DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF CALGARY
Course Information Sheet

PMAT 613 – Galois Theory – Fall Term 2007

Table 1: **Fall 2007 lecture times**

Monday	Tuesday	Wednesday	Thursday	Friday
	lecture: (10:00–11:30) in (TBA)		lecture: (10:00–11:30) in (TBA)	

1. **Professor:** Clifton Cunningham
   - **Email:** `cunning@math.ucalgary.ca`
   - **Office:** Mathematical Sciences Building, Room 528.
   - **Lectures:** Tuesday and Thursday, 10:00-11:30 (tentative)
   - **Office hour:** by appointment

   Course website: go to `http://blackboard.ucalgary.ca`

2. **Pre-requisite:** PMAT431 or approval from the Division of Pure Mathematics.

   It is the student's responsibility to ensure that they have the pre- and co-requisites for the course.

3. **Fee policy**: After the last day to drop/add courses, there will be no refund of tuition fees if a student withdraws from a course, courses or the session.

4. **Academic Accommodations:** It is the student's responsibility to request academic accommodations. A student with a documented disability who may require academic accommodation must register with the Disability Resource Centre to be eligible for formal academic accommodation. DRC registered students are required to discuss their needs with the instructor no later than fourteen (14) days after the start of this course.

5. **Evaluation:** All grading will use the GPA (grade point average) system, and subject to regulations as described in the current University Calendar, Academic Standings.
   - four assignments: $4 \times 20\% = 80\%$
   - final oral exam: 20%

6. **Missed Components of Term Work:** The regulations of the Faculty of Science pertaining to this matter are outlined in the current University Calendar, Faculty of Science, section 6A. It is the student's responsibility to familiarize herself/himself with these regulations.

7. **Academic misconduct** (cheating, plagiarism, or any other form) is a very serious offence that will be dealt with rigorously in all cases. A single offence may lead to disciplinary probation or suspension or expulsion. The Faculty of Science follows a zero tolerance policy regarding dishonesty. Please read the sections of the current University Calendar. See: http://www.ucalgary.ca/honesty/

8. There will be no out-of-class-time activity for this course. Regularly scheduled classes have precedence over any out-of-class-time activity.

Department Authorization: Date:

- **Recommended Reading:** Many excellent introductory treatments of Galois theory are available (by Cox, Dummit & Foote, Garling, Kaplansky, Stewart, to name but a few) but I have yet to find any text that takes the perspective of this course. The closest treatment is perhaps *Field and Galois Theory* by Patrick Morandi (Grad. Texts in Math., V. 167. Springer-Verlag, 1996), but even this is far from a perfect fit. Consequently, there is no textbook for this course. Course notes, called *An Invitation to Galois Theory*, will be posted during the term.

Table 2: **Schedule of lectures and assignments**

	Date	**Topic**	**Chapter**	**Ass't (posted) due**			
1	2007.09.11	categories, functors and adjunction	**Category**				
2	2007.09.13	limits and colimits	**Theory**				
3	2007.09.18	the category of fields	**Chapter 1:**				
4	2007.09.11	the Galois functors	**Galois**				
5	2007.09.25	algebraic extensions	**Adjunction**	(1)			
6	2007.09.27	galois extensions					
7	2007.10.02	finite extensions	**Chapter 2:**				
8	2007.10.04	finitely generated extensions	**Galois**	1			
9	2007.10.09	finite subgroups are galois	**Extensions**				
10	2007.10.11	splitting extensions					
11	2007.10.16	normal extensions					
12	2007.10.18	separable extensions		(2)			
13	2007.10.23	galois is normal and separable					
14	2007.10.25	profinite groups	**Chapter 3:**				
15	2007.10.30	the Krull topology	**The Fund.**	2			
16	2007.11.01	topological galois groups	**Theorem**				
17	2007.11.06	closed subgroups		(3)			
18	2007.11.08	the Fundamental Theorem					
19	2007.11.15	the Prüffer ring, p-adic integers	**Chapter 4:**				
20	2007.11.19	subgroups of $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$	**Important**	3			
21	2007.11.22	cyclotomic subgroups of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$	**Galois Groups**				
22	2007.11.27	decomposition subgroups		(4)			
23	2007.11.29	inertia subgroups					
24	2007.12.04	Ramification	**An Intro. to**				
25	2007.12.06	ℓ-adic cyclotomic characters	**Galois Rep'ns**	4			

Preface to the course notes

Number theorists used to boast that their subject was safe from the vagaries of mathematical fashion, since no applications of the subject could possibly be found. Of course, this absurd claim was always meant to be taken with a grain of salt, but it was rendered rediculous in a spectacular manner when the proof of Fermat's Last Theorem lead directly to new cryptosystems which are now, only a few years later, ubiquitous in information security.

Now, as cryptography research groups have sprung up in universities around the world, many graduate mathematics programs attract significant numbers of talented students who are interested in applications of algebraic number theory to information security, and consequently, need a course in Galois theory which will take them quickly to areas of current research in number theory.

On the other hand, many students of pure mathematics have been electrified by the spectacular progress in arithmetic geometry in recent years, including, but not limited to, the proof of the Shimura-Tanyama-Weil conjecture. Many of these students have sensed - correctly - that the Langlands Program provides a unifying framework with which to understand this work and from which to attack open problems, and consequently, they need a course in Galois theory which will bring them closer class field theory and thence to the Langlands Programme.

This course has been designed to reach both modern students interested in applications to algebraic number theory and also those students setting out toward the Langlands Programme. The result is a course with three defining features:

1. **Categorical.** The first defining feature of our treatment is that we take the point of view that Galois theory is an example of a subject which straddles two categories and therefore, at its heart, the subject concerns functors and natural transformations. In fact, Galois theory is probably the first deeply categorical subject, both historically and pedagogically. In this course, the term 'Galois' is defined in terms of an adjoint pair of functors. The course does not, however, assume any great familiarity with category theory apart from the very basic definitions, which are supplied in an appendix. Consequently, this course provides an introduction to category theory at the same time that it introduces Galois theory, by considering a highly important example of an adjoint pair of functors and making a fairly detailed of the properties of limits and colimits (over various categories) of these functors. Adjoint functors, limits and colimits are ubiquitous in modern mathematics, especially in algebraic geometry, and we believe students are well-served by early exposure to this seminal concept.

2. **Topological.** The second defining feature of our treatment of Galois theory is that, almost from the beginning, we remind the reader that a Galois group is a *topological* group. Consequently, the major results are stated and proved for arbitrary Galois extensions, not finite Galois extensions. Of course, there are many excellent elementary treatments of Galois theory which include infinite Galois extensions, but for the most part they begin by studying the finite theory and then treating infinite Galois extensions as a sort of add-on. By contrast, we incorporate infinite Galois extensions into the discussion from the very beginning, and the big theorems in this course (such as the 'Galois is Normal and Separable' Theorem, and the Fundamental Theorem of Galois Theory) are stated and proved in that context. This is made possible by extensive use limits and colimits throughout the course, in various categories, which simplifies many arguments.

3. **Number Theoretic.** Although this course is meant to serve as an introduction to Galois theory, our ultimate goal is to provide students, as efficiently as possible, with the machinery necessary to study Galois representations and associated L-functions, which are basic objects in modern number theory. This is the third and most important defining feature of our treatment of Galois theory. Consequently, this course introduces the inertia and decomposition subgroups of the absolute Galois group of the Rational numbers, and also provides some simple examples of ℓ-adic Tate modules for curves, together with the action of the absolute Galois group on these modules. In this way we produce explicit examples of one- and two-dimensional ℓ-adic representations of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.

Galois Theory is an old subject, with a dramatic history reaching back into antiquity and forward to the most active branches of modern mathematics. These notes, however, offer an approach to Galois theory that is completely ahistorical. In particular, the classical Galois theory is altogether absent, and the reader will find no treatment of the hallowed topics of Galois theory such as soluble polynomials and compass and straightedge constructions in this course. These are indeed lamentable lacunae, but we have sacrificed these topics in order to go directly to the central concepts and principles necessary to understand how Galois theory plays a central role in modern number theory through Galois representations and associated L-functions. The classical topics are treated very well in existing introductory textbooks; students in this course are strongly encouraged to supplement the lectures with their own reading of classical topics.