STAT 407

Solutions to problems in Chapters 7 and 10

CHAPTER 7

1. (a) Yes, (b) no, (c) no.

2. (a) S_n is Poisson with mean $n\mu$.

   (b) $P\{N(t) = n\} = P\{N(t) \geq n\} - P\{N(t) \geq n+1\}$

$$= P\{S_n \leq t\} - P\{S_{n+1} \leq t\}$$

$$= \sum_{k=0}^{[t]} e^{-n\mu}(n\mu)^k/k! - \sum_{k=0}^{[t]} e^{-(n+1)\mu}[(n+1)\mu]^k/k!,$$

where $[t]$ is the largest integer not exceeding t.

5. The random variable N is equal to $N(1)+1$ where $\{N(t)\}$ is the renewal process whose interarrival distribution is uniform on $(0, 1)$. By the results of Example 2c,

$$E[N] = m(1) + 1 = e.$$

7. Once every five months.

2

(9.) A job completion constitutes a renewal. Let T denote the time between renewals. To compute $E[T]$ start by conditioning on W, the time it takes to finish the next job.

$$E[T] = E[E[T|W]] \ .$$

Now, to determine $E[T|W = w]$ condition on S, the time of the next shock. This gives

$$E[T|W = w] = \int_0^\infty E[T|W = w, S = x] \ \lambda e^{-\lambda x}dx$$

Now, if the time to finish is less than the time of the shock then the job is completed at the finish time; otherwise everything starts over when the shock occurs. This gives

$$E[T|W = w, S = x] = \begin{array}{ll} x + E[T] & \text{if } x < w \\ w & \text{if } x \geq w \end{array} .$$

Hence,

$$E[T|W = w] = \int_0^w (x+E[T])\lambda e^{-\lambda x}dx + w\int_w^\infty \lambda e^{-\lambda x}dx$$

$$= E[T][1-e^{-\lambda w}] + 1/\lambda - we^{-\lambda w} - \frac{1}{\lambda}e^{-\lambda w} + we^{-\lambda w}$$

Thus,

$$E[T|W] = (E[T] + 1/\lambda)(1 - e^{-\lambda W}).$$

Taking expectations gives

$$E[T] = (E[T] + 1/\lambda)(1 - E[e^{-\lambda W}]),$$

and so

$$E[T] = \frac{1 - E[e^{-\lambda W}]}{\lambda E[e^{-\lambda W}]} \ .$$

In the above, W is a random variable having distribution F and so

$$E[e^{-\lambda W}] = \int_0^\infty e^{-\lambda w}f(w)dw \ .$$

(10) yes. ρ/μ

CHAPTER 10

2. The conditional distribution $X(s) - A$ given that $X(t_1) = A$ and $X(t_2) = B$ is the same as the conditional distribution of $X(s-t_1)$ given that $X(0) = 0$ and $X(t_2-t_1) = B - A$, which by (1.4) is normal with mean $\frac{s-t_1}{t_2-t_1}(B-A)$ and variance $\frac{(s-t_1)}{t_2-t_1}(t_2-s)$. Hence the desired conditional distribution is normal with mean $A + \frac{(s-t_1)(B-A)}{t_2-t_1}$ and variance $\frac{(s-t_1)(t_2-s)}{t_2-t_1}$.

6. The probability of recovering your purchase price is the probability that a Brownian motion goes up c by time t. Hence the desired probability is

$$1 - P\left\{\max_{0 \le s \le t} X(s) \ge c\right\} = 1 - \frac{2}{\sqrt{2\pi}} \int_{a/\sqrt{t}}^{\infty} e^{-y^2/2}\, dy.$$

7. Condition on $X(t_1)$:

$$(*) \quad P\left\{\max_{t_1 \le s \le t_2} X(s) > x\right\}$$

$$= \int_{-\infty}^{\infty} P\left\{\max_{t_1 \le s \le t_2} X(s) > x \mid X(t_1) = y\right\} \frac{1}{\sqrt{2\pi t_1}} e^{-y^2/2}\, dy.$$

$$\mathrm{Var}(X_i) = E\left[X_i^2\right] - (E[X_i])^2$$

$$= 1 - \mu^2 \Delta t \quad \text{since } X_i^2 = 1$$

we obtain that

$$E[X(t)] = \sqrt{\Delta t}\left[\frac{t}{\Delta t}\right]\mu\sqrt{\Delta t}$$

$$\longrightarrow \mu t \text{ as } \Delta t \longrightarrow 0$$

$$\mathrm{Var}(X(t)) = \Delta t\left[\frac{t}{\Delta t}\right](1-\mu^2\Delta t)$$

$$\longrightarrow t \text{ as } \Delta t \longrightarrow 0.$$

(continued)

7 (continued)

(b) By the gambler's ruin problem the probability of going up A before going down B is

$$\frac{1-(q/p)^B}{1-(q/p)^{A+B}}$$

when each step is either up 1 or down 1 with probabilities p and $q = 1 - p$. (This is the probability that a gambler starting with B will reach his goal of A + B before going broke.) Now, when $p = \frac{1}{2}(1+\mu\sqrt{\Delta t})$, $q = 1 - p = \frac{1}{2}(1-\mu\sqrt{\Delta t})$ and so $q/p = \frac{1-\mu\sqrt{\Delta t}}{1+\mu\sqrt{\Delta t}}$. Hence, in this case the probability of going up $A/\sqrt{\Delta t}$ before going down $B/\sqrt{\Delta t}$ (we divide

Now,

$$P\left\{\max_{t_1 \le s \le t_2} X(s) > x \mid X(t_1) = y\right\}$$

$$= \begin{cases} P\{\max\limits_{0<s<t_2-t_1} X(s) > x - y\} & \text{if } y < x \\ 1 & \text{if } x > y. \end{cases}$$

Substitution into the above equation (*) now gives the desired result [when one uses that

$$P\left\{\max_{0<s<t_2-t_1} X(s) > x-y\right\} = 2P\{X(t_2-t_1) > x-y\}\Big].$$

Ch 10, 9

Write $X(t) = \sigma B(t) + \mu t$, where $B(t)$ is standard Brownian motion. It was shown in class that, for $s < t$, the joint distribution of $B(s)$ and $B(t)$ is Bivariate Normal with parameters $(0, 0, s, t, \rho = \sqrt{s/t})$. Since $(X(s), X(t))$ is a simple linear transformation of $(B(s), B(t))$, the joint distribution of $(X(s), X(t))$ is also Bivariate Normal. We have

$$E(X(s)) = \sigma E\, B(s) + \mu s = \mu s, \quad Var(X(s)) = \sigma^2 Var(B(s)) = \sigma^2 s,$$

$$E[X(t)] = \mu t, \quad Var(X(t)) = \sigma^2 t.$$

Also $\rho_{X(s), X(t)} = \rho_{B(s), B(t)} = \sqrt{s/t}$, since correlation is invariant under linear transformations. So the joint distribution of $(X(s), X(t))$ is $BVN(\mu s, \mu t, \sigma^2 s, \sigma^2 t, \rho)$