STAT 407

Solutions to problems in Chapter 5

(15) $E[\text{Time}] = E[T_1 + T_2 + T_3]$ where T_i is the time between the (i-1) and the i^{th} departure. Now T_1 and T_2 are exponential with rate 2λ since they are both the minimum of two exponentials each of which has rate λ. Also, T_3 is exponential with rate λ. Thus,

$$E[T] = E[T_1] + E[T_2] + E[T_3]$$
$$= 1/(2\lambda) + 1/(2\lambda) + 1/\lambda = 2/\lambda.$$

(16) Using the same representation as in the solution to Problem 15 we have that

$$E[T] = E[T_1] + E[T_2] + E[T_3]$$

The random variables T_1 and T_2 are both exponential with rate $\lambda_1 + \lambda_2$, and so have mean $1/(\lambda_1 + \lambda_2)$. To determine $E[T_3]$ consider the time at which the first customer has departed and condition on which server completes the next service. This gives

$$E[T_3] = E[T_3|\text{server }1]\,[\lambda_1/(\lambda_1 + \lambda_2)]$$
$$+ E[T_3|\text{server }2]\,[\lambda_2/(\lambda_1 + \lambda_2)]$$
$$= (1/\lambda_2)[\lambda_1/(\lambda_1 + \lambda_2)] + (1/\lambda_1)[\lambda_2/(\lambda_1 + \lambda_2)].$$

Therefore,

$$E[\text{Time}] = 2/(\lambda_1+\lambda_2) + (1/\lambda_2)[\lambda_1/(\lambda_1 + \lambda_2)]$$
$$+ (1/\lambda_1)[\lambda_2/(\lambda_1 + \lambda_2)].$$

21 $E[\text{time}] =$

$E[\text{time waiting at 1}] + 1/\mu_1 + E[\text{time waiting at 2}] + 1/\mu_2$.

Now

$$E[\text{time waiting at 1}] = 1/\mu_1,$$
$$E[\text{time waiting at 2}] = (1/\mu_2)\frac{\mu_1}{\mu_1+\mu_2}.$$

The last equation follows by conditioning on whether or not the customer waits for server 2. Therefore,

$$E[\text{time}] = 2/\mu_1 + (1/\mu_2)[1 + \mu_1/(\mu_1+\mu_2)] .$$

22 $E[\text{time}] = E[\text{time waiting for server 1}] + 1/\mu_1$

$+ E[\text{time waiting for server 2}] + 1/\mu_2$.

Now, the time spent waiting for server 1 is the remaining service time of the customer with server 1 plus any additional time due to that customer blocking your entrance. If server 1 finishes before server 2 this additional time will equal the additional service time of the customer with server 2. Therefore,

$$E[\text{time waiting for server 1}] = 1/\mu_1 + E[\text{Additional}]$$
$$= 1/\mu_1 + (1/\mu_2)[\mu_1/(\mu_1+\mu_2)] .$$

Since when you enter service with server 1 the customer preceding you will be entering service with server 2, it follows that you will have to wait for server 2 if you finish service first. Therefore, conditioning on whether or not you finish first

$$E[\text{time waiting for server 2}] = (1/\mu_2)[\mu_1/(\mu_1+\mu_2)] .$$

Thus,

$$E[\text{time}] = 2/\mu_1 + (2/\mu_2)[\mu_1/(\mu_1+\mu_2)] + 1/\mu_2 .$$

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(a) 1/2.

(b) $(1/2)^{n-1}$: whenever battery 1 is in use and a failure occurs the probability is 1/2 that it is not battery 1 that has failed.

(c) $(1/2)^{n-i+1}$, $i > 1$.

(d) T is the sum of n−1 independent exponentials with rate 2μ (since each time a failure occurs the time until the next failure is exponential with rate 2μ).

(e) Gamma with parameters n−1 and 2μ.

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$$(a)\quad \frac{1}{\mu_1+\mu_2+\mu_3} + \sum_{i=1}^{3} \frac{\mu_i}{\mu_1+\mu_2+\mu_3}\frac{1}{\mu_i} = \frac{4}{\mu_1+\mu_2+\mu_3}$$

$$(b)\quad \frac{1}{\mu_1+\mu_2+\mu_3} + (a) = \frac{5}{\mu_1+\mu_2+\mu_3}$$

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$$(a)\quad \frac{\mu_1}{\mu_1+\mu_3}$$

$$(b)\quad \frac{\mu_1}{\mu_1+\mu_3}\frac{\mu_2}{\mu_2+\mu_3}$$

$$(c)\quad \sum_i \frac{1}{\mu_i} + \frac{\mu_1}{\mu_1+\mu_3}\frac{\mu_2}{\mu_2+\mu_3}\frac{1}{\mu_3}$$

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$$\sum_i \frac{1}{\mu_i} + \frac{\mu_1}{\mu_1+\mu_2}\left[\frac{1}{\mu_2} + \frac{\mu_2}{\mu_2+\mu_3}\frac{1}{\mu_3}\right] + \frac{\mu_2}{\mu_1+\mu_2}\frac{\mu_1}{\mu_1+\mu_3}\frac{\mu_2}{\mu_2+\mu_3}\frac{1}{\mu_3}$$

29 (a) $\dfrac{\lambda}{\lambda+\mu_A}$ (b) $\dfrac{\lambda+\mu_A}{\lambda+\mu_A+\mu_B}\cdot\dfrac{\lambda}{\lambda+\mu_B}$

(30) Condition on which animal died to obtain

$$E[\text{additional life}] = E[\text{additional life—dog died}]\frac{\lambda_d}{\lambda_c+\lambda_d} + E[\text{additional life—cat died}]\frac{\lambda_c}{\lambda_c+\lambda_d}$$
$$= \frac{1}{\lambda_c}\frac{\lambda_d}{\lambda_c+\lambda_d} + \frac{1}{\lambda_d}\frac{\lambda_c}{\lambda_c+\lambda_d}$$

(31) Condition on whether the 1 PM appointment is still with the doctor at 1:30, and use the fact that if she or he is then the remaining time spent is exponential with mean 30. This gives

$$E[\text{time spent in office}] = 30(1-e^{-30/30}) + (30+30)e^{-30/30}$$
$$= 30 + 30e^{-1}$$

(37) For $s = 2/60, 5/60, 10/60, 20/60$ minutes, we see that

$$e^{-3s} = e^{-1/10}, e^{-1/4}, e^{-1/2}, e^{-1}.$$

(38) $e^{-3s} + 3se^{-3s}$.

(42) a) $E[S_4] = 4/\lambda$.

(b) $E[S_4 | N(1) = 2] = 1 + E[\text{time for 2 more events}] = 1 + 2/\lambda$.

(c) $E[N(4) - N(2) | N(1) = 3] = E[N(4) - N(2)]$
$= 2\lambda$.

The first equality used the independent increments property.

(43) Let S_i denote the service time at server $i, i = 1, 2$, and let X denote the time until the next arrival. Then, with p denoting the proportion of customers that are served by both servers, we have

$$p = P\{X > S_1 + S_2\} = P\{X > S_1\}P\{X > S_1 + S_2 | X > S_1\} = \frac{\mu_1}{\mu_1+\lambda}\frac{\mu_2}{\mu_2+\lambda}$$

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Given T, the time until the next arrival, N, the number of busy servers found by the next arrival, is a binomial random variable with parameters n and $p = e^{-\mu T}$.

(a)

$$E[N] = \int E[N|T=t]\lambda e^{-\lambda t}dt = \int ne^{-\mu t}\lambda e^{-\lambda t}dt = \frac{n\lambda}{\lambda+\mu}$$

For (b) and (c), you can either condition on T, or use the approach of part (a) of Exercise 14 to obtain

$$P\{N=0\} = \prod_{j=1}^{n} \frac{(n-j+1)\mu}{\lambda+(n-j+1)\mu}$$

$$P\{N=n-i\} = \frac{\lambda}{\lambda+(n-i)\mu}\prod_{j=1}^{i} \frac{(n-j+1)\mu}{\lambda+(n-j+1)\mu}$$

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Let T denote the time until the next train arrives; and so T is uniform on (0, 1). Note that, conditional on T, X is Poisson with mean 7T.

(a) $E[X] = E[E[X|T]] = E[7T] = 7/2$.

(b) $E[X|T] = 7T$, $Var(X|T) = 7T$. **By the conditional variance formula**

$$Var(X) = 7E[T] + 49Var(T) = 7/2 + 49/12 = 91/12.$$

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(c) $1 - 5e^{-4}$.

(a) e^{-2t}. **(b)** $2t$.

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The unconditional probability that the claim is type 1 is 10/11. Therefore

$$P(1|4000) = \frac{P(4000|1)P(1)}{P(4000|1)P(1) + P(4000|2)P(2)} = \frac{e^{-4}10/11}{e^{-4}10/11 + .2e^{-.8}1/11}$$

61 (a) Poisson with mean $cG(t)$.

(b) Poisson with mean $c[1-G(t)]$.

(c) Independent.

63 . Let X and Y be respectively the number of customers in the system at time t+s that were present at time s, and the number in the system at t+s that were not in the system at time s. Since there are an infinite number of servers, it follows that X and Y are independent (even if given the number in the system at time s). Since the service distribution is exponential with rate μ, it follows that given that $X(s) = n$, X will be Binomial with parameters n and $p = e^{-\mu t}$. Also Y, which is independent of X(s), will have the same distribution as X(t). Therefore, Y is Poisson with mean $\lambda \int_0^t e^{-\mu y} dy = \lambda(1 - e^{-\mu t})/\mu$.

(a) $E[X(t+s)|X(s) = n] = E[X|X(s) = n] + E[Y|X(s) = n].$

$= ne^{-\mu t} + \lambda(1 - e^{-\mu t})/\mu$.

(b) $Var(X(t+s)|X(s) = n) = Var(X + Y|X(s) = n)$

$= Var(X|X(s) = n) + Var(Y)$

$= ne^{-\mu t}(1-e^{-\mu t}) + \lambda(1 - e^{-\mu t})/\mu$.

The above equation uses the formulas for the variances of a Binomial and a Poisson random variable.

77 $e^{-11}(11)^n/n!$