# STAT 407

## Solutions to Assignment #1

### Problems in Chapter 3:

9. $E[X|Y=y] = \sum_x xP\{X=x|Y=y\}$

$= \sum_x xP\{X=x\}$ by independence

$= E[X]$.

15. $$f_{X|Y=y}(x|y) = \frac{\frac{1}{y}\exp^{-y}}{f_y(y)} = \frac{\frac{1}{y}\exp^{-y}}{\int_0^y \frac{1}{y}\exp^{-y}\,dx}$$

$$= \frac{1}{y}, \quad 0 < x < y$$

$$E[X^2|Y=y] = \frac{1}{y}\int_0^y x^2 dx = \frac{y^2}{3}.$$

21. (a) $X = \sum_{i=1}^{N} T_i$.

(b) Clearly $N$ is geometric with parameter 1/3; thus, $E[N] = 3$.

(c) Since $T_N$ is the travel time corresponding to the choice leading to freedom it follows that $T_N = 2$, and so $E[T_N] = 2$.

(d) Given that $N = n$, the travel times $T_i i = 1, \ldots, n-1$ are each equally likely to be either 3 or 5 (since we know that a door leading back to the nine is selected), whereas $T_n$ is equal to 2 (since that choice led to safety). Hence,

$$E\left[\sum_{i=1}^{N} T_i | N=n\right] = E\left[\sum_{i=1}^{n-1} T_i | N=n\right] + E[T_n|N=n]$$
$$= 4(n-1) + 2.$$

(continued)

21 (continued)

(e) Since part (d) is equivalent to the equation

$$E\left[\sum_{i=1}^{N} T_i | N\right] = 4N - 2,$$

we see from parts (a) and (b) that

$$E[X] = 4E[N] - 2$$
$$= 10.$$

22. Letting $N_i$ denote the time until the same outcome occurs $i$ consecutive times we obtain, upon conditioning $N_{i-1}$, that

$$E[N_i] = E[E[N_i|N_{i-1}]].$$

Now,

$$E[N_i|N_{i-1}]$$
$$= N_{i-1} + \begin{cases} 1 \text{ with probability } 1/n \\ E[N_i] \text{ with probability}(n-1)/n \end{cases}$$

The above follows because after a run of $i-1$ either a run of $i$ is attained if the next trial is the same type as those in the run or else if the next trial is different then it is exactly as if we were starting all over at that point.

From the above equation we obtain that

$$E[N_i] = E[N_{i-1}] + 1/n + E[N_i](n-1)/n.$$

Solving for $E[N_i]$ gives

$$E[N_i] = 1 + nE[N_{i-1}].$$

Solving recursively now yields

$$E[N_i] = 1 + n\{1 + nE[N_{i-2}]\}$$
$$= 1 + n + n^2 E[N_{i-2}]$$
$$\cdot$$
$$\cdot$$
$$= 1 + n + \cdots + n^{k-1} E[N_1]$$
$$= 1 + n + \cdots + n^{k-1}.$$

27. Condition on the outcome of the first flip to obtain

$$E[X] = E[X|H]p + E[X|T](1-p)$$
$$= (1 + E[X])p + E[X|T](1-p)$$

Conditioning on the next flip gives

$$E[X|T] = E[X|TH]p + E[X|TT](1-p)$$
$$= (2 + E[X])p + (2 + 1/p)(1-p)$$

where the final equality follows since given that the first two flips are tails the number of additional flips is just the number of flips needed to obtain a head. Putting the preceding together yields

$$E[X] = (1 + E[X])p + (2 + E[X])p(1-p) + (2 + 1/p)(1-p)^2$$

or

$$E[X] = \frac{1}{p(1-p)^2}$$

40. Let $X$ denote the number of door chosen, and let $N$ be the total number of days spent in jail.

(a) Conditioning on $X$, we get

$$E[N] = \sum_{i=1}^{3} E\{N|X = i\}P\{X = 1\}.$$

The process restarts each time the prisoner returns to his cell. Therefore,

$$E(N|X = 1) = 2 + E(N)$$
$$E(N|X = 2) = 3 + E(N)$$
$$E(N|X = 3) = 0.$$

and

$$E(N) = (.5)(2 + E(N)) + (.3)(3 + E(N)) + (.2)(0),$$

or

$$E(N) = 9.5 \text{ days.}$$

(b) Let $N_i$ denote the number of additional days the prisoner spends after having initially chosen cell $i$.

$$E[N] = \frac{1}{3}(2 + E[N_1]) + \frac{1}{3}(3 + E[N_2]) + \frac{1}{3}(0)$$
$$= \frac{5}{3} + \frac{1}{3}(E[N_1] + E[N_2]).$$

Now,

$$E[N_1] = \frac{1}{2}(3) + \frac{1}{2}(0) = \frac{3}{2}$$
$$E[N_2] = \frac{1}{2}(2) + \frac{1}{2}(0) = 1$$

and so,

$$E[N] = \frac{5}{3} + \frac{1}{3}\frac{5}{2} = \frac{5}{2}.$$

41. Let $N$ denote the number of minutes in the maze. If $L$ is the event the rat chooses its left, and $R$ the event it chooses its right, we have by conditioning on the first direction chosen:

$$\begin{aligned} E(N) &= \frac{1}{2}E(N|L) + \frac{1}{2}E(N|R) \\ &= \frac{1}{2}\left[\frac{1}{3}(2) + \frac{2}{3}(5 + E(N))\right] + \frac{1}{2}[3 + E(N)] \\ &= \frac{5}{6}E(N) + \frac{21}{6} \\ &= 21. \end{aligned}$$

43. Using Examples 4d and 4e, mean $= \mu_1\mu_2$, variance $= \mu_1\sigma_2^2 + \mu_2^2\sigma_1^2$.