

(1)

## STAT 407

Solutions to Assignment #1Chapter 3

- ⑧ (a)  $X \sim \text{geometric}$  with  $p = 1/6$ .  
 So  $EX = 1/p = 6$ .

Another solution (deriving  $EX$  for  $X \sim \text{geometric}$ ):

$$\begin{aligned} EX &= \frac{1}{6} E[X \mid \text{first roll is } 6] + \frac{5}{6} E[X \mid \text{first roll is not } 6] \\ &= \frac{1}{6} \cdot 1 + \frac{5}{6} (1 + EX) \end{aligned}$$

Solve this equation to obtain  $EX = 6$ .

- (b) Cond. dist. of  $X \mid Y=1$

$$= 1 + [\text{Unconditional dist. of } X]$$

$$\text{So } E[X \mid Y=1] = 1 + EX = 1 + 6 = 7$$

- (c) For  $n = 1, 2, 3, 4$ , ~~the~~ the conditional distribution of the  $n$ th ~~roll~~ given  $Y=5$  (i.e., given first 5 on roll 5) is clearly uniform on  $\{1, 2, 3, 4, 5\}$ .

$\frac{x}{5}$	$P(X=x \mid Y=5)$
1	$(4/5)(1/5)$
2	$(4/5)^2(1/5)$
3	$(4/5)^3(1/5)$
4	$(4/5)^4(1/5)$

(continued)

(2)

(8)(c) (continued)

Also  $P[X=5|Y=5] = 0$ , clearly.

For  $n > 5$ , the condition  $Y=5$  no longer precludes obtaining a 6 on the  $n^{\text{th}}$  roll. So we have:

$x$	$P[X=x Y=5]$
6	$(\frac{4}{5})^4 \frac{1}{6}$
7	$(\frac{4}{5})^4 (\frac{5}{6}) \frac{1}{6}$
8	$(\frac{4}{5})^4 (\frac{5}{6})^2 \frac{1}{6}$
$\vdots$	$\vdots$
$x$	$(\frac{4}{5})^4 (\frac{5}{6})^{x-6} \frac{1}{6}$
	$\vdots$

Putting this all together we obtain

$$\begin{aligned}
 E[X|Y=5] &= \sum_{x=1}^{\infty} P[X=x|Y=5] \\
 &= 1(\frac{1}{5}) + 2(\frac{4}{5})\frac{1}{5} + 3(\frac{4}{5})^2 \frac{1}{5} + 4(\frac{4}{5})^3 \frac{1}{5} + 5(0) \\
 &\quad + \cancel{6(\frac{4}{5})^4 \frac{1}{5}} + (\frac{4}{5})^4 \sum_{x=6}^{\infty} x(\frac{5}{6})^{x-6} \frac{1}{6}.
 \end{aligned}$$

It is easy to get from here to a numerical answer. Just write the last term as  $(\frac{4}{5})^4 (\frac{5}{6})^{-5} \sum_{x=6}^{\infty} x(\frac{5}{6})^{x-1} \frac{1}{6} =$

$$(\frac{4}{5})^4 (\frac{5}{6})^{-5} \left[ 1 - \frac{1}{6} - 2(\frac{5}{6})(\frac{1}{6}) - 3(\frac{5}{6})^2 \frac{1}{6} - 4(\frac{5}{6})^3 \frac{1}{6} - 5(\frac{5}{6})^4 \frac{1}{6} \right],$$

then do the arithmetic.

(11) Fix  $y > 0$ .

$$\text{Then } E[X|Y=y] = \int_{x=-y}^y \frac{x f(x,y)}{f_Y(y)} dy$$

$$= C \int_{-y}^y x(y^2 - x^2) dx = 0,$$

a constant that does  
not depend on  $x$

$$(12) \text{ For } y > 0, f_Y(y) = \int_{x=0}^{\infty} \frac{e^{-x/y}}{y} e^{-y} dx$$

$$= e^{-y} \int_{x=0}^{\infty} \frac{1}{y} e^{-x/y} dx = e^{-y}.$$

So for  $x > 0, y > 0$ ,

$$f(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{e^{-x/y}}{y \cdot e^{-y}} = \frac{1}{y} e^{-x/y}.$$

So the conditional dist. of  $X$ , given  $Y=y$ , is exponential with mean  $y$ .

$$\text{Hence } E[X|Y=y] = y.$$

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$$(a) X = \sum_{i=1}^N T_i.$$

(b) Clearly  $N$  is geometric with parameter  $1/3$ ; thus  $E[N] = 3$ .

(c) Since  $T_N$  is the travel time corresponding to the choice leading to freedom it follows that  $T_N = 2$ , and so  $E[T_N] = 2$ .

(d) Given that  $N=n$ , the travel times  $T_i$ ,  $i=1, \dots, n-1$  are each equally likely to be either 3 or 5 (since we know that a door leading back to the mine is selected), whereas  $T_n$  is equal to 2 (since that choice led to safety). Hence,

$$\begin{aligned} E\left[\sum_{i=1}^N T_i | N = n\right] &= E\left[\sum_{i=1}^{n-1} T_i | N = n\right] + E[T_n | N = n] \\ &= 4(n-1) + 2. \end{aligned}$$

(e) Since part (d) is equivalent to the equation

$$E\left[\sum_{i=1}^N T_i | N\right] = 4N - 2,$$

we see from parts (a) and (b) that

$$\begin{aligned} E[X] &= 4E[N] - 2 \\ &= 10. \end{aligned}$$

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Letting  $N_i$  denote the time until the same outcome occurs  $i$  consecutive times we obtain, upon conditioning upon  $N_{i-1}$ , that

$$E[N_i] = E[E[N_i | N_{i-1}]].$$

Now,

$$E[N_i | N_{i-1}] = N_{i-1} + \begin{cases} 1 & \text{with probability } \frac{1}{m} \\ E[N_i] & \text{with probability } \frac{m-1}{m}. \end{cases}$$

The above follows because after a run of  $i-1$  either a run of  $i$  is attained if the next trial is the same type as those in the run or else if the next trial is different then it is exactly as if we were starting all over at that point.

(continued)

(22) (continued)

From the above equation, we obtain that

$$E(N_i) = E(N_{i-1}) + \frac{1}{m} + E(N_i) \frac{m-1}{m}.$$

Solving for  $E(N_i)$  yields

$$E(N_i) = 1 + m E(N_{i-1}).$$

Solving recursively now yields

$$E(N_k) = 1 + m [1 + m E(N_{k-1})]$$

$$= 1 + m + m^2 E(N_{k-1})$$

$$= 1$$

$$\vdots$$

$$= 1 + m + m^2 + \dots + m^{k-2} + m^{k-1} E(N_1)$$

$$= 1 + m + m^2 + \dots + m^{k-1} \quad (\text{since } N_1 = 1 \text{ with prob 1})$$

$$= \frac{m^k - 1}{m - 1}$$

In the numerical example  $m = 10, k = 9$

$$\text{yields } E(N_9) \approx \frac{10^9}{9} \approx 111,111,111$$

Having a run of length 9 after 24 million digits when one expects the first such run to occur at about 111 million digits, seems a bit early to me but not really too far off, alone with the null hypothesis.

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Let  $X$  denote the first time a head appears. Let us obtain an equation for  $E[N|X]$  by conditioning on the next two flips after  $X$ . This gives

$$E[N|X] = E[N|X, h, h]p^2 + E[N|X, h, t]pq + E[N|X, t, h]pq + E[N|X, t, t]q^2$$

where  $q = 1-p$ . Now

$$E[N|X, h, h] = X+1, \quad E[N|X, h, t] = X+1$$

$$E[N|X, t, h] = X+2, \quad E[N|X, t, t] = X+2+E[N].$$

Substituting back gives

$$E[N|X] = (X+1)(p^2 + pq) + (X+2)pq + (X+2+E[N])q^2$$

Taking expectations, and using the fact that  $X$  is geometric with mean  $1/p$ , we obtain

$$E[N] = 1 + p + q + 2pq + q^2/p + 2q^2 + q^2E[N]$$

Solving for  $E[N]$  yields

$$E[N] = \frac{2 + 2q + q^2/p}{1 - q^2}.$$

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In all parts, let  $X$  denote the random variable whose expectation is desired, and start by conditioning on the result of the first flip. Also,  $h$  stands for heads and  $t$  for tails.

(a)

$$\begin{aligned} E[X] &= E[X|h]p + E[X|t](1-p) \\ &= \left(1 + \frac{1}{1-p}\right)p + \left(1 + \frac{1}{p}\right)(1-p) \\ &= 1 + p/(1-p) + (1-p)/p \end{aligned}$$

(b)

$$\begin{aligned} E[X] &= (1 + E[\text{number of heads before first tail}])p + 1(1-p) \\ &= 1 + p(1/(1-p) - 1) = 1 + p/(1-p) - p \end{aligned}$$

(c) Interchanging  $p$  and  $1-p$  in (b) gives result:  $1 + (1-p)/p - (1-p)$   
 (d)

$$\begin{aligned} E[X] &= (1 + \text{answer from (a)})p + (1 + 2/p)(1-p) \\ &= (2 + p/(1-p) + (1-p)/p)p + (1 + 2/p)(1-p) \end{aligned}$$

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27. Condition on the outcome of the first flip to obtain

$$\begin{aligned} E[X] &= E[X|H]p + E[X|T](1-p) \\ &= (1+E[X])p + E[X|T](1-p) \end{aligned}$$

Conditioning on the next flip gives

$$\begin{aligned} E[X|T] &= E[X|TH]p + E[X|TT](1-p) \\ &= (2+E[X])p + (2+1/p)(1-p) \end{aligned}$$

where the final equality follows since given that the first two flips are tails the number of additional flips is just the number of flips needed to obtain a head. Putting the preceding together yields

$$E[X] = (1+E[X])p + (2+E[X])p(1-p) + (2+1/p)(1-p)^2$$

or

$$E[X] = \frac{1}{p(1-p)^2}$$

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Let  $N$  denote the number of minutes in the maze. If  $L$  is the event the rat chooses its left, and  $R$  the event it chooses its right, we have by conditioning on the first direction chosen:

$$\begin{aligned} E(N) &= \frac{1}{2}E(N|L) + \frac{1}{2}E(N|R) \\ &= \frac{1}{2}\left[\frac{1}{3}(2) + \frac{2}{3}(5+E(N))\right] + \frac{1}{2}[3+E(N)] \\ &= \frac{5}{6}E(N) + \frac{21}{6} \\ &= 21. \end{aligned}$$