PMAT 607 FINAL EXAMINATION April 17, 2005

There are ten questions, each is worth 10 points. The notation $\mathbb{Z}/n$ will be used to denote the integers modulo n. For any abelian group G, G^n will denote its n-fold direct sum (equivalently direct product) with itself.

The solutions must be turned in on a separate examination booklet(s), written in order and using only one side of the page (right hand side).

1. Let $(X, <)$ be a simply ordered set, and give it the order topology. Show that it is T_3 (Hausdorff and regular)

2. Give a full proof (i.e. exhibit an isomorphism) of Corollary 10.10(b), namely
$$(\pi_1(T_s, x_0))_{\text{ab}} \approx \mathbb{Z}^{s-1} \oplus \mathbb{Z}/2 \ .$$

3. Let $\{x_1, x_2, ...\}$ be a sequence of points in a Hausdorff space X, converging to $x \in X$. Show that $K := \{x\} \bigcup \{x_n : n \in \mathbb{N}\}$ is compact.

4. (a) Find spaces with fundamental groups respectively
$$\mathbb{Z}/2 \oplus \mathbb{Z}/3, \quad \mathbb{Z}/2 * \mathbb{Z}/3 \ .$$

   (b) Find closed 3-manifolds with fundamental groups respectively
$$\{0\}, \quad \mathbb{Z}, \quad \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \ .$$

5. (a) Determine $\pi_1(\mathbb{R}^3 \setminus \{P\})$, for any $P \in \mathbb{R}^3$.

   (b) Generalize (a) to $\mathbb{R}^3 \setminus \{P_1, ..., P_k\}$, for any distinct $P_i \in \mathbb{R}^3$.

   (c) Explain briefly why your answers in (a), (b), will generalize to $\mathbb{R}^n$, $n \geq 3$.

6. Let X be a k-space and Y any space. Prove that $f : X \to Y$ is continuous iff $f|K$ is continuous for all compact subsets $K \subseteq X$.

7. Generalize 5(c) to the fundamental group of any n-manifold M with finitely many points removed, where $n \geq 3$. You need only state the appropriate theorem, and indicate a sketch of the proof (which will be fairly similar to the proof of 5(b)).

8. (a) Let Y be a metric space, and X Hausdorff. Show that the uniform topology on Y^X is in general finer (more open sets) than the compact-open topology.

   (b) If in (a) X is also compact, then show the two topologies agree.

9. (a) The universal cover of a space X has already been defined (7.15). State the appropriate theorems pertaining to the existence and uniqueness of universal covers, first carefully defining all terms in the theorems (all this can be found in Munkres as well as other texts).

   (b) Give an example of a path connected space X that has no universal cover.

10. For $r > 1$ show that the fundamental group of the genus r Riemann surface S_r is not abelian (this is Munkres p.454-6, and a hint is given there).