

(1)

STAT 407

Solutions to Assignment # 4Chapter 6

- (9) Since the death rate is constant, it follows that as long as the system is nonempty, the number of deaths in any interval of length t will be a Poisson random variable with mean μt . Hence,

$$P_{ij}(t) = e^{-\mu t} (\mu t)^{i-j} / (i-j)!, \quad 0 < j \leq i$$

$$P_{i,0}(t) = \sum_{k=i}^{\infty} e^{-\mu t} (\mu t)^k / k!.$$

- (12) (a) If the state is the number of individuals at time t , we get a birth and death process with

$$\lambda_n = n\lambda + \theta \quad n < N$$

$$\lambda_n = n\lambda \quad n \geq N$$

$$\mu_n = n\mu.$$

- (b) Let P_i be the long-run probability that the system is in state i . Since this is also the proportion of time the system is in state i , we are looking for $\sum_{i=3}^{\infty} P_i$.

We have $\lambda_k P_k = \mu_{k+1} P_{k+1}$.

This yields

$$P_1 = \frac{\theta}{\mu} P_0$$

$$P_2 = \frac{\lambda + \theta}{2\mu} P_1 = \frac{\theta(\lambda + \theta)}{2\mu^2} P_0$$

$$P_3 = \frac{2\lambda + \theta}{2\mu} P_2 = \frac{\theta(\lambda + \theta)(2\lambda + \theta)}{6\mu^3} P_0.$$

For $k \geq 4$, we get

$$P_k = \frac{(k-1)\lambda}{k\mu} P_{k-1},$$

(continued)

(2)

(12) (continued)

which implies

$$P_k = \frac{(k-1)(k-2)\cdots(3)}{(k)(k-1)\cdots(4)} \left[\frac{\lambda}{\mu} \right]^{k-3}$$

$$P_3 = \frac{3}{k} \left[\frac{\lambda}{\mu} \right]^{k-3} P_3;$$

$$\text{therefore } \sum_{k=3}^{\infty} P_k = 3 \left[\frac{\mu}{\lambda} \right]^3 P_3 \sum_{k=3}^{\infty} \frac{1}{k} \left[\frac{\lambda}{\mu} \right]^k,$$

$$\begin{aligned} \text{but } \sum_{k=1}^{\infty} \frac{1}{k} \left[\frac{\lambda}{\mu} \right]^k &= \log \left[\frac{1}{1 - \frac{\lambda}{\mu}} \right] \\ &= \log \left[\frac{\mu}{\mu - \lambda} \right] \text{ if } \frac{\lambda}{\mu} < 1. \end{aligned}$$

$$\begin{aligned} \text{So } \sum_{k=3}^{\infty} P_k &= 3 \left[\frac{\mu}{\lambda} \right]^3 P_3 \left[\log \left[\frac{\mu}{\mu - \lambda} \right] \right. \\ &\quad \left. - \frac{\lambda}{\mu} - \frac{1}{2} \left[\frac{\lambda}{\mu} \right]^2 \right] \end{aligned}$$

$$\begin{aligned} \sum_{k=3}^{\infty} P_k &= 3 \left[\frac{\mu}{\lambda} \right]^3 \left[\log \left[\frac{\mu}{\mu - \lambda} \right] - \frac{\lambda}{\mu} - \frac{1}{2} \left[\frac{\lambda}{\mu} \right]^2 \right] \\ &\quad \frac{\theta(\lambda + \theta)(2\lambda + \theta)}{6\mu^3} P_0. \end{aligned}$$

Now $\sum_0^{\infty} P_i = 1$ implies

$$\begin{aligned} P_0 &= \left[1 + \frac{\theta}{\mu} + \frac{\theta(\lambda + \theta)}{2\mu^2} + \frac{1}{2\lambda^3} \theta(\lambda + \theta)(2\lambda + \theta) \right. \\ &\quad \times \left. \left[\log \left[\frac{\mu}{\mu - \lambda} \right] - \frac{\lambda}{\mu} - \frac{1}{2} \left[\frac{\lambda}{\mu} \right]^2 \right] \right]^{-1}. \end{aligned}$$

And finally,

$$\begin{aligned} \sum_{k=3}^{\infty} P_k &= \left[\left[\frac{1}{2\lambda^3} \right] \left[\log \left[\frac{\mu}{\mu - \lambda} \right] - \frac{\lambda}{\mu} - \frac{1}{2} \left[\frac{\lambda}{\mu} \right]^2 \right] \right. \\ &\quad \left. \theta(\lambda + \theta)(2\lambda + \theta) \right] / \left[1 + \frac{\theta}{\mu} + \frac{\theta(\lambda + \theta)}{2\mu^2} \right. \\ &\quad \left. + \frac{\theta(\lambda + \theta)(2\lambda + \theta)}{2\lambda^3} \right. \\ &\quad \left. \times \left[\log \left[\frac{\mu}{\mu - \lambda} \right] - \frac{\lambda}{\mu} - \frac{1}{2} \left[\frac{\lambda}{\mu} \right]^2 \right] \right]. \end{aligned}$$

13. With the number of customers in the shop as the state, we get a birth and death process with

$$\lambda_0 = \lambda_1 = 3 \quad \mu_1 = \mu_2 = 4.$$

Therefore

$$P_1 = \frac{3}{4} P_0 \quad P_2 = \frac{3}{4} \quad P_1 = \left[\frac{3}{4} \right]^2 P_0.$$

And since $\sum_0^2 P_i = 1$, we get

$$P_0 = \left[1 + \frac{3}{4} + \left[\frac{3}{4} \right]^2 \right]^{-1} = \frac{16}{37}.$$

- (a) The average number of customers in the shop is

$$\begin{aligned} P_1 + 2P_2 &= \left[\frac{3}{4} + 2 \left[\frac{3}{4} \right]^2 \right] P_0 \\ &= \frac{30}{16} \left[1 + \frac{3}{4} + \left[\frac{3}{4} \right]^2 \right]^{-1} = \frac{30}{37}. \end{aligned}$$

- (b) The proportion of customers that enter the shop is

$$\frac{\lambda(1 - P_2)}{\lambda} = 1 - P_2 = 1 - \frac{9}{16} \cdot \frac{16}{37} = \frac{28}{37}.$$

- (c) Now $\mu = 8$, and so

$$P_0 = \left[1 + \frac{3}{8} + \left[\frac{3}{8} \right]^2 \right]^{-1} = \frac{64}{97}.$$

So the proportion of customers who now enter the shop is

$$1 - P_2 = 1 - \left[\frac{3}{8} \right]^2 \frac{264}{97} = 1 - \frac{9}{97} = \frac{88}{97}.$$

The rate of added customers is therefore

$$\lambda \left[\frac{88}{97} \right] - \lambda \left[\frac{28}{37} \right] = 3 \left[\frac{88}{97} - \frac{28}{37} \right] = 0.45.$$

The business he does would improve by 0.45 customers per hour.

- (14) Letting the number of cars in the station be the state, we have a birth and death process with

$$\lambda_0 = \lambda_1 = \lambda_2 = 20, \quad \lambda_i = 0, \quad i > 2,$$

$$\mu_1 = \mu_2 = 12.$$

Hence,

$$P_1 = \frac{5}{3}P_0, \quad P_2 = \frac{5}{3}P_1 = \left[\frac{5}{3}\right]^2 P_0,$$

$$P_3 = \frac{5}{3}P_2 = \left[\frac{5}{3}\right]^3 P_0$$

and as $\sum_0^3 P_i = 1$, we have

$$P_0 = \left[1 + \frac{5}{3} + \left[\frac{5}{3}\right]^2 + \left[\frac{5}{3}\right]^3\right]^{-1} = \frac{27}{272}.$$

- (15) With the number of customers in the system as the state, we get a birth and death process with

$$\lambda_0 = \lambda_1 = \lambda_2 = 3 \quad \lambda_i = 0, \quad i \geq 4,$$

$$\mu_1 = 2 \quad \mu_2 = \mu_3 = 4.$$

Therefore, the balance equations reduce to

$$P_1 = \frac{3}{2}P_0 \quad P_2 = \frac{3}{4}P_1 = \frac{9}{8}P_0 \quad P_3 = \frac{3}{4}P_2 = \frac{27}{32}P_0.$$

And therefore,

$$P_0 = \left[1 + \frac{3}{2} + \frac{9}{8} + \frac{27}{32}\right]^{-1} = \frac{32}{143}.$$

- (a) The fraction of potential customers that enter the system is

$$\frac{\lambda(1 - P_3)}{\lambda} = 1 - P_3 = 1 - \frac{27}{32} \times \frac{32}{143} = \frac{116}{143}.$$

- (b) With a server working twice as fast we would get

$$P_1 = \frac{3}{4}P_0 \quad P_2 = \frac{3}{4}P_1 = \left[\frac{3}{4}\right]^2 P_0 \quad P_3 = \left[\frac{3}{4}\right]^3 P_0,$$

$$\text{and } P_0 = \left[1 + \frac{3}{4} + \left[\frac{3}{4}\right]^2 + \left[\frac{3}{4}\right]^3\right]^{-1} = \frac{64}{175}.$$

So that now

$$1 - P_3 = 1 - \frac{27}{64} = 1 - \frac{64}{175} = \frac{148}{175}.$$

(5)

- 17) Say the state is 0 if the machine is up, say it is i when it is down due to a type i failure, $i = 1, 2$. The balance equations for the limiting probabilities are as follows.

$$\lambda P_0 = \mu_1 P_1 + \mu_2 P_2$$

$$\mu_1 P_1 = \lambda p P_0$$

$$\mu_2 P_2 = \lambda(1-p) P_0$$

$$P_0 + P_1 + P_2 = 1.$$

These equations are easily solved to give the results

$$P_0 = (1 + \lambda p / \mu_1 + \lambda(1-p) / \mu_2)^{-1}$$

$$P_1 = \lambda p P_0 / \mu_1, \quad P_2 = \lambda(1-p) P_0 / \mu_2.$$

- 23.) Let the state denote the number of machines that are down. This yields a birth and death process with

$$\lambda_0 = \frac{3}{10}, \quad \lambda_1 = \frac{2}{10}, \quad \lambda_2 = \frac{1}{10}, \quad \lambda_i = 0, \quad i \geq 3$$

$$\mu_1 = \frac{1}{8}, \quad \mu_2 = \frac{2}{8}, \quad \mu_3 = \frac{2}{8}.$$

The balance equations reduce to

$$P_1 = \frac{3/10}{1/8} P_0 = \frac{12}{5} P_0$$

$$P_2 = \frac{2/10}{2/8} P_1 = \frac{4}{5} P_1 = \frac{48}{25} P_0$$

$$P_3 = \frac{1/10}{2/8} P_2 = \frac{4}{10} P_3 = \frac{192}{250} P_0.$$

Hence, using $\sum_0^3 P_i = 1$, yields

$$P_0 = \left[1 + \frac{12}{5} + \frac{48}{25} + \frac{192}{250} \right]^{-1} = \frac{250}{1522}.$$

- (a) Average number not in use

$$= P_1 + 2P_2 + 3P_3 = \frac{2136}{1522} = \frac{1068}{761}.$$

- (b) Proportion of time both repairmen are busy

$$= P_2 + P_3 = \frac{672}{1522} = \frac{336}{761}.$$

(6) ~~6~~

Additional details

(23) Let "state" = "number of machines not in use"

Let λ = failure rate, μ = repair rate.

State rate at which leave = rate at which enter

0

$$3\lambda P_0 = \mu P_1$$

1

$$(2\lambda + \mu) P_1 = 3\lambda P_0 + 2\mu P_2$$

2

$$(\lambda + 2\mu) P_2 = 2\lambda P_1 + 2\mu P_3$$

3

$$2\mu P_3 = \lambda P_2$$

$$\text{So } P_1 = \frac{3\lambda}{\mu} P_0$$

$$2\lambda P_1 = 2\mu P_2 \Rightarrow P_2 = \frac{\lambda}{\mu} P_1 = \frac{3\lambda^2}{\mu^2} P_0$$

$$P_3 = \frac{\lambda}{2\mu} P_2 = \frac{3\lambda^3}{2\mu^3} P_0$$

$$P_0 \left[1 + \frac{3\lambda}{\mu} + \frac{3\lambda^2}{\mu^2} + \frac{3\lambda^3}{2\mu^3} \right] = 1$$

$$\text{So } P_0 = 1 / \left[1 + \frac{3\lambda}{\mu} + \frac{3\lambda^2}{\mu^2} + \frac{3\lambda^3}{2\mu^3} \right],$$

$$P_1 = \frac{3\lambda}{\mu} P_0, \quad P_2 = \frac{3\lambda^2}{\mu^2} P_0, \quad P_3 = \frac{3\lambda^3}{2\mu^3} P_0$$

Insert $\lambda = \frac{1}{10}$, $\mu = \frac{1}{8}$ into the above to compute P_0, P_1, P_2, P_3 .

(a) average # of machines not in use is $\sum_{n=0}^3 P_n = P_0 + 2P_1 + 3P_2$

(b) proportion of time that both repairmen are busy = $P_2 + P_3$.

(7) ~~7~~

Chapter 7

(1) We know that $N(t) \geq n \Leftrightarrow S_n \leq t$,

$$\text{so } \overline{\{N(t) \geq n\}} \Leftrightarrow \overline{\{S_n \leq t\}},$$

that is, $N(t) < n \Leftrightarrow S_n > t$. So (a) is true.

(b) $N(t) \leq n \Leftrightarrow N(t) < n+1 \Leftrightarrow S_{n+1} > t$.

But $\{S_{n+1} > t\}$ is not equivalent to $\{S_n \geq t\}$, so (b) is false.

(c) $N(t) > n \Leftrightarrow N(t) \geq n+1 \Leftrightarrow S_{n+1} \leq t$

But $\{S_{n+1} \leq t\}$ is not equivalent to $S_n < t$, so (c) is false.

(2) (a) S_n is Poisson with mean $n\mu$.

(b) $P\{N(t) = n\}$

$$= P\{N(t) \geq n\} - P\{N(t) \geq n+1\}$$

$$= P\{S_n \leq t\} - P\{S_{n+1} \leq t\}$$

$$= \sum_{k=0}^{[t]} e^{-n\mu} (n\mu)^k / k!$$

$$- \sum_{k=0}^{[t]} e^{-(n+1)\mu} [(n+1)\mu]^k / k!,$$

where $[t]$ is the largest integer not exceeding t .

(8)

②

$$(a) S_n = \sum_{i=1}^n X_i, \text{ where } X_i \stackrel{iid}{\sim} \text{Poisson}(\mu).$$

The mgf of X_i is $\phi(t) = E[e^{tX}] = e^{\mu(e^t - 1)}$

So the mgf of S_n is $[\phi(t)]^n = e^{n\mu(e^t - 1)}$

Hence $S_n \sim \text{Poisson}$ with mean $n\mu$.

$$(b) P[N(t)=n] = P[S_n < t] - P[S_{n+1} < t]$$

WLOG, take t to be an integer, since S_n is non-negative integer-valued

$$\begin{aligned} P[N(t)=n] &= \sum_{x=0}^t e^{-n\mu} \frac{(n\mu)^x}{x!} - \sum_{x=0}^{t-1} e^{-(n+1)\mu} \frac{((n+1)\mu)^x}{x!} \\ &= e^{-n\mu} \sum_{x=0}^t \frac{\mu^x}{x!} [n^x - e^{-\mu} (n+1)^x] \end{aligned}$$

5. The random variable N is equal to $N(1) + 1$ where $\{N(t)\}$ is the renewal process whose interarrival distribution is uniform on $(0, 1)$. By the results of Example 2c,

$$E[N] = a(1) + 1 = c.$$

(9)

- (5) Consider U_1, U_2, \dots to be the iid interarrival times for a renewal process. Let $N(t)$ denote the number of renewals up to time t .

Then the random variable $N = \min\{n : U_1 + U_2 + \dots + U_n \geq t\}$ is just $N(1) + 1$, since $\{n\}$ is the min. integer for which $U_1 + \dots + U_n \geq t \Leftrightarrow \{n-1\}$ is the largest integer for which $U_1 + U_2 + \dots + U_{n-1} \leq t$.

It was shown in Example 7.3 on p368, that for $U[0,1]$ interarrival times, $m(t) \stackrel{\text{def}}{=} E[N(t)] = e^t - 1$ for $0 \leq t \leq 1$.

In particular $E[N(1)] = e - 1$, so

$$E(N) = E[N(1) + 1] = e \approx 2.71828 \dots$$

(10)

6. (a) Consider a Poisson process having rate λ and say that an event of the renewal process occurs when ever one of the events numbered $r, 2r, 3r, \dots$ of the Poisson process occur. Then

$$\begin{aligned} P\{N(t) \geq n\} &= P\{\text{nr or more Poisson events by } t\} \\ &= \sum_{i=nr}^{\infty} e^{-\lambda t} (\lambda t)^i / i!. \end{aligned}$$

$$\begin{aligned} (b) \quad E[N(t)] &= \sum_{n=1}^{\infty} P\{N(t) \geq n\} = \sum_{n=1}^{\infty} \sum_{i=nr}^{\infty} e^{-\lambda t} (\lambda t)^i / i! \\ &= \sum_{i=r}^{\infty} \sum_{n=1}^{\lfloor i/r \rfloor} e^{-\lambda t} (\lambda t)^i / i! = \sum_{i=r}^{\infty} [i/r] e^{-\lambda t} (\lambda t)^i / i!. \end{aligned}$$

10. Yes, ρ/μ

$$11. \frac{N(t)}{t} = \frac{1}{t} + \frac{\text{number of renewals in } (X_1, t)}{t}$$

Since $X_1 < \infty$, Proposition 3.1 implies that

$$\frac{\text{number of renewals in } (X_1, t)}{t} \rightarrow \frac{1}{\mu} \text{ as } t \rightarrow \infty.$$

(10)

- (a) YES — each time there is a "counted event", the process starts over, as though from time 0, until the next "counted event" occurs.

$$(b) \lim_{t \rightarrow \infty} \frac{N(t)}{t} = \frac{1}{\bar{M}_c}, \text{ where } \bar{M}_c \text{ is the mean time between counted events.}$$

Let X_1, X_2, X_3, \dots be iid interarrival times between "ordinary" events (i.e., before counting is considered). Then $M_c = \text{mean of lengths of } X_i$

(11)

⑩ (continued)

The time between "counted" events is a random variable taking values

X_1	with prob	p
$X_1 + X_2$	" "	$(1-p)p$
$X_1 + X_2 + X_3$	" "	$(1-p)^2 p$
,		
$X_1 + X_2 + \dots + X_n$	" "	$(1-p)^{n-1} p$
:		

$$\text{So } M_c = E[X]p + E(X_1 + X_2)(1-p)p + E(X_1 + X_2 + X_3)(1-p)^2 p + \dots$$

$$= \mu p + 2\mu(1-p)p + 3\mu(1-p)^2 p + \dots + n\mu(1-p)^{n-1} p + \dots$$

$$= \mu [p + 2(1-p)p + 3(1-p)^2 p + \dots + n(1-p)^{n-1} p + \dots]$$

$$= \mu E[Y] \quad (\text{where } Y \sim \text{geometric with parameter } p) =$$

$$= \mu \cdot \frac{1}{p} = \frac{\mu}{p}$$

$$\text{So } \lim_{t \rightarrow \infty} \frac{N_c(t)}{t} = \frac{1}{M_c} = \frac{1}{\mu/p} = \underline{\underline{\frac{p}{\mu}}}$$

(12)

Chapter 10

(2)

$B(t_1)$, $B(s) - B(t_1)$ and $B(t_2) - B(s)$ are mutually indep r.v's

with $B(t_1) \sim N(0, t_1)$, $B(s) - B(t_1) \sim N(0, s-t_1)$ and

$B(t_2) - B(s) \sim N(0, t_2-s)$ by the axioms of the standard B.M. process,

With a change of variable $(x_1, x_2, x_3) \rightarrow (x_1, x_2-x_1, x_3-x_2)$ with [problem 1],

the joint density function of $B(t_1), B(s), B(t_2)$ is seen to be

$$f(x_1, x_2, x_3) = \text{const.} \times \exp \left\{ -\frac{1}{2} \left[\frac{x_1^2}{t_1} + \frac{(x_2-x_1)^2}{s-t_1} + \frac{(x_3-x_2)^2}{t_2-s} \right] \right\},$$

So the conditional density of $B(s)$ given that $B(t_1)=A$ and

$B(t_2)=B$ is

(continued)

(B)

② (continued)

$$\begin{aligned}
 f_{(x_2|B(t_1)=A, B(t_2)=B)} &= \text{const.} \cdot \exp \left\{ -\frac{1}{2} \left[\frac{A^2}{t_1} + \frac{(x_2-A)^2}{s-t_1} + \frac{(B-x_2)^2}{t_2-s} \right] \right\} \\
 &= \text{const.} \cdot \exp \left\{ -\frac{1}{2} \left[\frac{(t_2-s)[x_2^2 - 2Ax_2]}{(s-t_1)(t_2-s)} + (s-t_1)(x_2^2 - 2Bx_2) \right] \right\} = \\
 &= \text{const.} \cdot \exp \left\{ -\frac{1}{2} \left[\frac{x_2^2(t_2-t_1) - 2x_2[A(t_2-s) + B(s-t_1)]}{(s-t_1)(t_2-s)} \right] \right\} \\
 &= \text{const.} \cdot \exp \left\{ -\frac{1}{2} \frac{t_2-t_1}{(s-t_1)(t_2-s)} \left[x_2^2 - 2x_2 \left(\frac{A(t_2-s) + B(s-t_1)}{t_2-t_1} \right) \right] \right\} \\
 &= \text{const.} \cdot \exp \left\{ -\frac{1}{2} \frac{1}{(s-t_1)(t_2-s)/(t_2-t_1)} \left[x_2 - 2 \left(A \frac{t_2-s}{t_2-t_1} + B \frac{s-t_1}{t_2-t_1} \right) \right]^2 \right\} \\
 &\quad \text{(by "completing the square" in the exponent)}
 \end{aligned}$$

In the conditional dist. of $B(s)$ given $B(t_1)=A$ and $B(t_2)=B$
is normal, with mean $A \frac{t_2-s}{t_2-t_1} + B \frac{s-t_1}{t_2-t_1}$
and variance $\frac{(s-t_1)(t_2-s)}{t_2-t_1}$.

(14)

6. The probability of recovering your purchase price is the probability that a Brownian motion goes up c by time t . Hence the desired probability is

$$1 - P\{\max_{0 \leq s \leq t} X(s) \geq c\} = 1 - \frac{2}{\sqrt{2\pi t}} \int_c^{\infty} e^{-y^2/2} dy$$

(6) Let t_0 be the present time.

Given $B(t_0) = b$, you do not recover your purchase price $\Leftrightarrow B(v) < b + c$ for all $v \in [t_0, t_0 + t]$

$\Leftrightarrow \max_{t_0 \leq v \leq t_0 + t} B(v) < b + c$.

$$P\left[\max_{t_0 \leq v \leq t_0 + t} B(v) < b + c \mid B(t_0) = b\right] =$$

$$= P\left[\max_{t_0 \leq v \leq t_0 + t} B(v) < b + c \text{ and } B(t_0) = b \mid B(t_0) = b\right]$$

$$= P\left[\max_{t_0 \leq v \leq t_0 + t} [B(v) - B(t_0)] < c \mid B(t_0) = b\right]$$

$$= P\left[\max_{t_0 \leq v \leq t_0 + t} [B(v) - B(t_0)] < c\right] \text{ since } B(t_0) \text{ and } (B(v) - B(t_0)) \text{ are i.i.d. (independent increments)}$$

$$= P\left[\max_{t_0 \leq v \leq t_0 + t} B(v - t_0) < c\right] \text{ (by stationarity of B.M. process)}$$

$$= P\left[\max_{0 \leq v^* \leq t} B(v^*) < c\right] \text{ by change of variable } v^* = v - t_0$$

$$= 1 - P\left[\max_{0 \leq v^* \leq t} B(v^*) \geq c\right] = 1 - \frac{2}{\sqrt{2\pi t}} \int_c^{\infty} e^{-x^2/2} dx$$

(15)

7. Let $M = \{\max_{t_1 \leq s \leq t_2} X(s) > x\}$. Condition on $X(t_1)$ to obtain

$$P(M) = \int_{-\infty}^{\infty} P(M|X(t_1) = y) \frac{1}{\sqrt{2\pi t_1}} e^{-y^2/2t_1} dy$$

Now, use that

$$P(M|X(t_1) = y) = 1 \quad y \geq x$$

and, for $y < x$

$$\begin{aligned} P(M|X(t_1) = y) &= P\left\{\max_{0 \leq s \leq t_2 - t_1} X(s) > x - y\right\} \\ &= 2P\{X(t_2 - t_1) > x - y\} \end{aligned}$$

First, fix $x, \epsilon(-\infty, \infty)$ and compute $P\left\{\max_{t_1 \leq s \leq t_2} B(s) > x \mid B(t_1) = x_1\right\}$

$$P\left\{\max_{t_1 \leq s \leq t_2} B(s) > x \mid B(t_1) = x_1\right\} =$$

$$= P\left\{\max_{t_1 \leq s \leq t_2} [B(t_1) + (B(s) - B(t_1))] > x \mid B(t_1) = x_1\right\}$$

$$= P\left\{x_1 + \max_{t_1 \leq s \leq t_2} (B(s) - B(t_1)) > x \mid B(t_1) = x_1\right\}$$

$$= P\left\{\max_{t_1 \leq s \leq t_2} [B(s) - B(t_1)] > x - x_1\right\}, \text{ by indep of } B(t_1) \text{ and } [B(s) - B(t_1)]$$

$$= P\left\{\max_{0 \leq s - t_1 \leq t_2 - t_1} B(s - t_1) > x - x_1\right\} \text{ by stationarity}$$

$$= P\left\{\max_{0 \leq v \leq t_2 - t_1} B(v) > x - x_1\right\} = \cancel{\text{Wishart}} \cancel{\text{Ansatz}}$$

$$= 2 \left[1 - \Phi\left(\frac{|x - x_1|}{\sqrt{t_2 - t_1}}\right) \right], \text{ where } \Phi(v) = \int_{-\infty}^v \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

$$\text{L} \quad P\left\{\max_{t_1 \leq s \leq t_2} B(s) > x\right\} = \int_{x_1 = -\infty}^{\infty} P\left\{\max_{t_1 \leq s \leq t_2} B(s) > x \mid B(t_1) = x_1\right\} f(x_1) dx_1,$$

Since the density of x_1 , $f(x_1)$, is normal $(0, \sigma^2 t_1)$, i.e. $f(x_1) = \frac{1}{\sqrt{2\pi t_1}} e^{-x_1^2/(2t_1)}$,

we have

$$P\left\{\max_{t_1 \leq s \leq t_2} B(s) > x\right\} = 2 \int_{x_1 = -\infty}^{\infty} \left[1 - \Phi\left(\frac{|x - x_1|}{\sqrt{t_2 - t_1}}\right)\right] \frac{1}{\sqrt{2\pi t_1}} e^{-x_1^2/(2t_1)} dx_1.$$

(16)

Ques, #9.

I showed in class that if $\{B(t), t \geq 0\}$ is standard Brownian motion, then for $s < t$, the joint dist. of $B(s)$ and $B(t)$ is bivariate normal $(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$.

and $B(t)$ is bivariate normal $(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$
with $\mu_1 = \mu_2 = 0$, $\sigma_1^2 = s$, $\sigma_2^2 = t$ and $\rho = \sqrt{\frac{s}{t}}$.

With $X(t) = \sigma B(t) + \mu t$, $t \geq 0$,

Then since linear transformations of bivariate normal r.v.'s are known to be also bivariate normal, we have that the jt. dist. of $X(s)$ and $X(t)$ is bivariate normal with

$$\text{parameters } E[X(s)] = E[\sigma B(s) + \mu s] = \mu s,$$

$$\text{Var}[X(s)] = \text{Var}[\sigma B(s) + \mu s] = \sigma^2 \text{Var}(B(s)) = \sigma^2 s,$$

$$E[X(t)] = \mu t, \quad \text{Var}[X(t)] = \sigma^2 t$$

$$\text{and } \rho_{[X(s), X(t)]} = \rho_{[B(s), B(t)]} = \sqrt{\frac{s}{t}}.$$

So the jt density fns of $X(s)$ and $X(t)$ is [see formula in prob 16
on page 147 of text]

~~$$f(x, y) = \frac{1}{2\pi\sigma^2 s t \sqrt{1 - \frac{s}{t}}} \exp\left\{-\frac{1}{2(1-\frac{s}{t})} \left[\frac{x^2}{s} - 2\frac{\sqrt{\frac{s}{t}}}{\sigma\sqrt{s}\sqrt{t}} xy + \frac{y^2}{t} \right]\right\}$$~~

$\rightarrow \infty \propto \infty$
 $\infty \propto \infty$