PMAT 613 L01 Fall 2009
Final Examination Due December 11

Guidelines : The text and the lecture notes may be used, and only these materials.

1. Show that the polynomial $f(t) = t^7 - 10t^5 + 15t + 5$ cannot be solved by radicals over $\mathbb{Q}$.

2. Answer True or False for each of the following, and give a counterexample (or disproof) when False.

   (i) Every quartic equation over a subfield of $\mathbb{C}$ can be solved by radicals.

   (ii) Every radical extension is finite.

   (iii) Every finite extension is radical.

   (iv) The order of the Galois group of a polynomial of degree n always divides $n!$.

   (v) Any reducible quintic polynomial can be solved by radicals.

   (vi) There exist quartic polynomials with Galois group $\mathfrak{S}_4$.

   (vii) An irreducible polynomial over $\mathbb{Q}$ of degree 11 with exactly two nonreal zeros has Galois group $\mathfrak{S}_{11}$.

   (viii) The normal closure of a radical extension over a field K (of any characteristic) is also radical.

   (ix) $\mathfrak{A}_6$, which has order 360, has no subgroup of order 180.

   (x) The Galois group of any cubic polynomial over $\mathbb{Q}$ is non-trivial.

3. If $K \subseteq L \subseteq M$ and $M : K$ is radical, show that in general $L : K$ need not be radical. [Hint : For a counterexample some hard work is necessary! Here is an outline. Try $M = \mathbb{Q}(\omega) \subset \mathbb{C}$, where $\omega = e^{2\pi i/7}$ is a primitive 7'th root of unity. Now let $\alpha = \omega^2 + \omega^5$ and take $L = \mathbb{Q}(\alpha) \subset \mathbb{R}$, since $\alpha = 2\cos(4\pi/7) \in \mathbb{R}$. Show $M : \mathbb{Q}$ is radical with degree 6. Next show that α has $m_\alpha(t) = t^3 + t^2 - 2t - 1$, so

$[L : \mathbb{Q}] = 3$. The extension $L : \mathbb{Q}$ is also normal, indeed writing α, β, γ for the three zeros of m_α, show that $\beta = \alpha^2 - 2, \quad \gamma = \beta^2 - 2$, whence $\alpha, \beta, \gamma \in L$ and thus L is a splitting field for m_α, which implies normality.

Finally, show $L : \mathbb{Q}$ cannot be a radical extension. To see this, first show that $L : \mathbb{Q}$ radical must imply that $L = \mathbb{Q}(x)$ for some $x \in L$ with $x^3 = a \in \mathbb{Q}$. Then normality would imply that all three zeros of $t^3 - a$ must be in L, and show that this will lead to a contradiction.] Note that this example explains Remark 18.3(d) in the lecture notes.

4. Verify the Thompson-Feit theorem, that any finite group having odd order is solvable, for all groups of order less than 26. You are free to use the three Sylow theorems, and the fact that any group of order p^2, p prime, is abelian.

5. Demonstrate that a finite extension $L : K$ of prime degree is necessarily simple.

6. Let $L = \mathbb{Q}(\sqrt{3}, \sqrt{-7})$. Determine $\Gamma(L, \mathbb{Q})$, exhibiting each automorphism. Also show whether or not this extension is normal.

7. Show that the group of quaternion units $G = \{\pm 1, \pm i, \pm j, \pm k\}$ is solvable.