STAT 407

Solutions to Assignment #3

Chapter 4:

64 (a) $E\left[\sum_{k=0}^{\infty} X_k | X_0 = 1\right] = \sum_{k=0}^{\infty} E[X_k | X_0 = 1]$

$$= \sum_{k=0}^{\infty} \mu^k = \frac{1}{1-\mu}.$$

(b) $E\left[\sum_{k=0}^{\infty} X_k | X_0 = n\right] = \frac{n}{1-\mu}.$

65 $r \geq 0 = P\{X_0 = 0\}$. Assume that $r \geq P\{X_{n-1} = 0\}$.

$$\begin{aligned} P\{X_n = 0 &= \sum_j P\{X_n = 0 | X_1 = j\} P_j \\ &= \sum_j [P\{X_{n-1} = \}]^j P_j \\ &\leq \sum_j r^j P_j \\ &= r. \end{aligned}$$

66. (a) $r_0 = \frac{1}{3}$

(b) $r_0 = 1.$

(c) $r_0 = \left(\sqrt{3} - 1\right)/2.$

Chapter 5:

2. Let T be the time you spend in the system; let S_i be the service time of the i person in queue: let R be the remaining service time of the person in service; let S be your service time. Then,

$$\begin{aligned} E[T] &= E[R + S_1 + S_2 + S_3 + S_4 + S] \\ &= E[R] + \sum_{i=1}^{4} E[S_i] + E[S] = 6/\mu \end{aligned}$$

Where we have used the lack of memory property to conclude that R is also exponential with rate μ.

4. (a) 0, (b) $\frac{1}{27}$, (c) $\frac{1}{4}$.

6. Condition on which server initially finishes first. Now,

$$
\begin{aligned}
&P\{\text{Smith is last}|\text{server 1 finishes first}\}\\
&\quad = P\{\text{server 1 finishes before server 2}\}\\
&\qquad \text{by lack of memory}\\
&\quad = \frac{\lambda_1}{\lambda_1+\lambda_2}.
\end{aligned}
$$

Similarly,

$$P\{\text{Smith is last}|\text{server 2 finished first}\} = \frac{\lambda_2}{\lambda_1+\lambda_2}$$

and thus

$$P\{\text{Smith is last}\} = \left[\frac{\lambda_1}{\lambda_1+\lambda_2}\right]^2 + \left[\frac{\lambda_2}{\lambda_1+\lambda_2}\right]^2.$$

9. Condition on whether machine 1 is still working at time t, to obtain the answer,

$$1 - e^{-\lambda_1 t} + e^{-\lambda_1 t}\frac{\lambda_1}{\lambda_1+\lambda_2}$$

12. (a) $P\{X_1 < X_2 < X_3\}$

$$
\begin{aligned}
&= P\{X_1 = \min(X_1, X_2, X_3)\}\\
&\quad P\{X_2 < X_3 | X_1 = \min(X_1, X_2, X_3)\}\\
&= \frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3} P\{X_2 < X_3 | X_1\\
&\qquad\qquad = \min(X_1, X_2, X_3)\}\\
&= \frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3}\frac{\lambda_2}{\lambda_2+\lambda_3}
\end{aligned}
$$

where the final equality follows by the lack of memory property.

14. (a) The conditional density of X gives that $X < c$ is

$$f(x|X < c) = \frac{f(x)}{P\{x < c\}} = \frac{\lambda e^{-\lambda x}}{1 - e^{-\lambda c}}, \quad 0 < x < c$$

Hence,

$$E[X|X < c] = \int_0^c x\lambda e^{-\lambda x} dx/(1 - e^{-\lambda c}).$$

Integration by parts yields that

$$\int_0^c x\lambda e^{-\lambda x}\, dx = -xe^{-\lambda x}\Big|_0^c + \int_0^c e^{-\lambda x} dx$$
$$= -ce^{-\lambda c} + (1 - e^{-\lambda c})/\lambda.$$

Hence,

$$E[X|X < c] = 1/\lambda - ce^{-\lambda c}/(1 - e^{-\lambda c}).$$

(b) $1/\lambda = E[X|X < c](1 - e^{-\lambda c}) + (c + 1/\lambda)e^{-\lambda c}$
This simplifies to the same answer as given in part (a).

15. Let T_i denote the time between the $(i-1)^{th}$ and the i^{th} failure. Then the T_i are independent with T_i being exponential with rate $(101 - i)/200$. Thus,

$$E[T] = \sum_{i=1}^{5} E[T_i] = \sum_{i=1}^{5} \frac{200}{101 - i}$$

$$Var(T) = \sum_{i=1}^{5} Var(T_i) = \sum_{i=1}^{5} \frac{(200)^2}{(101 - i)^2}$$

20. (a) $P_A = \dfrac{\mu_1}{\mu_1 + \mu_2}$

(b) $P_B = 1 - \left(\dfrac{\mu_2}{\mu_1 + \mu_2}\right)^2$

(c) $E[T] = 1/\mu_1 + 1/\mu_2 + P_A/\mu_2 + P_B/\mu_2$

23. (a) $1/2$.

(b) $(1/2)^{n-1}$: whenever battery 1 is in use and a failure occurs the probability is $1/2$ that it is not battery 1 that has failed.

(c) $(1/2)^{n-i+1}, \quad i > 1$.

(d) T is the sum of $n - 1$ independent exponentials with rate 2μ (since each time a failure occurs the time until the next failure is exponential with rate 2μ).

(e) Gamma with parameters $n - 1$ and 2μ.

25. Parts (a) and (b) follow upon integration. For part (c), condition on which of X or Y is larger and use the lack of memory property to conclude that the amount by which it is larger is exponential rate λ. For instance, for $x < 0$,

$$f x - y(x)dx$$
$$= P\{X < Y\}P\{-x < Y - X < -x + dx | Y > X\}$$
$$= \frac{1}{2}\lambda e^{\lambda x}dx$$

For (d) and (e), condition on I.

30. Condition on which animal died to obtain

$$E[\text{additional life}]$$
$$= E[\text{additional life} \mid \text{dog died}]$$
$$\frac{\lambda_d}{\lambda_c + \lambda_d} + E[\text{additional life} \mid \text{cat died}]\frac{\lambda_c}{\lambda_c + \lambda_d}$$
$$= \frac{1}{\lambda_c}\frac{\lambda_d}{\lambda_c + \lambda_d} + \frac{1}{\lambda_d}\frac{\lambda_c}{\lambda_c + \lambda_d}$$

34. (a) $\dfrac{\lambda}{\lambda + \mu_A}$

(b) $\dfrac{\lambda + \mu_A}{\lambda + \mu_A + \mu_B} \cdot \dfrac{\lambda}{\lambda + \mu_B}$

42. (a) $E[S_4] = 4/\lambda$.

(b) $E[S_4 | N(1) = 2]$
$= 1 + E[\text{time for 2 more events}] = 1 + 2/\lambda$.

(c) $E[N(4) - N(2) | N(1) = 3] = E[N(4) - N(2)]$
$= 2\lambda$.

The first equality used the independent increments property.

5

44. (a) $e^{-\lambda T}$

(b) Let W denote the waiting time and let X denote the time until the first car. Then

$$E[W] = \int_0^\infty E[W|X = x]\lambda e^{-\lambda x} dx$$

$$= \int_0^T E[W|X = x]\lambda e^{-\lambda x} dx$$

$$+ \int_T^\infty E[W|X = x]\lambda e^{-\lambda x} dx$$

$$= \int_0^T (x + E[W|)\lambda e^{-\lambda x} dx + Te^{-\lambda T}$$

Hence,

$$E[W] = T + e^{\lambda T} \int_0^T x\lambda e^{-\lambda x} dx$$

50. Let T denote the time until the next train arrives; and so T is uniform on $(0, 1)$. Note that, conditional on T, X is Poisson with mean $7T$.

(a) $E[X] = E[E[X|T]] = E[7T] = 7/2$.

(b) $E[X|T] = 7T$, $Var(X|T) = 7T$. By the conditional variance formula $Var(X) = 7E[T] + 49Var[T] = 7/2 + 49/12 = 91/12$.

52. This is the gambler's ruin probability that, starting with k, the gambler's fortune reaches $2k$ before 0 when her probability of winning each bet is $p = \lambda_1/(\lambda_1 + \lambda_2)$. The desired probability is $\dfrac{1 - (\lambda_2/\lambda_1)^k}{1 - (\lambda_2/\lambda_1)^{2k}}$.

56. (a) It is a binomial (n, p) random variable.

(b) It is geometric with parameter p.

(c) It is a negative binomial with parameters r, p.

(d) Let $0 < i_1 < i_2, \cdots < i_r < n$. Then,

$$P\{\text{events at } i_1, \ldots, i_r | N(n) = r\}$$

$$= \frac{P\{\text{events at } i_1, \ldots, i_r, N(n) = r\}}{P\{N(n) = r\}}$$

$$= \frac{P^r(1-p)^{n-r}}{\binom{n}{r} P^r(1-p)^{n-r}}$$

$$= \frac{1}{\binom{n}{r}}$$

59. The unconditional probability that the claim is type 1 is 10/11. Therefore,

$$P(1|4000) = \frac{P(4000|1)P(1)}{P(4000|1)P(1) + P(4000|2)P(2)}$$
$$= \frac{e^{-4}10/11}{e^{-4}10/11 + .2e^{-.8}1/11}$$

61. (a) Poisson with mean $cG(t)$.

(b) Poisson with mean $c[1 - G(t)]$.

(c) Independent.

63. Let X and Y be respectively the number of customers in the system at time $t + s$ that were present at time s, and the number in the system at $t + s$ that were not in the system at time s. Since there are an infinite number of servers, it follows that X and Y are independent (even if given the number is the system at time s). Since the service distribution is exponential with rate μ, it follows that given that $X(s) = n, X$ will be binomial with parameters n and $p = e^{-\mu t}$. Also Y, which is independent of $X(s)$, will have the same distribution as $X(t)$.

Therefore, Y is Poisson with mean $\lambda \int_0^t e^{-\mu y} dy$

$= \lambda(1 - e^{-\mu t})/\mu$.

(a) $E[X(t + s)|X(s) = n]$

$= E[X|X(s) = n] + E[Y|X(s) = n]$.

$= ne^{-\mu t} + \lambda(1 - e^{-\mu t})/\mu$.

(b) $Var(X(t + s)|X(s) = n)$

$= Var(X + Y|X(s) = n)$

$= Var(X|X(s) = n) + Var(Y)$

$= ne^{-\mu t}(1 - e^{-\mu t}) + \lambda(1 - e^{-\mu t})/\mu$.

The above equation uses the formulas for the variances of a binomial and a Poisson random variable.

(continued)

63 (continued)

(c) Consider an infinite server queuing system in which customers arrive according to a Poisson process with rate λ, and where the service times are all exponential random variables with rate μ. If there is currently a single customer in the system, find the probability that the system becomes empty when that customer departs.

Condition on R, the remaining service time:

$P\{\text{empty}\}$

$$= \int_0^\infty P\{\text{empty}|R = t\}\mu e^{-\mu t}dt$$

$$= \int_0^\infty \exp\{-\lambda \int_0^t e^{-\mu y}dy\}\mu e^{-\mu t}dt$$

$$= \int_0^\infty \exp\{-\frac{\lambda}{\mu}(1 - e^{-\mu t})\}\mu e^{-\mu t}dt$$

$$= \int_0^1 e^{-\lambda x/\mu}dx$$

$$= \frac{\mu}{\lambda}(1 - e^{-\lambda/\mu})$$

where the preceding used that $P\{\text{empty}| R = t\}$ is equal to the probability that an $M/M/\infty$ queue is empty at time t.

77. $e^{-11}(11)^n/n!$

80. (i) No.

(ii) No.

(iii) $P\{T_1 > t\} = P\{N(t) = 0\} = e^{-m(t)}$ where

$$m(t) = \int_0^t \lambda(s)ds.$$

85. \$ 40,000 and \$1.6 $\times$ 10^8.

86. (a) $P\{N(t) = n\} = .3\,e^{-3t}(3t)^n n! + .7e^{-5t}(5t)^n n!$.

(b) No!

(c) Yes! The probability of n events in any interval of length t will, by conditioning on the type of year, be as given in (a).

(d) No! Knowing how many storms occur in an interval changes the probability that it is a good year and this affects the probability distribution of the number of storms in other intervals.

(e) $P\{\text{good}|N(1) = 3\}$

$$= \frac{P\{N(1) = 3|\text{good}\}\, P\{\text{good}\}}{P\{N(1) = 3|\text{good}\}P\{\text{good}\} + P\{N(1) = 3|\text{bad}\}P\{\text{bad}\}}$$

$$= \frac{(e^{-3}3^3/3!).3}{(e^{-3}3^3/3!).3 + e^{-5}5^3/3!).7}$$

87. $Cov[X(t), X(t+s)]$

$= Cov[X(t), X(t) + X(t+s) - X(t)]$

$= Cov[X(t), X(t)] + Cov[X(t), X(t+s) - X(t)]$

$= Cov[X(t), X(t)]$ by independent increments

$= Var[X(t)] = \lambda t E[Y^2]$.

Chapter 6:

3. This is not a birth and death process since we need more information than just the number working. We also must know which machine is working. We can analyze it by letting the states be

b : both machines are working

1 : 1 is working, 2 is down

2 : 2 is working, 1 is down

0_1: both are down, 1 is being serviced

0_2: both are down, 2 is being serviced.

$$v_b = \mu_1 + \mu_2,\ v_1 = \mu_1 + \mu,\ v_2 = \mu_2 + \mu,$$

$$v_{0_1} = v_{0_2} = \mu$$

$$P_{b,1} = \frac{\mu_2}{\mu_2 + \mu_1} = 1 - P_{b,2}, \quad P_{1,b} = \frac{\mu}{\mu + \mu_1}$$

$$= 1 - P_{1,0_2}$$

$$P_{2,b} = \frac{\mu}{\mu + \mu_2} = 1 - P_{2,0_1}, \quad P_{0_1,1} = P_{0_2,2} = 1.$$

86. (a) $P\{N(t)=n\} = .3\,e^{-3t}(3t)^n n! + .7e^{-5t}(5t)^n n!.$

(b) No!

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$$= \frac{P\{N(1)=3|\text{good}\}\,P\{\text{good}\}}{P\{N(1)=3|\text{good}\}P\{\text{good}\}+P\{N(1)=3|\text{bad}\}P\{\text{bad}\}}$$

$$= \frac{(e^{-3}3^3/3!).3}{(e^{-3}3^3/3!).3 + e^{-5}5^3/3!).7}$$

87. $Cov[X(t), X(t+s)]$

$= Cov[X(t), X(t) + X(t+s) - X(t)]$

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$$P_{b,1} = \frac{\mu_2}{\mu_2+\mu_1} = 1 - P_{b,2}, \quad P_{1,b} = \frac{\mu}{\mu+\mu_1}$$

$$= 1 - P_{1,0_2}$$

$$P_{2,b} = \frac{\mu}{\mu+\mu_2} = 1 - P_{2,0_1}, \quad P_{0_1,1} = P_{0_2,2} = 1.$$

5. (a) Yes.
   (b) It is a pure birth process.
   (c) If there are i infected individuals then since a contact will involve an infected and an uninfected individual with probability $i(n-i)/\binom{n}{2}$, it follows that the birth rates are $\lambda_i = \lambda i(n-i)/\binom{n}{2}$, $i = 1, \ldots, n$. Hence,

$$E[\text{time all infected}] = \frac{n(n-1)}{2\lambda} \sum_{i=1}^{n} 1/[i(n-i)].$$

6. Starting with $E[T_0] = \dfrac{1}{\lambda_0} = \dfrac{1}{\lambda}$, employ the identity

$$E[T_i] = \frac{1}{\lambda_i} + \frac{\mu_i}{\lambda_i} E[T_{i-1}]$$

to successively compute $E[T_i]$ for $i = 1, 2, 3, 4$.

(a) $E[T_0] + \cdots + E[T_3]$.
(b) $E[T_2] + E[T_3] + E[T_4]$.