PMAT 613 L01 Fall 2009
Midterm Solutions

The solutions are outlined here, most but not all details are given.

1. Express $x^3y^3z + y^3z^3x + z^3x^3y$ as a polynomial in the elementary symmetric polynomials $\sigma_1, \sigma_2, \sigma_3$ in x, y, z.

   Answer : $p(x, y, z) = \sigma_3(\sigma_2^2 - 2\sigma_1\sigma_3)$.

2. Answer True or False for each of the following, and give a counterexample when False.

   (i) Every polynomial irreducible over $\mathbb{Z}$ is irreducible over $\mathbb{Q}$.

   True - Gauss' Lemma

   (ii) Every polynomial irreducible over $\mathbb{Q}$ is irreducible over $\mathbb{R}$.

   False, e.g. $t^2 - 2$.

   (iii) Every polynomial irreducible over $\mathbb{R}$ is irreducible over $\mathbb{C}$.

   False, e.g. $t^2 + 1$.

   (iv) Every non-constant polynomial over $\mathbb{Q}$ has a zero in the algebraic numbers $\mathcal{A}$.

   True

   (v) All simple algebraic extensions of a field K are isomorphic.

   False, e.g. $\mathbb{Q}(2^{1/2})$ and $\mathbb{Q}(2^{1/3})$ are simple algebraic extensions of $\mathbb{Q}$ but cannot be isomorphic since they have different degrees over $\mathbb{Q}$.

   (vi) All simple transcendental extensions of a field K are isomorphic.

   True

   (vii) Two field extensions over K having the same finite degree are isomorphic.

   False, e.g $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{3})$ both have degree 2 but are not isomorphic, which has to be proved by showing that no isomorphism $\phi : \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(\sqrt{3})$ can exist (where ϕ is the identity on $\mathbb{Q}$). For one would have to have $\phi(\sqrt{2}) = a + b\sqrt{3}$ for some $a, b \in \mathbb{Q}$, and squaring both sides will lead to a contradiction.

(viii) Two field extensions over K that are isomorphic have the same degree.

True

(ix) The degree of any minimum polynomial is a prime number.

False, e.g. $m_\alpha(t) = t^4 - 2$ has degree 4, and this is a monic irreducible (by Eisenstein) polynomial, hence is the minimal polynomial of a simple algebraic extension of $\mathbb{Q}$.

(x) Two simple algebraic extensions of a field K with different minimal polynomials cannot be isomorphic.

False, e.g. $\mathbb{Q}(\alpha) = \mathbb{Q}(\beta)$ where $\alpha = \sqrt{3}$ and $\beta = \sqrt{3}+1$, but $m_\alpha \neq m_\beta$.

3. Let $L = \mathbb{Q}(\alpha)$, where $\alpha = \sqrt{1+\sqrt{5}}$. Find $[L : \mathbb{Q}]$, giving the necessary details.

   Solution : The degree is 4. To see this one shows first that $\alpha^4 - 2\alpha^2 - 4 = 0$. So consider $p(t) = t^4 - 2t^2 - 4$. This is monic, and to show that it equals the minimal polynomial m_α it remains to show it is irreducible (over $\mathbb{Q}$, equivalently over $\mathbb{Z}$). Unfortunately there seems to not be any quick way to do this, but a little work shows it cannot have a linear factor, or a quadratic factor.

4. Let $L = \mathbb{Q}(\alpha)$, where α has minimum polynomial $M_\alpha(t) = t^3 - t^2 + 1$. Express $\dfrac{\alpha^5}{\alpha^3 + \alpha + 3}$ in the form $a\alpha^2 + b\alpha + c$ for some $a, b, c \in \mathbb{Q}$.

   Solution : $(1/5)(\alpha^2 - \alpha - 2)$

5. (a) The dihedral group D_{2n} of order $2n$ is defined as

$$D_{2n} = \langle a, b \mid a^n = b^2 = e,\ bab = a^{-1} \rangle.$$

Show that D_{2n} is a solvable group.

(b) Let $x \in \mathfrak{S}_n$ have odd order. Show that $x \in \mathfrak{A}_n$.

Solution (a) $A := \langle a \rangle$ is clearly a cyclic subgroup of order n. So it has index 2, and is therefore a normal subgroup. Then $\{e\} \lhd A \lhd D_{2n}$ is a normal series and the quotients are abelian, being respectively A and $\mathbb{Z}_2$. By definition, then, D_{2n} is solvable.

(b) Recall that there is a homomorphism $\varepsilon : \mathfrak{S}_n \to \mathbb{Z}_2 = \{\pm 1\}$, where $\varepsilon(\sigma) = \mathrm{sgn}(\sigma)$. We are given that $\sigma^{2k+1} = e$. It follows that

$$+1 = \varepsilon(e) = \varepsilon(\sigma^{2k+1}) = (\varepsilon(\sigma))^{2k+1} = (\varepsilon(\sigma)^{2k} \cdot \varepsilon(\sigma) = \varepsilon(\sigma),$$

and thus $\sigma \in \mathfrak{A}_n$.