STAT 407

Solutions to assignment #2

Chapter 4

(8) Let the state on any day be the number of the coin that is flipped on that day.

$$\mathbf{P} = \begin{bmatrix} .7 & .3 \\ .6 & .4 \end{bmatrix}$$

and so,

$$\mathbf{P}^2 = \begin{bmatrix} .67 & .33 \\ .66 & .34 \end{bmatrix}$$

and

$$\mathbf{P}^3 = \begin{bmatrix} .667 & .333 \\ .666 & .334 \end{bmatrix}$$

Hence,

$$\tfrac{1}{2}\left[P_{11}^3 + P_{21}^3\right] \equiv .6665.$$

(9) To answer this question, we do not need to keep track of which coin is tossed. So let the state be 0 when the most recent flip lands heads and let it equal 1 when it lands tails. Then the sequence of states is a Markov chain with transition probability matrix

$$\begin{array}{c|cc} & 0 & 1 \\ \hline 0 & .7 & .3 \\ 1 & .6 & .4 \end{array}$$

(continued)

9 (continued)

So the desired probability is $P^4_{0,0}$.

$$P^4 = P^3 \cdot P = \begin{Vmatrix} .667 & .333 \\ .666 & .334 \end{Vmatrix} \cdot \begin{Vmatrix} .7 & .3 \\ .6 & .4 \end{Vmatrix}$$

So $P^4_{0,0} = .667(.7) + (.333)(.6) = \underline{\underline{0.6667}}$.

14 (i) $P = \begin{Vmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{Vmatrix}$

So $0 \to 1$, $0 \to 2$, $1 \to 0$, $1 \to 2$, $2 \to 0$, $2 \to 1$

Clearly $0 \leftrightarrow 1$, so $0, 1$ are in the same class.
Also
$0 \to 2$, $2 \to 0$, so $0 \leftrightarrow 2$, so 2 is in the same class as 0. Hence $\{0, 1, 2\}$ is a single class, which is recurrent (because there are a finite # of states).

(ii) $P_2 = \begin{Vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{Vmatrix}$

$0 \to 3$
$1 \to 3$
$2 \to 0$, $2 \to 1$
$3 \to 2$

$0 \to 3 \to 2 \to 1$. shows $0 \to 1$.
Also $1 \to 3 \to 2 \to 0$ shows $1 \to 0$. Hence $0 \leftrightarrow 1$. So $0, 1$ in same class.
Also $2 \to 1$, and $1 \to 3 \to 2$ shows $1 \leftrightarrow 2$. So $0, 1, 2$ are in same class.
Finally $3 \to 2$ and $2 \to 0 \to 3$ shows $2 \leftrightarrow 3$. So $\{0, 1, 2, 3\}$ is one class, necessarily recurrent.

14 (continued)

(iii) $P_3 = \left\| \begin{matrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{matrix} \right\|$

So $0 \to 0, 0 \to 2$
$1 \to 0, 1 \to 1, 1 \to 2$
$2 \to 0, 2 \to 2$
$3 \to 3, 3 \to 4$
$4 \to 3, 4 \to 4$

Clearly $3 \leftrightarrow 4$. Since neither 3 nor 4 are accessible from any other state, $\{3,4\}$ must constitute a class, necessarily recurrent.

Now $0 \to 2$, $2 \to 0$, so $0 \leftrightarrow 2$, hence 0 and 2 are in the same class. It remains to be determined whether 1 is the class with 0, 2 or is in a class by itself.

Although $1 \to 0$, $1 \to 2$, we do not have either $2 \to 1$ or $0 \to 1$.

Hence $\{0,2\}$ is a class, and $\{1\}$ is a class. Since the M.C. chain eventually with prob 1 leaves state 1, $\{1\}$ is a transient class.

$\{0,2\}$ is a recurrent class, because the corresponding submatrix of transition probabilities is $\begin{array}{c|c} & \begin{matrix} 0 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 2 \end{matrix} & \left\| \begin{matrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{matrix} \right\| \end{array}$.

(iv) By similar arguments, the classes are: $\{0,1\}$ recurrent, $\{2\}$ recurrent, $\{3\}$ transient, $\{4\}$ transient.

18 If the state at time n is the n^{th} coin to be flipped then sequence of consecutive states constitute a two state Markov chain with transition probabilities

$$P_{1,1} = .6 = 1 - P_{1,2}, \quad P_{2,1} = .5 = P_{2,2}$$

(a) The stationary probabilities satisfy

$$\begin{aligned} \pi_1 &= .6\pi_1 + .5\pi_2 \\ \pi_1 + \pi_2 &= 1 \end{aligned}$$

Solving yields that $\pi_1 = 5/9$, $\pi_2 = 4/9$. So the proportion of flips that use coin 1 is 5/9.
(b) $P^4_{1,2} = .44440$

19 The limiting probabilities are obtained from

$$\pi_0 = .7\pi_0 + .5\pi_1$$

$$\pi_1 = .4\pi_2 + .2\pi_3$$

$$\pi_2 = .3\pi_0 + .5\pi_1$$

$$\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1,$$

and the solution is

$$\pi_0 = \frac{1}{4}, \pi_1 = \frac{3}{20}, \pi_2 = \frac{3}{20}, \pi_3 = \frac{9}{20}.$$

The desired result is thus

$$\pi_0 + \pi_1 = \frac{2}{5}.$$

24 Let the state be the color of the last ball selected, call it 0 if that color was red, 1 if white, and 2 if blue. The transition probability matrix of this Markov chain is

$$\mathbf{P} = \begin{bmatrix} 1/5 & 0 & 4/5 \\ 2/7 & 3/7 & 2/7 \\ 3/9 & 4/9 & 2/9 \end{bmatrix}$$

Solve for the stationary probabilities to obtain the solution. (Omitted — straightforward but tedious).

27 **The limiting probabilities are obtained from**

$$\pi_0 = \frac{1}{9}\pi_1$$

$$\pi_1 = \pi_0 + \frac{4}{9}\pi_1 + \frac{4}{9}\pi_2$$

$$\pi_2 = \frac{4}{9}\pi_1 + \frac{4}{9}\pi_2 + \pi_3$$

$$\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1,$$

and the solution is $\pi_0 = \pi_3 = \frac{1}{20}$. $\pi_1 = \pi_2 = \frac{9}{20}$.

28 **There are 4 states:** **1 = success on last 2 trials;**

2 = success on last, failure on next to last;

3 = failure on last, success on next to last;

4 = failure on last 2 trials.

(continued)

Transition probabilities are:

$$P_{1,1} = \frac{3}{4}, P_{1,3} = \frac{1}{4}$$

$$P_{2,1} = \frac{2}{3}, P_{2,3} = \frac{1}{3}$$

$$P_{3,2} = \frac{2}{3}, P_{3,4} = \frac{1}{3}$$

$$P_{4,2} = \frac{1}{2}, P_{4,4} = \frac{1}{2}.$$

Limiting probabilities are given by

$$\Pi_1 = \frac{3}{4}\Pi_1 + \frac{2}{3}\Pi_2$$

$$\Pi_2 = \frac{2}{3}\Pi_3 + \frac{1}{2}\Pi_4$$

$$\Pi_3 = \frac{1}{4}\Pi_1 + \frac{1}{3}\Pi_2$$

$$\Pi_1 + \Pi_2 + \Pi_3 + \Pi_4 = 1,$$

and the solution is $\Pi_1 = 1/2$, $\Pi_2 = 3/16$, $\Pi_3 = 3/16$, $\Pi_4 = 1/8$. Hence, the desired answer is $\Pi_1 + \Pi_2 = 11/16$.

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Each employee moves according to a Markov chain whose limiting probabilities are the solution of

$$\Pi_1 = .7\Pi_1 + .2\Pi_2 + .1\Pi_3$$
$$\Pi_2 = .2\Pi_1 + .6\Pi_2 + .4\Pi_3$$

$$\Pi_1 + \Pi_2 + \Pi_3 = 1.$$

Solving yields $\Pi_1 = 6/17$, $\Pi_2 = 7/17$, $\Pi_3 = 4/17$. Hence, if N is large, it follows from the law of large numbers that approximately 6, 7, and 4 of each 17 employees are in categories 1, 2, and 3.