UNIVERSITY OF CALGARY

DEPARTMENT OF MATHEMATICS AND STATISTICS

STAT 407 — Test no. 1 — Oct. 20, 2006

Marks

NAME: Solutions

10 1. The joint density of X and Y is given by

$$f(x,y) = \frac{e^{-y}}{y}, \quad 0 < x < y, \quad 0 < y < \infty.$$

Compute $E[X|Y=y]$.

3 marks

For $y>0$, $f(x|y) = \frac{f(x,y)}{f(y)}$, $\quad f(y) = \int_0^y \frac{e^{-y}}{y}\,dx = \frac{e^{-y}}{y}\cdot y = e^{-y},\ y>0$

$f(x|y) = \frac{e^{-y}}{y}/e^{-y} = \frac{1}{y}, \ 0<x<y$ ← ③

So $\underbrace{E[X|Y=y] = \int_{x=0}^{y} x f(x|y)\,dx}_{②} = \int_{x=0}^{y} x \frac{1}{y}\,dy = \frac{1}{y}\left.\frac{x^2}{2}\right]_0^y = \frac{1}{y}\frac{y^2}{2} = \frac{y}{2}, \ 0 \le x < y.$

②

10 2. A rat is trapped in a maze. Initially he has to choose one of four directions. If he goes North, then he will depart the maze after one minute of travelling. If he goes East, he will wander around in the maze for three minutes and will then return to his initial position. If he goes South, he will wander around in the maze for ten minutes and will then return to his initial position. If he goes West, he will depart the maze after five minutes of travelling. Assume that each time the rat has a choice of the four directions, he is equally likely to make any one of the four choices. Find the expected number of minutes that the rat will be trapped in the maze.

① → $E(T) = E[T|N]\,P(N) + E[T|E]\,P(E) + E[T|S]\,P(S) + E[T|W]\,P(W)$

$P(N) = P(E) = P(S) = P(N) = \frac{1}{4}$

$E[T|N] = 1$, $E[T|E] = 3 + ET$, $E[T|S] = 10 + ET$, $E[T|W] = 5$

① ① ① ①

③ So $E(T) = \frac{1}{4}[1 + 3 + E(T) + 10 + E(T) + 5]$

$= \frac{1}{4}[19 + 2E(T)]$

So $4E(T) = 19 + 2E(T)$, or $2E(T) = 19$, or $E(T) = \frac{19}{2} = 9.5$ min

②

ID number: ______________

10

3. Consider the Markov chain consisting of five states 0, 1, 2, 3, 4 — and having transition probability matrix

$$P = \left\| \begin{array}{ccccc} 0.6 & 0 & 0.4 & 0 & 0 \\ 0 & 0.9 & 0 & 0 & 0.1 \\ 0.2 & 0 & 0.8 & 0 & 0 \\ 0.1 & 0.2 & 0.3 & 0.3 & 0.1 \\ 0 & 0.3 & 0 & 0 & 0.7 \end{array} \right\|$$

Find the classes of this Markov chain, and determine which classes are recurrent and which are transient.

$0 \to 0$, $0 \to 2$

$1 \to 1$, $1 \to 4$

$2 \to 0$, $2 \to 2$

$3 \to 0$, $3 \to 1$, $3 \to 2$, $3 \to 3$, $3 \to 4$

$4 \to 1$, $4 \to 4$

$0 \to 2$, $2 \to 0 \Rightarrow 0 \leftrightarrow 2$

By inspection, states 1, 3, 4 are not accessible from 0 or 2.

So $\{0,2\}$ is a class.

$1 \leftrightarrow 4$, since $4 \to 1$ and $1 \to 4$, so 1 and 4 are in the same class.

$3 \to 1$ but $1 \not\to 3$, so $\{1,4\}$ is a class and $\{3\}$ is a class.

Clearly $\{0,2\}$ and $\{1,4\}$ are recurrent classes, since whenever the M.C. enters one of these classes, it never leaves after that.

$\{3\}$ is transient, since $P\{X_{n+1} = 0,1,2,4 \mid X_n = 3\} > 0$, hence the M.C. will eventually (with prob 1) leave state 3 and never return.

So: $\{0,2\}$ recurrent, $\{1,4\}$ recurrent, $\{3\}$ transient.

2 1 2 1 2 2

10 4. Suppose that you have a fair coin and a fair die. Consider a sequence of coin-tosses and die-rollings according to the following rules.

If you toss the coin at time n and Heads comes up, then you toss the coin at time $(n+1)$; otherwise you roll the die at time $(n+1)$.

If you roll the die at time n, and either a 1 or a 2 comes up, then you toss the coin at time $(n+1)$; otherwise (i.e. if a 3,4,5 or 6 comes up) , you roll the die at time $(n+1)$.

For $n = 0, 1, 2, \ldots$, let $X_n = 0$ if the coin is tossed, and let $X_n = 1$ if the die is rolled.

4 (a) Find the transition probability matrix for this Markov chain.

$$\begin{array}{c|cc|} & 0 & 1 \\ \hline 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & \frac{1}{3} & \frac{2}{3} \end{array}$$

(1 mark per correct entry in matrix).

3 (b) Find the probability that the die is rolled at time 7, given that the coin is tossed at time 5.

The answer is P^2_{01}. $\leftarrow$ ①

$$P^2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{4}+\frac{1}{6} & \frac{1}{4}+\frac{2}{6} \\ \frac{1}{6}+\frac{2}{9} & \frac{1}{6}+\frac{4}{9} \end{pmatrix} = \begin{pmatrix} \frac{5}{12} & \frac{7}{12} \\ \frac{7}{18} & \frac{11}{18} \end{pmatrix} \leftarrow ①$$

So $P^2_{01} = \underline{\underline{\frac{7}{12}}} = 0.5333$

3 (c) Find the limiting proportion of the times (as the number of transitions approaches infinity) that the coin is tossed.

$$① \rightarrow \left.\begin{array}{l} \pi_0 = \frac{1}{2}\pi_0 + \frac{1}{3}\pi_1 \\ \left(\pi_1 = \frac{1}{2}\pi_0 + \frac{2}{3}\pi_1\right) \end{array}\right\} \Rightarrow \frac{1}{2}\pi_0 = \frac{1}{3}\pi_1 \Rightarrow \pi_1 = \frac{3}{2}\pi_0$$

$$① \rightarrow \pi_0 + \pi_1 = 1.$$

So $\pi_0 + \frac{3}{2}\pi_0 = 1$ OR $\frac{5}{2}\pi_0 = 1$, so $\pi_0 = \underline{\underline{\frac{2}{5}}} = 0.4$

$\uparrow$ ①