STAT 407

Solutions to Assignment #2

Chapter 4

(4.6) First we check that the formula for $P^{(n)}$ is correct when $n=1$:

$$P^{(1)} = \begin{Vmatrix} \frac{1}{2}+\frac{1}{2}(2p-1) & \frac{1}{2}-\frac{1}{2}(2p-1) \\ \frac{1}{2}-\frac{1}{2}(2p-1) & \frac{1}{2}+\frac{1}{2}(2p-1) \end{Vmatrix} = \begin{Vmatrix} p & (1-p) \\ 1-p & p \end{Vmatrix} \overset{\text{def.}}{=} P \quad \checkmark$$

Now suppose the formula for $P^{(n)}$ is true for some fixed n. Then

$$P^{(n+1)} = P^{(n)} \cdot P = \begin{Vmatrix} \frac{1}{2}+\frac{1}{2}(2p-1)^n & \frac{1}{2}-\frac{1}{2}(2p-1)^n \\ \frac{1}{2}-\frac{1}{2}(2p-1)^n & \frac{1}{2}+\frac{1}{2}(2p-1)^n \end{Vmatrix} \cdot \begin{Vmatrix} p & 1-p \\ 1-p & p \end{Vmatrix}$$

Let's just check the upper left hand element of $P^{(n+1)}$: It is

$$\left[\tfrac{1}{2}+\tfrac{1}{2}(2p-1)^n\right]p + \left[\tfrac{1}{2}-\tfrac{1}{2}(2p-1)^n\right](1-p) =$$

$$\tfrac{1}{2}p + \tfrac{1}{2}p(2p-1)^n + \tfrac{1}{2}(1-p) - \tfrac{1}{2}(1-p)(2p-1)^n =$$

$$\tfrac{1}{2} + (2p-1)^n\left[\tfrac{1}{2}p - \tfrac{1}{2}(1-p)\right] = \tfrac{1}{2} + (2p-1)^n \tfrac{1}{2}(2p-1) = \tfrac{1}{2}+\tfrac{1}{2}(2p-1)^{n+1}.$$

This calculation, and 3 more like it, yields that

(continued)

4.6 (continued)

$$P^{(n+1)} = P^{(n)} \cdot P = \left\| \begin{matrix} \frac{1}{2} + \frac{1}{2}(2p-1)^{n+1} & \frac{1}{2} - \frac{1}{2}(2p-1)^{n+1} \\ \frac{1}{2} - \frac{1}{2}(2p-1)^{n+1} & \frac{1}{2} + \frac{1}{2}(2p-1)^{n+1} \end{matrix} \right\|.$$

This completes the general inductive step. It follows that $P^{(n)}$ has the given form for $n = 1, 2, 3, \ldots$

14. (i) $\{0,1,2\}$ recurrent.
    (ii) $\{0,1,2,3\}$ recurrent.
    (iii) $\{0,2\}$ recurrent, $\{1\}$ transient, $\{3,4\}$ recurrent.
    (iv) $\{0,1\}$ recurrent, $\{2\}$ recurrent, $\{3\}$ transient, $\{4\}$ transient.

Some explanation is required:

(i) By definition $i \leftrightarrow i$ for all i (even when $P_{ii} = 0$).
For $i \neq j$, $P_{ij} > 0$. Hence $i \leftrightarrow j$ $i = 0,1,2$; $j = 0,1,2$.

$\therefore$ Since all states communicate, there is a single class $\{0,1,2\}$.
In a finite-state Markov Chain, this single class must be recurrent

(ii) Inspection of P_2 shows that, starting in State 0, the transitions will always be

$$0 \to 3 \to 2 \to \begin{matrix} 0 \\ \text{or} \\ 1 \end{matrix} \to 3 \to 2 \to \begin{matrix} 0 \\ \text{or} \\ 1 \end{matrix} \to 3 \to 2, \text{etc.}$$

In fact, starting in any of the states, the above pattern keeps repeating with prob. 1. Hence all states communicate. The single class $\{0,1,2,3\}$ is necessarily recurrent.

(iii) States 3 and 4 clearly communicate with each other, but with no other states. So $\{3,4\}$ is a class. It is necessarily recurrent since $0<P_{34}<1$ and $0<P_{43}<1$. Similar $\{0,2\}$ is a recurrent class. So state 1 must be in a class by itself. [Note that $1\to 0$ and $1\to 2$, but $0\not\to 1$, $2\not\to 1$.]

$\{1\}$ is a transient class, because recurrence of $\{1\}$ would imply the pattern of transitions $1\to 1\to 1\to 1\to \cdots$, which has probability $\lim_{n\to\infty}\left(\frac{1}{2}\right)^n = 0$.

(iv) $\{0,1\}$ is clearly a class, necessarily recurrent.
"2" is an absorbing state, hence $\{2\}$ is a recurrent class.

The only ~~state~~ other state accessible from state 3 is state 4 (i.e. $3\to$
But $4\not\to 3$. So $\{3\}$ is a class. It is transient because a Markov chain starting in state 3 must eventually (with prob 1) enter state 4.
State $4\to 0$, but $0\not\to 4$. So $\{4\}$ is a class. It is transient, since a M.C. starting in State 4 makes an immediate transition to state 0 and never returns to State 4.

18. If the state at time n is the n^{th} coin to be flipped then sequence of consecutive states constitute a two state Markov chain with transition probabilities

$$P_{1,1} = .6 = 1 - P_{1,2}, \quad P_{2,1} = .5 = P_{2,2}$$

(a) The stationary probabilities satisfy

$$\pi_1 = .6\pi_1 + .5\pi_2$$
$$\pi_1 + \pi_2 = 1$$

Solving yields that $\pi_1 = 5/9, \pi_2 = 4/9$. So the proportion of flips that use coin 1 is 5/9.

(b) $P^4_{1,2} = .44440$

19. The limiting probabilities are obtained from

$$r_0 = .7r_0 + .5r_1$$
$$r_1 = .4r_2 + .2r_3$$
$$r_2 = .3r_0 + .5r_1$$
$$r_0 + r_1 + r_2 + r_3 = 1,$$

and the solution is

$$r_0 = \frac{1}{4}, \quad r_1 = \frac{3}{20}, \quad r_2 = \frac{3}{20}, \quad r_3 = \frac{9}{20}.$$

The desired result is thus

$$r_0 + r_1 = \frac{2}{5}.$$

23. Let the state be 0 if the last two trials were both successes. 1 if the last trial was a success and the one before it a failure. 2 if the last trial was a failure. The transition probability matrix of this Markov chain is

$$P = \begin{bmatrix} .8 & 0 & .2 \\ .5 & 0 & .5 \\ 0 & .5 & .5 \end{bmatrix}$$

This gives $\pi_0 = 5/11, \pi_1 = 2/11, \pi_2 = 4/11$. Consequently, the proportion of trials that are successes is $.8\pi_0 + .5(1 - \pi_0) = 7/11$.

25

I will work out the special case $\underline{\underline{n=8}}$. Then the transition prob matrix is:

	0	1	2	3	4	5	6	7	8
0	$\frac{1}{2}$	0	0	0	0	0	0	0	$\frac{1}{2}$
1	$\frac{1}{4}$	$\frac{1}{4}$	0	0	0	0	0	$\frac{1}{4}$	$\frac{1}{4}$
2	0	$\frac{1}{4}$	$\frac{1}{4}$	0	0	0	$\frac{1}{4}$	$\frac{1}{4}$	0
3	0	0	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{4}$	0	0
4	0	0	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0	0	0
5	0	0	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0	0	0
6	0	0	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{4}$	0	0
7	0	$\frac{1}{4}$	$\frac{1}{4}$	0	0	0	$\frac{1}{4}$	$\frac{1}{4}$	0
8	$\frac{1}{4}$	$\frac{1}{4}$	0	0	0	0	0	$\frac{1}{4}$	$\frac{1}{4}$

Limiting probability equations: (1) $\pi_0 = \frac{1}{2}\pi_0 + \frac{1}{4}\pi_1 + \frac{1}{4}\pi_8$

(2) $\pi_1 = \frac{1}{4}\pi_1 + \frac{1}{4}\pi_2 + \frac{1}{4}\pi_7 + \frac{1}{4}\pi_8$

(3) $\pi_2 = \frac{1}{4}\pi_2 + \frac{1}{4}\pi_3 + \frac{1}{4}\pi_6 + \frac{1}{4}\pi_7$

(4) $\pi_3 = \frac{1}{4}\pi_3 + \frac{1}{4}\pi_4 + \frac{1}{4}\pi_5 + \frac{1}{4}\pi_6$

(5) $\pi_4 = \frac{1}{2}\pi_4 + \frac{1}{2}\pi_5$ (continued)

Note that (5) $\Rightarrow \pi_4 = \pi_5$. If we consider similarly equations (6) through (9), we notice (by finding matching columns of $\mathbb{P}$) that $\pi_5 = \pi_3$, $\pi_6 = \pi_2$, $\pi_7 = \pi_1$ and $\pi_8 = \pi_0$.

Substituting these identities into (1) through (4), as well as (0) $\sum_{i=1}^{\infty} \pi_i = 1$ yields:

(0′) $\pi_0 + \pi_1 + \pi_2 + \pi_3 + \frac{1}{2}\pi_4 = \cancel{1}\ 1/2$

(1′) $\pi_0 = \frac{1}{2}\pi_0 + \frac{1}{4}\pi_1 + \frac{1}{4}\pi_0$

(2′) $\pi_1 = \frac{1}{4}\pi_1 + \frac{1}{4}\pi_2 + \frac{1}{4}\pi_1 + \frac{1}{4}\pi_0$

(3′) $\pi_2 = \frac{1}{4}\pi_2 + \frac{1}{4}\pi_3 + \frac{1}{4}\pi_2 + \frac{1}{4}\pi_1$

(4′) $\pi_3 = \frac{1}{4}\pi_3 + \frac{1}{4}\pi_4 + \frac{1}{4}\pi_3 + \frac{1}{4}\pi_2$

(1′) $\Rightarrow \pi_1 = \pi_0$; and this plus (2′) $\Rightarrow \pi_2 = \pi_1$. Similarly $\pi_3 = \pi_2$ and $\pi_4 = \pi_3$.

So $\pi_0 = \pi_1 = \pi_2 = \pi_3 = \pi_4$. This plus (0′) implies $\frac{9}{2}\pi_0 = \frac{1}{2}$.

Hence $\pi_i = \frac{1}{9}$ for $i = 0, 1, \ldots, 9$.

(With a bit of care, this proof for ~~[illegible]~~ $n = 8$ could be generalized to arbitrary n, although a bit of care is necessarily. I think you need to divide into the case n = even and n = odd to take care of different pattern for the 1/4's and 1/2's appearing in the transition prob. matrix).

28. There are 4 states: 1 = success on last 2 trials;
2 = success on last, failure on next to last;
3 = failure on last, success on next to last;
4 = failure on last 2 trials.

Transition probabilities are:

$$P_{1,1} = \frac{3}{4}, \quad P_{1,3} = \frac{1}{4}$$
$$P_{2,1} = \frac{2}{3}, \quad P_{2,3} = \frac{1}{3}$$
$$P_{3,2} = \frac{2}{3}, \quad P_{3,4} = \frac{1}{3}$$
$$P_{4,2} = \frac{1}{2}, \quad P_{4,4} = \frac{1}{2}.$$

Limiting probabilities are given by

$$\Pi_1 = \frac{3}{4}\Pi_1 + \frac{2}{3}\Pi_2$$
$$\Pi_2 = \frac{2}{3}\Pi_3 + \frac{1}{2}\Pi_4$$
$$\Pi_3 = \frac{1}{4}\Pi_1 + \frac{1}{3}\Pi_2$$
$$\Pi_1 + \Pi_2 + \Pi_3 + \Pi_4 = 1,$$

and the solution is $\Pi_1 = 1/2, \Pi_2 = 3/16, \Pi_3 = 3/16, \Pi_4 = 1/8$. Hence, the desired answer is $\Pi_1 + \Pi_2 = 11/16$.

33. Consider the Markov chain whose state at time n is the type of exam number n. The transition probabilities of this Markov chain are obtained by conditioning on the performance of the class. This gives the following.

$$P_{11} = .3(1/3) + .7(1) = .8$$
$$P_{12} = P_{13} = .3(1/3) = .1$$
$$P_{21} = .6(1/3) + .4(1) = .6$$
$$P_{22} = P_{23} = .6(1/3) = .2$$
$$P_{31} = .9(1/3) + .1(1) = .4$$
$$P_{32} = P_{33} = .9(1/3) = .3$$

(continued)

Let r_i denote the proportion of exams that are type $i, i = 1, 2, 3$. The r_i are the solutions of the following set of linear equations.

$$r_1 = .8\, r_1 + .6\, r_2 + .4\, r_3$$

$$r_2 = .1\, r_1 + .2\, r_2 + .3\, r_3$$

$$r_1 + r_2 + r_3 = 1$$

Since $P_{i2} = P_{i3}$ for all states i, it follows that $r_2 = r_3$. Solving the equations gives the solution

$$r_1 = 5/7, \quad r_2 = r_3 = 1/7.$$

34. (a) $\pi_i,\ i = 1, 2, 3$, which are the unique solutions of the following equations.

$$\pi_1 = q_2\pi_2 + p_3\pi_3$$

$$\pi_2 = p_1\pi_1 + q_3\pi_3$$

$$\pi_1 + \pi_2 + \pi_3 = 1$$

(b) The proportion of time that there is a counterclockwise move from i which is followed by 5 clockwise moves is $\pi_i q_i p_{i-1} p_i p_{i+1} p_{i+2} p_{i+3}$, and so the answer to (b) is $\sum_{i=1}^{3} \pi_i q_i p_{i-1} p_i p_{i+1} p_{i+2} p_{i+3}$. In the preceding, $p_0 = p_3, p_4 = p_1, p_5 = p_2, p_6 = p_3$.

35. The equations are

$$r_0 = r_1 + \frac{1}{2}\, r_2 + \frac{1}{3}\, r_3 + \frac{1}{4}\, r_4$$

$$r_1 = \frac{1}{2}\, r_2 + \frac{1}{3}\, r_3 + \frac{1}{4}\, r_4$$

$$r_2 = \frac{1}{3}\, r_3 + \frac{1}{4}\, r_4$$

$$r_3 = \frac{1}{4}\, r_4$$

$$r_4 = r_0$$

$$r_0 + r_1 + r_2 + r_3 + r_4 = 1$$

The solution is

$$r_0 = r_4 = 12/37, \quad r_1 = 6/37, \quad r_2 = 4/37, \quad r_3 = 3/37.$$

(9)

(46) Uses the same logic as in problem 25 — but is a bit simpler.

Following the hint in the text, define

X_n = # of umbrellas at his location (home or work) at the beginning of the n^{th} trip

Then $\{X_0, X_1, X_2, \ldots\}$ is a Markov chain with states $\{0, 1, 2, \ldots, r\}$

The transition prob's are: $P_{0,r} = 1$

and for $i = 1, 2, \ldots, r$, $\underbrace{P_{i,r-i} = q = 1-p}$ and $\underbrace{P_{i,r-i+1} = p}$.

If it doesn't rain, the # of umbrellas at each location stays the same

If it does rain, the # of umbrellas at the destination location will go up by 1.

The transition matrix looks like this:

$$
\begin{array}{c|cccccc}
 & 0 & 1 & 2 & \cdots & r-1 & r \\
\hline
0 & & & & & & 1 \\
1 & & & & & q & p \\
2 & & & & q & p & \\
\vdots & & & \iddots & \iddots & & \\
r-1 & & q & p & & & \\
r & q & p & & & &
\end{array}
$$

(continued)

46 (continued)

So the limiting prob's are determined by

$\pi_0 = q\pi_r$

$$\left.\begin{aligned}
\pi_1 &= q\pi_{r-1} + p\pi_r \\
\pi_2 &= q\pi_{r-2} + p\pi_{r-1} \\
&\ \vdots \\
\pi_{r-1} &= q\pi_1 + p\pi_2
\end{aligned}\right\} \leftarrow \text{that is, for } i = 1, 2, \ldots, r-1, \quad \pi_i = q\pi_{r-i} + p\pi_{r-i+1}$$

$\pi_r = \pi_0 + p\pi_1$

and $\sum_{j=0}^{r} \pi_j = 1$

To solve, start with $\boxed{\pi_0 = q\pi_r}$, substitute this into $\pi_r = \pi_0 + p\pi_1$ to obtain $\pi_r = q\pi_r + p\pi_1$ OR $\pi_r(1-q) = p\pi_1$ OR $p\pi_r = p\pi_1$, so $\boxed{\pi_r = \pi_1}$. Substitute this into $\pi_1 = q\pi_{r-1} + p\pi_r$ to get $\pi_1 = q\pi_{r-1} + p\pi_1$ OR $(1-p)\pi_1 = q\pi_{r-1}$ OR $\boxed{\pi_{r-1} = \pi_1}$.

Substitute this into $\pi_{r-1} = q\pi_1 + p\pi_2$ to get $\pi_1(1-q) = p\pi_2$ OR $\boxed{\pi_2 = \pi_1}$.

You should now see where this argument is headed: Alternating between both ends of the list of equations, we can (continued) prove, in turn, that each of the following π's are equal to the next one: $\pi_r, \pi_1, \pi_{r-1}, \pi_2, \pi_{r-2}, \pi_3, \pi_{r-3}, \pi_4 \ldots$ and so on

[I should really write out the formal general inductive step, but I am afraid I might screw up the general notation.]

So we have

$$\pi_0 = q\pi_r \quad \text{and} \quad \pi_1 = \pi_2 = \pi_3 = \cdots = \pi_{r-1} = \pi_r .$$

Since $\sum_{j=0}^{r} \pi_j = 1$, we have $\pi_r(q+r) = 1$ OR $\pi_r = \frac{1}{q+r}$.

So $\pi_0 = \frac{q}{q+r}$ and $\pi_1 = \pi_2 = \cdots = \pi_r = \frac{1}{q+r}$.

(iii) What fraction of the time does our man get wet?

Answer = M.C. must be in state 0 and it rains,

so answer is $p\pi_0 = \frac{pq}{r+q} = \frac{p(1-p)}{r+1-p}$

(continued)

47 (continued)

(iv) When $r=3$, we need to find the value of $p \in [0,1]$ which maximizes $\frac{p-p^2}{4-p}$.

This is a continuous function of p which must take its maximum at some interior point of $[0,1]$ (since it is $=0$ at both $p=0$ and $p=1$).

$$0 = \frac{d}{dp}\left[\frac{p-p^2}{4-p}\right] \Rightarrow (4-p)(1-2p) - (p-p^2)(-1) = 0$$

$$\Rightarrow 4 - 9p + 2p^2 + p - p^2 = 0 \Rightarrow p^2 - 8p + 4 = 0.$$

Solutions to this quadratic eqn are $p = \frac{8 \pm \sqrt{48}}{2}$ OR $4 \pm 2\sqrt{3}$

OR 4 ± 3.464. The only solution in $(0,1)$ is $4 - 3.464 = 0.536$

So the maximizing value of p is $\hat{=} 0.\underline{\underline{536}}$.