UNIVERSITY OF CALGARY

DEPARTMENT OF MATHEMATICS AND STATISTICS

STAT 407 — Test no. 1 — Oct. 22, 2007 — TIME: 50 min.

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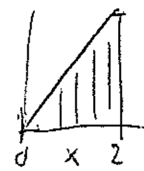
10

1. The joint density of X and Y is given by

$$f(x,y) = \begin{cases} \frac{1}{2}xy, & \text{when } 0 < x < 2 \text{ and } 0 < y < x \\ 0, & \text{otherwise.} \end{cases}$$



5 (a) Find the marginal density of Y.

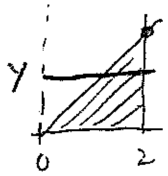
For $0<y<2$ $\quad f_2(y) = \int_{x=y}^{2} \frac{1}{2}xy\,dx = \frac{1}{2}y\int_{x=y}^{2} x\,dx$

(1)

$= \frac{1}{2}y\, \frac{x^2}{2}\Big]_y^2 = \frac{1}{4}y(4-y^2) = y - \frac{y^3}{4}$

(4)

3 (b) Find the conditional density of X given $Y = y$, for $0 < y < 2$.



$f(x|y) = \frac{f(x,y)}{f_2(y)} = \frac{\frac{1}{2}xy}{\frac{1}{4}y(4-y^2)} = \frac{2x}{4-y^2}$ when $y \le x < 2$ (1) (1)

$= 0$ $\quad 0<x<y$

(1)

[check. $\int f(x|y)\,dx = \frac{1}{4-y^2}\int_{x=y}^{2} 2x\,dx = \frac{1}{4-y^2}(4-y^2) = 1.$ ✓]

2 (c) Compute $E[X|Y = y]$ for $0 < y < 2$.

(1)

$E[X|Y=y] = \int x f(x|y)\,dx = \frac{1}{4-y^2}\int_{x=y}^{2} 2x^2\,dx$

$= \frac{1}{4-y^2}\, \frac{2}{3}x^3\Big]_y^2 = \frac{2(8-y^3)}{3(4-y^2)}$ (1)

[Note: If you integrated from $x=0$ to 2, instead of $x=y$ to 2, you can get at most 3 marks. But this error is very serious, and I think it deserves a harsh penalty].

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10 2. A coin has probability p of coming up Heads, and probability $1-p$ of coming up tails, on each toss, where p is a fixed value with $0 < p < 1$. Suppose that the coin is tossed repeatedly until the first time a Tail is followed immediately by a Head. Let X denote the number of tosses required for this to occur. For example, if the sequence of outcomes is HHTTH..., then $X = 5$.

8 (a) Find the expected value of X.

[**Hint 1.** Use a conditioning argument.]
[**Hint 2.** You are allowed to use the fact, without providing a proof, that the mean of a geometric random variable with parameter p is $1/p$]

Let $H_1 = \{\text{Heads on first toss}\}$, $T_1 = \{\text{tails on first toss}\}$

2 $\rightarrow$ $EX = P(H_1)E[X|H_1] + P(T_1)E[X|T_1] = pE[X|H_1] + qE[X|T_1]$ where $q = 1-p$.

But $X|H_1 \sim 1 + X$
and $X|T_1 \sim 1 + Y$, where $Y = \#$ of coin tosses required to obtain first Head

So $E[X|H_1] = 1 + EX$ (2) and $E[X|T_1] = 1 + EY = 1 + 1/p$ (2) since $Y \sim \text{geom}(p)$.

Hence
1 $\rightarrow$ $EX = p[1 + EX] + q\left[1 + \frac{1}{p}\right]$

or $EX(1-p) = p + q + pq = 1 + \frac{q}{p}$

or $EX \cdot q = 1 + \frac{q}{p}$ or $\underline{\underline{EX}} = \frac{1}{q} + \frac{1}{p} = \frac{p+q}{pq} = \underline{\frac{1}{pq}} = \underline{\underline{\frac{1}{p(1-p)}}}$ (1)

2 (b) Find the value of p in (0,1) which minimizes the corresponding value of the expected value of X.

EX is minimized over $(0,1)$ where $pq = p(1-p)$ is maximized, namely at $p = 1/2$.

[*Note*: full credit was given for more complicated, but correct, solution such as conditioning on the 4 events HH, HT, TH, TT]

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3. Consider the Markov chain consisting of five states 0, 1, 2, 3, 4 — and having transition probability matrix

$$P = \left\| \begin{matrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 2/5 & 3/5 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 1/4 & 3/4 \end{matrix} \right\|$$

Find the classes of this Markov chain, and determine which classes are recurrent and which are transient.

$0 \to 4$

$1 \to 1$

$1 \to 2$

$2 \to 2$

$3 \to 3$, $3 \to 4$

$4 \to 3$, $4 \to 4$

$0 \to 4$ but $4 \not\to 0$, so $\{0\}$ is a class by itself.

$1 \to 2$ but $2 \not\to 1$, so $\{1\}$ is a class.

Clearly $\{2\}$ is a class.

3 & 4 communicate, so $\{3,4\}$ is a class.

$\{3,4\}$ is clearly recurrent.

$\{0\}$ is transient, because it leads to the recurrent class $\{3,4\}$ with pos prob.

$\{1\}$ is transient, because it leads to the absorbent class $\{2\}$ with pos. prob.

$\{2\}$ is absorbent, hence recurrent.

2 2 3 3

In summary, the classes are: $\{0\}$, $\{1\}$, $\{2\}$, $\{3,4\}$.

$\{0\}$ ↑ transient, $\{1\}$ ↑ transient, $\{2\}$ ↑ recurrent, $\{3,4\}$ ↑ recurrent

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4. Consider the following model for a weather system: If it rains today, then it will rain tomorrow with probability 0.6 (and so will not rain tomorrow with probability 0.4). If it does not rain today, then it will rain tomorrow with probability 0.2 (and so will not rain tomorrow with probability 0.8).

(a) Write down the transition probability matrix for this Markov chain,with the states defined by 0=[no rain], 1= [it rains].

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$$P = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{Vmatrix} .8 & .2 \\ .4 & .6 \end{Vmatrix} \end{array}$$

(b) Find the probability that it rains on Thursday, given that it does not rain on Tuesday.

2

$$P^2 = \begin{vmatrix} .8 & .2 \\ .4 & .6 \end{vmatrix} \begin{vmatrix} .8 & .2 \\ .4 & .6 \end{vmatrix} = \begin{vmatrix} .72 & .28 \\ .56 & .44 \end{vmatrix}$$

So the answer is $P^2_{01} = \underline{\underline{0.28}}$.

(c) Find the probability that it does not rain on Friday, given that it rains on Tuesday.

2

$$P^3 = P^2 \cdot P = \begin{vmatrix} .72 & .28 \\ .56 & .44 \end{vmatrix} \cdot \begin{vmatrix} .8 & .2 \\ .4 & .6 \end{vmatrix} = \begin{vmatrix} .688 & .312 \\ .624 & .376 \end{vmatrix},$$

so the answer is $P^3_{10} = \underline{\underline{0.624}}$.

(c) Find the limiting proportion of days on which it rains.

3

$$\left.\begin{array}{l} \pi_0 = .8\,\pi_0 + .4\,\pi_1 \\ (\pi_1 = .2\,\pi_0 + .6\,\pi_1) \\ \pi_0 + \pi_1 = 1 \end{array}\right\} \Rightarrow .2\pi_0 = .4\pi_1 \text{ OR } \pi_0 = 2\pi_1$$

So $2\pi_1 + \pi_1 = 1$ OR $3\pi_1 = 1$

so $\underline{\underline{\pi_1 = 1/3}}$.

40 marks total