ for some break point t because the elements of T can pairwise not be embedded into each other. But T equal to $\mathrm{T}| \leq t$ is impossible because $t \leq$ contains at least two elements. □

Lemma 9.5. *Let $\mathcal{T}$ be a free boundary set and U a general unary relation on σ and β a finite element of $\mathrm{cut}_U(\mathcal{T})$. Then there exists an orbit* X *of $\mathrm{H}_{\mathcal{T}}$ so that* $\mathrm{broken}(\beta_{\mathrm{X}}) = \mathrm{broken}(\beta)$.

Proof. Identify the roots of the elements of $\text{broken}_d(\beta)$ into the element a. Let D be the free amalgam of the elements in $\text{broken}_d(\beta)$ along a. Then $\mathrm{D} \in \text{age}(\mathrm{H}_{\mathcal{T}})$. Hence there is an embedding f of D into $\mathrm{H}_{\mathcal{T}}$. Let X be the orbit with $\mathbb{F}(\mathrm{X}) = f[\langle \mathrm{D} \rangle - \{a\}]$ and with $f(a) \in X$. It follows that $\text{broken}(\beta_{\mathrm{X}}) = \text{broken}(\beta)$.

□

Let $\mathcal{T}$ be a free boundary set with signature σ. Let $\text{cut}^f(\mathcal{T})$ be the set of finite elements in $\text{cut}(\mathcal{T})$. Then

$$\mathbf{orb}(\mathcal{T}) := \{\text{Forb}\big(\text{broken}_u(\beta)\big) \mid \beta \in \text{cut}^f(\mathcal{T})\}.$$

The set $\mathbf{orb}(\mathcal{T})$ forms a partial order $(\mathbf{orb}(\mathcal{T}); \subseteq)$ under set inclusion $\subseteq$.

Theorem 9.1. *Let $\mathcal{T}$ be a free boundary set in the binary signature σ and $\mathrm{H}_{\mathcal{T}}$ the $\mathcal{T}$-free homogeneous structure.*

Then equality is an isomorphism between the partial orders $(\mathbf{orb}(\mathrm{H}_{\mathcal{T}}); \subseteq)$ and $(\mathbf{orb}(\mathcal{T}); \subseteq)$.

Proof. The Theorem follows from Lemma 9.2 and Lemma 9.3 and Lemma 9.5.

□

Corollary 9.1. *Let $\mathcal{T}$ be a finite free boundary set in the binary signature σ and $\mathrm{H}_{\mathcal{T}}$ the $\mathcal{T}$-free homogeneous structure.*

Then there is a finite algorithm to construct the partial order $(\mathbf{orb}(\mathrm{H}_{\mathcal{T}}); \subseteq)$.

For every ω-enumeration $\{b_i \mid i \in \omega\}$ of $\langle \mathrm{H}_{\mathcal{T}} \rangle$ and every chain $\mathbf{C}$ of $\mathbf{orb}(\mathrm{H}_{\mathcal{T}})$ and every element b_i there is a finite algorithm to decide if $b_i \in \mathbf{C}(\mathrm{H}_{\mathcal{T}})$. Hence a canonical partition of $\langle \mathrm{H}_{\mathcal{T}} \rangle$ can be finitely constructed.

Proof. According to Theorem 9.1 it suffices to construct the partial order $(\mathbf{orb}(\mathcal{T}); \subseteq)$.

The second part of the assertion follows from Lemma 9.2 and Theorem 9.1.

□

Theorem 8.2 and Theorem 8.3 and Corollary 8.2 together with Theorem 9.1 imply the following theorems:

Theorem 9.2. *Let $\mathcal{T}$ be a free boundary set and $\mathrm{H}_{\mathcal{T}}$ the $\mathcal{T}$-free homogeneous structure in the binary signature σ so that $\mathbf{orb}(\mathcal{T})$ is finite. Let $\mathfrak{C}$ be a set of maximal chains of $(\mathbf{orb}(\mathcal{T}); \subseteq)$ with $\bigcup \mathfrak{C} = \mathbf{orb}(\mathcal{T})$ and so that $|\mathfrak{C}|$ is minimal under those conditions.*

Let $\mathcal{Q} := \{S_{\mathbf{C}} \mid \mathbf{C} \in \mathfrak{C}\}$ be a partition of $\langle \mathrm{H}_{\mathcal{T}} \rangle$ into finitely many classes so that $S_{\mathbf{C}} \subseteq \mathbf{C}(\mathrm{H}_{\mathcal{T}})$ for every $\mathbf{C} \in \mathfrak{C}$.

Then $\mathcal{Q}$ is a canonical partition of $\mathrm{H}_{\mathcal{T}}$.

Let U be a unary relation symbol of the signature σ. An element $\mathbf{c}$ of $\mathbf{orb}(\mathcal{T})$ is a *U-element* of $\mathbf{orb}(\mathcal{T})$ if $U(a)$ for every structure $\mathrm{A} \in \mathbf{c}$ and every element $a \in \langle \mathrm{A} \rangle$.

Theorem 9.3. *Let $\mathcal{T}$ be a free boundary set and $\mathrm{H}_{\mathcal{T}}$ the $\mathcal{T}$-free homogeneous structure in the binary signature σ so that the partial order of U-elements of $(\mathbf{orb}(\mathcal{T}); \subseteq)$ is a chain for every unary relation U of σ.*

Then the partition of $\mathrm{H}_{\mathcal{T}}$ into unary sets is a canonical partition.

Theorem 9.4. *Let $\mathcal{T}$ be a free boundary set and $\mathrm{H}_{\mathcal{T}}$ the $\mathcal{T}$-free homogeneous structure in the binary signature σ so that the partial order $(\mathbf{orb}\mathcal{T}, \subseteq)$ is a chain.*

Then $\mathrm{H}_{\mathcal{T}}$ is indivisible.

If $\mathrm{H}_{\mathcal{T}}$ is indivisible then the partial order $(\mathbf{orb}(\mathcal{T}); \subseteq)$ is a chain.

10 Examples

Let $\mathcal{T}$ be a free boundary set and $\mathrm{H}_{\mathcal{T}}$ the $\mathcal{T}$-free homogeneous structure. It is customary to not list all of the elements of $\mathcal{T}$ but instead postulate some axioms for the structure $\mathrm{H}_{\mathcal{T}}$. For example, the Rado graph is the free homogeneous graph, so $\mathcal{T}$ is empty. This is equivalent to saying that the Rado graph is the $\mathcal{T}$-free homogeneous structure in signature σ where σ contains exactly one binary relation symbol E and $\mathcal{T}$ consists of the structures T and R. The structure T has two elements a and b so that $E(a,b)$ but not $E(b,a)$ and the structure R contains one element a so that $E(a,a)$.

If the structures T and R are in $\mathcal{T}$ then we will say that E is a *graph relation* and not list T and R in $\mathcal{T}$. The following requirements can be similarly translated into the presence of particular complete structures in $\mathcal{T}$. Two binary relations E and F are *disjoint* if $E(a,b)$ implies not $F(a,b)$ and not $F(b,a)$. A binary relation is an *oriented graph relation* if not $E(a,a)$ and if $E(a,b)$ implies not $E(b,a)$.

Example 10.1. Let σ be the binary relational signature containing the graph relations E_r and E_b of "red" and "blue" edges. We consider only such σ-structures in which the set of red edges is disjoint from the set of blue edges. Let $\mathcal{T} = \{\mathrm{T}_r, \mathrm{T}_b\}$ consist of a red triangle, T_r, and a blue triangle, T_b.

Let R be the structure consisting of a single red edge and B be the structure consisting of a single blue edge and $\mathrm{H}_{\mathcal{T}}$ the $\mathcal{T}$-free homogeneous structure.

It follows that $\mathbf{orb}(\mathrm{H}_{\mathcal{T}})$ consists of the elements $\mathrm{Forb}(\mathrm{T}_r, \mathrm{T}_b)$, $\mathrm{Forb}(\mathrm{R}, \mathrm{T}_b)$, $\mathrm{Forb}(\mathrm{T}_r, \mathrm{B})$ and $\mathrm{Forb}(\mathrm{R}, \mathrm{B})$. (The last set of ages being the one which contains all finite structures without any relations.)

Let $\{b_i \mid i \in \omega\}$ be an ω-enumeration of $\mathrm{H}_{\mathcal{T}}$. Let Q be the subset of all elements $b_n \in \langle \mathrm{H}_{\mathcal{T}} \rangle$ so that there is no $i \in n$ with $E_r(b_i, b_n)$. Let P be the subset of all elements $b_n \in \langle \mathrm{H}_{\mathcal{T}} \rangle$ so that there is at least one $i \in n$ with $E_r(b_i, b_n)$. Then (Q, P) is a canonical partition of $\mathrm{H}_{\mathcal{T}}$.

That is, if γ is a coloring of the elements of $\mathrm{H}_{\mathcal{T}}$ with $n \in \omega$ colors, then there is an embedding f of $\mathrm{H}_{\mathcal{T}}$ into $\mathrm{H}_{\mathcal{T}}$ so that $f[Q] \subseteq Q$ and all of the elements of $f[Q]$ are colored with one color only and $f[P] \subseteq p$ and all of the elements of $f[P]$ are colored with one color only.

No embedding f of $\mathrm{H}_{\mathcal{T}}$ into $\mathrm{H}_{\mathcal{T}}$ maps all elements of $\mathrm{H}_{\mathcal{T}}$ into Q or maps all elements of $\mathrm{H}_{\mathcal{T}}$ into P. Hence $\mathrm{H}_{\mathcal{T}}$ is not indivisible and the given partition is the best possible partition result for $\mathrm{H}_{\mathcal{T}}$.

Example 10.2. Let $P = \{a_i \mid i \in \omega\} \cup \{b_i \mid i \in \omega\}$ and $\leq$ a partial order relation on P which is the transitive closure of the relations $a_i > a_{i+1}$, $b_i > b_{i+1}$, $a_i > b_{i+1}$ for all $i \in \omega$. We will construct a free boundary set $\mathcal{T}$ so that the partial order $(\mathbf{orb}(\mathcal{T}); \subseteq)$ is isomorphic to the partial order $(P; \leq)$ and discuss the divisibility properties of the homogeneous structure $\mathrm{H}_{\mathcal{T}}$.

The signature σ of the structures in $\mathcal{T}$ will consist of the binary relation symbols E_0, E_1, E_2, E_3 and E_4. We will restrict the structures under consideration to structures in which the relations $(E_i; i \in 4)$ are graph relations and the relation E_4 is an oriented graph relation and in which all five relations are pairwise disjoint.

The structure W_n for $n \in \omega$ consists of the elements $\{w_i; i \in m = n+5\}$, it is a complete structure in the graph relations E_2 and E_3 and $E_2(w_i, w_{i+1})$ mod m. There are no other pairs in E_2 except for the circle $w_0, w_1, \dots, w_{m-1}$. All of the other pairs being in the relation E_3.

The structure $\mathrm{W}_n \oplus \mathrm{W}_m$ consists of the two disjoint structures W_n and W_m. For every element y of W_m there are edges of E_4 oriented from y to-wards all of the odd indexed elements of W_n and oriented from all of the even indexed elements of W_n. (The idea being that any two elements of W_m are of the same type over the elements of W_n and the split of $W_n \oplus W_m$ into W_n and W_m is the only such non trivial split.)

The structure $\mathrm{T}_{0,m,n}$ consists of the structure $\mathrm{W}_m \oplus \mathrm{W}_n$ and the element x not an element of $\mathrm{W}_m \oplus \mathrm{W}_n$. The element x is adjacent via the graph relation E_0 with every element in $\mathrm{W}_n \oplus \mathrm{W}_m$. The structure $\mathrm{T}_{1,m,n}$ consists of the structure $\mathrm{W}_m \oplus \mathrm{W}_n$ and the element x not an element of $\mathrm{W}_m \oplus \mathrm{W}_n$. The element x is adjacent via the graph relation E_1 with every element in $\mathrm{W}_n \oplus \mathrm{W}_m$.

Then $\mathcal{T} = \{\mathrm{T}_{i,m,n} \mid i \in 2 \text{ and } m, n \in \omega \text{ and } n \leq m\} \cup \{T\}$ where T is the triangle in the relation E_1.

The only elements of $\mathrm{ord}(\mathrm{T}_{i,m,n})$ with two non trivial break points are the ones with x on the bottom then the elements of W_m above and then above that the elements of W_n. The lowest elements in W_m and W_n will be the break points. The break point in W_n is the *crucial break point* of $\mathrm{T}_{i,m,n}$.

Let $\beta \in \mathrm{cut}(\mathcal{T})$ be finite. The set $\mathrm{broken}(\beta)$ depends on largest numbers m_0 and m_1 so that T_{0,m_0,n_0} and T_{1,m_1,n_1} for some n_0 and n_1 are in the domain of β and β maps them to their crucial break points. Let this largest number m_0 be the 0-*index* of β and the number m_1 the 1-*index* of β.

If β does not map any boundary structure of the form T_{1,m_1,n_1} to its crucial break point and has 0-index m_0 and maps some boundary structure of the form T_{0,m_0,n_0} to its crucial break point, then associate the set $\mathrm{broken}(\beta)$ with the element a_{m_0} of the partial order $(P; \leq)$.

If β does map a boundary structure of the form T_{1,m_1,n_1} to its crucial break point and has 0-index m_0 and 1-index m_1 and maps some boundary structure of the form T_{1,m_1,n_1} to its crucial break point, then associate the set $\mathrm{broken}(\beta)$ with the element b_i of the partial order $(P; \leq)$ where $i = \max\{m_0, m_1\}$.

The association described above is an isomorphism between the partial orders $(\mathbf{orb}(\mathcal{T}); \subseteq)$ and $(P; \leq)$. To actually work out the details is surprisingly tedious but straight forward. Because of the notation involved it is actually easier think it through oneself than to follow a written out argument.

It follows from Theorem 9.1 that there is an isomorphism f of the partial order $(P; \leq)$ to the partial order $(\mathbf{orb}(\mathrm{H}_{\mathcal{T}}); \subseteq)$. Hence $(\mathbf{orb}(\mathrm{H}_{\mathcal{T}}); \subseteq)$ has width two and the chains $\{f(a_i) \mid i \in \omega\}$ and $\{f(b_i) \mid i \in \omega\}$ contain all of the elements of the partial order $(\mathbf{orb}(\mathrm{H}_{\mathcal{T}}); \subseteq)$.

The chain $\mathbf{C} := \{f(a_i) \mid i \in \omega\}$ satisfies the conditions of Lemma 8.1 and hence Theorem 8.1 applies with $\mathfrak{C} = \{\mathbf{C}\}$. That is, for every partition $(S_i; i \in n)$ of the elements of $\mathbf{C}(\mathrm{H}_{\mathcal{T}})$ into finitely many parts, there is $r \in n$ and an embedding g of $\mathrm{H}_{\mathcal{T}}$ into $\mathrm{H}_{\mathcal{T}}$ so that $g[\mathbf{C}(\mathrm{H}_{\mathcal{T}})] \subseteq S_r$.

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