χ^2 Tests of $H_0: \sigma = \sigma_0$ Using $\chi^2 = \frac{(n-1)s^2}{\sigma_0^2}$

Alternative	(P-value) Reject H_0 if	(Critical-value) Reject H_0 if
$H_a: \sigma < \sigma_0$	$P(\chi^2_{n-1} \le \chi^2) \le \alpha$	$\chi^2 \le \chi^2_{1-\alpha, n-1}$
$H_a: \sigma > \sigma_0$	$P(\chi^2_{n-1} \ge \chi^2) \le \alpha$	$\chi^2 \ge \chi^2_{\alpha, n-1}$
$H_a: \sigma \ne \sigma_0$	See next slide	$\chi^2 \le \chi^2_{1-\alpha/2, n-1}$ or $\chi^2 \ge \chi^2_{\alpha/2, n-1}$

27

To compute the P-value in the two-tail test, first compare the value of χ^2 with n – 1.

$$If \quad \chi^2 < n-1, \quad P-value = 2P(\chi^2_{n-1} \le \chi^2).$$

$$If \quad \chi^2 > n-1, \quad P-value = 2P(\chi^2_{n-1} \ge \chi^2).$$

28

Example 2.19

The errors of altimeters produced from an old production line have mean zero and standard deviation 43.7 ft. After updating the production line, it is concerned that the standard deviation of the new altimeters has changed. An SRS of 30 new altimeters are tested to give an s = 54.7 ft. Assess the above concern at 5% level.

29

Solution. Let σ be the standard deviation of the errors of the new altimeters. We want to test H_0: σ = 43.7 vs H_a: σ ≠ 43.7. This is a two-tail test and n = 30, α = 0.05, so we reject H_0 if $\chi^2 \le \chi^2_{1-0.05/2,\ 30-1} = \chi^2_{0.975,\ 29} = 16.05$,

$or \quad \chi^2 \ge \chi^2_{0.05/2,\ 30-1} = \chi^2_{0.025,\ 29} = 45.72.$

Now s = 54.7 gives

$$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2} = \frac{(30-1)(54.7)^2}{(43.7)^2} = 45.44,$$

so we barely fail to reject H_0 at 5% level, and we should keep monitoring the new production line.

30

Since $\chi^2 = 45.44 > 29 = 30 - 1$, the P-value is estimated to be between 5% and 10% from Table VII because

$$2P(\chi^2_{29} \ge 42.56) = 2(0.05) = 0.10,$$
$$2P(\chi^2_{29} \ge 45.72) = 2(0.025) = 0.05.$$

Using Minitab, the exact P-value is found to be

$$2P(\chi^2_{29} \ge 45.44) = 0.053304.$$

31

Example 2.20

A medical researcher believes that the standard deviation of the temperatures of newborn infants is greater than 0.6 degree. A sample of 15 infants were measured to give s = 0.8 degree. At 10% level, does the evidence support the researcher's belief?

32

Solution. Let σ be the temperature standard deviation of all newborn infants. We want to test H_0: σ = 0.6 vs H_a: σ > 0.6. At 10% level, we reject H_0 if

$$\chi^2 \geq \chi^2_{\alpha,\, n-1} = \chi^2_{0.10,\, 14} = 21.06.$$

Now s = 0.8 gives

$$\chi^2 = \frac{(15-1)(0.8)^2}{(0.6)^2} = 24.89 > 21.06,$$

so we reject H_0 and conclude that the researcher's belief is supported.

33

Them P-value of $\chi^2 = 24.89$ is estimated to be between 0.025 and 0.05.

From Minitab, the exact P-value is $P(\chi^2_{14} \geq 24.89) = 0.035669.$

34

Practice Problems

8.104, 8.105, 8.106, 8.107, 8.109, 8.113, 8.117, 8.121 [do a 95% confidence interval for question 8.121, the answer is (0.224, 9.685)]

35

The Power (Function) of a Test

Example 2.21

In Example 2.4 (tires tread wear) we tested H_0: μ = 0.0625 vs H_a: μ < 0.0625 with known σ = 0.006 inch at α = 0.05 level. The rejection region is

$$z = \frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}} \leq -z_\alpha \quad \text{or}$$

$$\bar{x} \leq \mu_0 - z_\alpha \frac{\sigma}{\sqrt{n}} = 0.0625 - 1.645\frac{0.006}{\sqrt{100}} \quad \text{or}$$

$$\bar{x} \leq 0.061513.$$

36

If H_0 is wrong, say μ ≠ 0.0625 but μ = 0.0605 (a specific value of H_a), our test has the probability of

$$P(\text{Reject } H_0 \,|\, H_0 \text{ is false}) = P(\text{Reject } H_0 \,|\, H_a \text{ is true})$$

$$= P(\bar{x} \leq 0.061513 \,|\, \mu = 0.0605)$$

$$= P\left(\frac{\bar{x}-0.0605}{0.006/\sqrt{100}} \leq \frac{0.061513-0.0605}{0.006/\sqrt{100}}\right)$$

$$= P(Z \leq 1.688) = 0.9543$$

rejecting H_0.

37

We call the probability

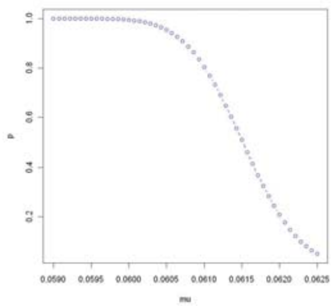
$$P(\text{Reject } H_0 \,|\, H_0 \text{ is false}) = P(\text{Reject } H_0 \,|\, H_a \text{ is true})$$

the **power of our test**. When H_a is assuming a range of values, the power of our test assumes a range of probabilities, giving a **power function**.

For μ in the range H_a: μ < 0.0625, our power function is plotted on the next slide.

38

Power Function (one-sided test)



Example 2.22 In Example 2.6 (pilot escape system) we tested H_0: $\mu = 50$ vs H_a: $\mu \neq 50$ with σ = 2 cm/s at α = 0.05 level. The rejection region is

$$z = \frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}} \leq -z_{\alpha/2} \quad \text{or} \quad z = \frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}} \geq z_{\alpha/2},$$

$$\bar{x} \leq \mu_0 - z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \quad \text{or} \quad \bar{x} \geq \mu_0 + z_{\alpha/2} \frac{\sigma}{\sqrt{n}},$$

$$\bar{x} \leq 50 - 1.96 \frac{2}{\sqrt{30}} \quad \text{or} \quad \bar{x} \geq 50 + 1.96 \frac{2}{\sqrt{30}},$$

$$\bar{x} \leq 49.28 \quad \text{or} \quad \bar{x} \geq 50.72.$$

For $\mu = 48$, the power of our test is

$$P(\bar{x} \leq 49.28 \text{ or } \bar{x} \geq 50.72 \mid \mu = 48)$$

$$= P\left(\frac{\bar{x} - 48}{2/\sqrt{30}} \leq \frac{49.28 - 48}{2/\sqrt{30}}\right) + P\left(\frac{\bar{x} - 48}{2/\sqrt{30}} \geq \frac{50.72 - 48}{2/\sqrt{30}}\right)$$

$$= P(Z \leq 3.505) + P(Z \geq 7.449) = 0.999772$$

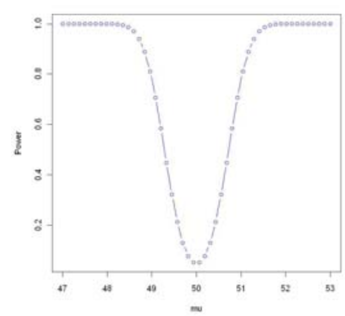
For $\mu = 51$, the power of our test is

$$P(\bar{x} \leq 49.28 \text{ or } \bar{x} \geq 50.72 \mid \mu = 51)$$

$$= P\left(\frac{\bar{x} - 51}{2/\sqrt{30}} \leq \frac{49.28 - 51}{2/\sqrt{30}}\right) + P\left(\frac{\bar{x} - 51}{2/\sqrt{30}} \geq \frac{50.72 - 51}{2/\sqrt{30}}\right)$$

$$= P(Z \leq -4.710) + P(Z \geq -0.767) = 0.778459.$$

Power Function (two-sided test)



For a given test, power and Type II error probability β are related through

$$\begin{aligned} \text{Power} &= P(\text{Reject } H_0 \mid H_a \text{ is true}) \\ &= P(\text{Reject } H_0 \mid H_0 \text{ is false}) \\ &= 1 - P(\text{Not Reject } H_0 \mid H_0 \text{ is false}) \\ &= 1 - \beta. \end{aligned}$$

Example 2.23

For testing H_0: $\mu = \mu_0$ vs H_a: $\mu > \mu_0$ at α level, we find

$$\begin{aligned} 1 - \beta = \text{Power} &= P(\text{Reject } H_0 \mid H_a \text{ is true}) \\ &= P(\bar{x} \geq \mu_0 + z_\alpha \sigma / \sqrt{n} \mid \mu = \mu_a) \ (\mu_a > \mu_0) \\ &= P\left(\frac{\bar{x} - \mu_a}{\sigma / \sqrt{n}} \geq \frac{\mu_0 + z_\alpha \sigma / \sqrt{n} - \mu_a}{\sigma / \sqrt{n}}\right) \\ &= P\left(Z \geq z_\alpha - \frac{\mu_a - \mu_0}{\sigma / \sqrt{n}}\right). \end{aligned}$$

We see that the power of a test will increase if

(1) alternative μ_a is further away from μ_0,
(2) sample size n increases,
(3) significance level α increases.

Between α and β, the relationship is reciprocal, namely, when α increases β decreases, and vice versa.

Practice Problems

8.92, 8.93, 8.94, 8.95, 8.97, 8.99, 8.103