Stat 213: Intro to Statistics 3 and 4

H. Kim

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`random phenomena`

- we know that what outcomes could happen, but not which particular value will happen

- `trial`: a single attempt or realization of a random phenomenon
- `outcome`: the value measured, observed, or reported for an individual instance of that trial
- `event`: a collection of outcomes

examples

- example:
  - trial: tossing a die and observing the face that appears
  - outcomes: $1, 2, 3, 4, 5, 6$
  - event: odd number, $1, 3, 5$
- example:
  - trial: a toss of a coin
  - outcomes: head(H); tail (T)
  - event A: observe a head
- example:
  - trial: three tosses of a coin
  - outcomes:
  - event A: observe exactly two heads

probability

- probability is used as a tool: it allows you to evaluate the reliability of inference about the population when you have only sample information
- when the population is known, the probability is used to describe the likelihood of observing a particular outcome
- when the population is unknown and only a sample from the population is available, probability is used in making statements about makeup of population, i.e. in making statistical inferences

probability

- that reports the likelihood of the event's occurrence
- derived from equally likely outcomes
- the long-run relative frequency of the event's occurrence
  - law of large numbers (Jacob Bernoulli, 1713): the long-run relative frequency of repeated independent events settles down to the true relative frequency as the number of trials increases
- the collection of all possible outcomes: sample space (S)
  - $P(A)$: the probability of the event A
  - $0 \leq P(A) \leq 1$
  - $P(S) = 1$

how to calculate the probability of event A ?

$$P(A) = \frac{\text{count of outcomes in } A}{\text{count of all possible outcomes}}$$

- example:
  - trial: a toss of a coin
  - outcomes: head(H); tail (T) $\Longrightarrow S = \{H, T\}$
  - event A: observe a head $= \{H\}$
  - event B: observe a tail $= \{T\}$
  - $P(A) = \frac{1}{2}$; $P(B) = \frac{1}{2}$; $P(S) = 1$

example

- trial: three tosses of a coin
  - outcomes:
  - event A: observe exactly two heads: $= \{HHT, HTH, THH\}$
    $P(A) = \frac{3}{8}$
  - event B: observe exactly two tails:
  - event C: observe at least one tails:
  - check that $P(S) = P(\text{all possible outcomes}) = 1$

complement rule

- the **complement** of an event A, A^c, consists of all the evens in S that are not in A event
- $P(A^c) = 1 - P(A)$
- complement rule: $P(A) = 1 - P(A^c)$

disjoint; mutually exclusive

- when one event occurs, the other cannot and vice versa
- no outcomes in common

additive rule

- the **union** of events A and B, $A \cup B$, is the event that A or B or both occur
- for two **disjoint** events A and B, the probability that one or the other occurs is the sum of the probabilities of the two events:
- $P(A \cup B) = P(A) + P(B)$

intersection

- the `intersection` of events A and B, $A \cap B$, is the event that both A and B occur
  - $P(A \cap B)$

general additive rule

- $P(A \cup C) = P(A) + P(C) - P(A \cap C)$

laws

- DeMorgan's laws:

$$P[A^c \cup B^c] = P[(A \cap B)^c] \text{ and } P[A^c \cap B^c] = P[(A \cup B)^c]$$

- total law of probability:

$$P(A) = P(A \cap B) + P(A \cap B^c)$$

multiplicative rule

- for two independent events A and B, the probability that both A and B occur is the product of the probabilities of the two events: $P(A \cap B) = P(A) \times P(B)$

tree diagram

- example: three tosses of a coin

1^{st}	2^{nd}	3^{rd}	event
H			
T			

permutation

- an ordered arrangement
  - suppose the outcome is a $(1, 2)$.
  - there are 2 different ways: $(1, 2)$ and $(2, 1)$
- the number of ways to arrange n-distinct objects is given by:

$$n! = n \times (n - 1) \times (n - 2) \times \cdots \times 2 \times 1$$

  - the number of ways with two district values: $2! = 2 \times 1 = 2$
  - $0! = 1$

combination

- the order of the selection is not important
  - suppose the outcome is a $(1, 2)$.
  - select 2-distinct objects: $(1, 2)$ or $(2, 1)$
- the number of ways to select r-distinct objects from n-distinct objects is given by:

$$\left(\begin{array}{c} n \\ r \end{array} \right) = \frac{n!}{r!(n-r)!}$$

  - $\left(\begin{array}{c} 2 \\ 2 \end{array} \right) = 1$
  - $\left(\begin{array}{c} 2 \\ 1 \end{array} \right) = 2$

example

- A boxcar contains 7 electronic systems of which 2 are defective. 3 are selected at random for testing. What is the probability that

  a. one of the 3 is defective ?

  b. none are defective ?

example

- Two commissioners are to be selected from a total of 5 commissioners C_1, C_2, C_3, C_4, C_5 to form a committee. Let S be the sample space, A be the event that C_2 will be in the committee and B be the event that C_5 will be the committee.

  a. List the events of S, A, B, $A \cap B$ and $A \cup B$.

  b. Calculate the probabilities of A, B, $A \cap B$, $A \cup B$ and $(A \cup B)^c$.

  c. Are A and B mutually exclusive ? Why ?

conditional probability

- probability that take into account a given condition
- $P(A|B)$: probability of A given B

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

general multiplicative rule

- $P(A \cap B) = P(A) \times P(B|A) = P(B) \times P(A|B)$

independence

- events A and B are **independent** whenever $P(A|B) = P(A)$ or $P(B|A) = P(B)$

- if A and B are independent,

  $P(A \cap B) = P(A) \times P(B|A) = P(A) \times P(B)$

- **disjoint events can't be independent**: the probability of event A is the same as the probability of event A after the event B occurs ?

example

$P(A) = 0.4$, $P(B) = 0.2$, $P(C) = 0.1$, $P(A|B) = 0.3$ and $P(A \cap C) = 0.04$

- $P(A \cap B) =$
- $P(A \cup B) =$
- $P(A|C) =$
- $P(C^c) =$
- are A and B mutually exclusive? why ?
- are A and B independent ? why ?
- are A and C independent ? why?

example

- the probability of rain today is 0.2 and same for tomorrow; the probability of rain on both days is 0.1; what is the probability of some rain tomorrow if it is raining today ?

contingency table and conditional probability

	A	B	C	total
M F	n_1 n_4	n_2 n_5	n_3 n_6	$\mathbf{n_M} = n_1 + n_2 + n_3$ $\mathbf{n_F} = n_4 + n_5 + n_6$
total	$\mathbf{n_A}$ $= n_1 + n_4$	$\mathbf{n_B}$ $= n_2 + n_5$	$\mathbf{n_C}$ $= n_3 + n_6$	$\mathbf{n} = n_A + n_B + n_C$ $= n_M + n_F$

- $P(A) = \frac{n_A}{n}$, $P(B) = \frac{n_B}{n}$, $P(C) = \frac{n_C}{n}$
- $P(M) = \frac{n_M}{n}$, $P(F) = \frac{n_F}{n}$
- $P(M \cap B) = \frac{n_2}{n}$, $P(C \cap F) = \frac{n_6}{n}$
- $P(M|B) = \frac{P(M \cap B)}{P(B)} = \frac{n_2/n}{n_B/n} = \frac{n_2}{n_B}$
- $P(C|F) = \frac{P(C \cap F)}{P(F)} = \frac{n_6/n}{n_F/n} = \frac{n_6}{n_F}$

example

- A group of 200 voters in a certain electoral poll are classified according to gender and candidate preference. Suppose that a voter is selected from this group.

preferred candidate

gender	A	B	C	total
male (M)	20	36	24	80
female (F)	40	44	36	120
total	60	80	60	200

- Find $P(F)$, $P(B)$, $P(B^c)$, $P(B \cap F)$, $P(B \cup F)$ and $P(B|F)$.

- Are F and B independent ? Why ?

reversing the conditioning

- suppose the sample space can be partitioned into 2 sub-populations, S_1 and S_2, and they are mutually exclusive

$$
\begin{aligned}
A &= (A \cap S_1) \cup (A \cap S_2) \\
P(A) &= P(A \cap S_1) + P(A \cap S_2) \\
&= P(A|S_1)P(S_1) + P(A|S_2)P(S_2)
\end{aligned}
$$

$$
P(S_1|A) = \frac{P(S_1 \cap A)}{P(A)} = \frac{P(S_1 \cap A)}{P(A|S_1)P(S_1) + P(A|S_2)P(S_2)}
$$

example

- A sample is selected from one of two population, S_1 and S_2, with probability $P(S_1) = 0.7$ and $P(S_2) = 0.3$. If the sample has been selected from S_1, the probability of observing an event A is $P(A|S_1) = 0.2$. Similarly, if the sample has been selected from S_2, the probability of observing A is $P(A|S_2) = 0.3$.
  - If a sample is randomly selected from one of the two populations, what is the probability that event A occurs ?
  - If a sample is randomly selected and event A is observed, what is the probability that the sample was selected from population S_1?

Bayes's rule

- want to find the probability of S_i given the information the event A is observed with prior probability $P(S_i)$

$$\begin{aligned} P(S_i|A) &= \frac{P(S_i \cap A)}{P(A)} \\ &= \frac{P(A|S_i)P(S_i)}{P(A|S_1)P(S_1) + P(A|S_2)P(S_2)} \end{aligned}$$

example

- suppose that in a particular city, airport **A** handles 50% of all airline traffic, and airports **B** and **C** handle 30% and 20%, respectively; the detection rate for weapons at the three airports are 0.9, 0.5 and 0.4, respectively; if a passenger at one of the airports is found to be **carrying a weapon** through the boarding gate, what is the **probability that the passenger is using airport B** ?

- $P(\text{airport A handles airline traffic}) = 0.5$
- $P(\qquad\qquad) = 0.9$
- $P(\text{a passenger is carrying a weapon}|\text{airport B}) = (\qquad\qquad)$
- $P(\text{airport B}|\text{a passenger is carrying a weapon}) = (\qquad\qquad)$

total law of probability

- suppose the sample space can be partitioned into k sub-populations, $S_1, S_2, \cdots, S_k$ (let say, $k = 3$) and they are mutually exclusive

$$
\begin{aligned}
A &= (A \cap S_1) \cup (A \cap S_2) \cup (A \cap S_3) \\
P(A) &= P(A \cap S_1) + P(A \cap S_2) + P(A \cap S_3) \\
&= P(A|S_1)P(S_1) + P(A|S_2)P(S_2) + P(A|S_3)P(S_3)
\end{aligned}
$$

Bayes's rule

- want to find the probability of S_i given the information the event A is observed with prior probability $P(S_i)$

$$
\begin{aligned}
P(S_i|A) &= \frac{P(S_i \cap A)}{P(A)} \\
&= \frac{P(A|S_i)P(S_i)}{P(A|S_1)P(S_1) + P(A|S_2)P(S_2) + P(A|S_3)P(S_3)}
\end{aligned}
$$

- general form for $i = 1, \cdots, k$

$$
P(S_i|A) = \frac{P(A|S_i)P(S_i)}{P(A|S_1)P(S_1) + \cdots + P(A|S_k)P(S_k)}
$$

the previous example

- suppose that in a particular city, airport A handles 50% of all airline traffic, and airports B and C handle 30% and 20%, respectively; the detection rate for weapons at the three airports are $0.9, 0.5$ and 0.4, respectively; if a passenger at one of the airports is found to be carrying a weapon through the boarding gate, what is the probability that the passenger is using airport B ?

- S_1= airport A handles airline traffic; $P(S_1) = 0.5$
- S_2= airport B handles airline traffic; $P(S_2) = 0.3$
- S_3= airport C handles airline traffic; $P(S_3) = 0.2$
- D = detection for weapons at an airport
  $P(D|S_1) = 0.9$, $P(D|S_2) = 0.5$ and $P(D|S_3) = 0.4$
- $P(S_1|D) = \frac{P(D|S_i)P(S_i)}{P(D|S_1)P(S_1)+P(D|S_2)P(S_2)+P(D|S_3)P(S_3)}$