UNIVERSITY OF CALGARY

DEPARTMENT OF MATHEMATICS AND STATISTICS

STAT 407 —- Test no. 2 —- Nov. 21, 2007 — TIME: 50 min.

NAME: Solutions

Marks

10 1. Consider a branching process, initially consisting of a single individual. Let P_j denote the probability that each individual in each generation has j offspring, for $j = 0, 1, 2, \ldots$.Let π_0 denote the probability that the population eventually dies out.

3 (a) Find π_0 when $P_0 = \frac{1}{2}, P_1 = \frac{1}{4}, P_3 = \frac{1}{4}$.

$$\mu = 0(\tfrac{1}{2}) + 1(\tfrac{1}{4}) + 3(\tfrac{1}{4}) = 1,$$

so $\pi_0 = 1$.

4 (b) Find π_0 when $P_0 = \frac{1}{3}, P_1 = \frac{1}{6}, P_2 = \frac{1}{2}$.

$$\mu = 0(\tfrac{1}{3}) + 1(\tfrac{1}{6}) + 2(\tfrac{1}{2}) = \tfrac{7}{6} > 1 \leftarrow (1)$$

So π_0 is smallest positive solution to $\pi_0 = \frac{1}{3} + \frac{1}{6}\pi_0 + \frac{1}{2}\pi_0^2 \leftarrow (1)$

OR $3\pi_0^2 - 5\pi_0 + 2 = 0$ OR $(3\pi_0 - 2)(\pi_0 - 1) = 0$

Solutions are 2/3 and 1. So $\pi_0 = \frac{2}{3}$. $\leftarrow (2)$

3 (c) What is the answer to part(b) if initially the population consisted of four individuals (instead of one individual)?

The pop. dies out $\Leftrightarrow$ the population of each of the 4 independent branches dies out.

So the answer is $\left(\frac{2}{3}\right)^4 = \frac{16}{81}$

ID number: ______________________

10 2. A system consists of five transistors of Type A and three transistors of Type B. The five Type A transistors each have exponentially distributed lifetimes with rate λ_1 failures per hour; and the three Type B transistors each have exponentially distributed lifetimes with rate λ_2 failures per hour.

3 (a) Find the probablity that all eight transistors fail within the first 10 hours of system operation.

$$\left(1-e^{-10\lambda_1}\right)^5\left(1-e^{-10\lambda_2}\right)^3$$

4 (b) Find the probability that exactly two of the eight transistors fail within the first 10 hours.

P[2 A's fail] + P[2 B's fail] + P[1 A and 1 B fail]

$$= \binom{5}{2}\left(1-e^{-10\lambda_1}\right)^2\left(e^{-10\lambda_1}\right)^3\left(e^{-10\lambda_2}\right)^3 + \left(e^{-10\lambda_1}\right)^5\binom{3}{2}\left(1-e^{-10\lambda_2}\right)^2 e^{-10\lambda_2}$$

$$+ 5\left(1-e^{-10\lambda_1}\right)\left(e^{-10\lambda_1}\right)^4 \cdot 3\left(1-e^{-10\lambda_2}\right)\left(e^{-10\lambda_2}\right)^2$$

$$= 10\left(1-e^{-10\lambda_1}\right)^2 e^{-30\lambda_1} e^{-30\lambda_2} + 3e^{-50\lambda_1}\left(1-e^{-10\lambda_2}\right)^2 e^{-10\lambda_2}$$

$$+ 15\left(1-e^{-10\lambda_1}\right)e^{-40\lambda_1}\left(1-e^{-10\lambda_2}\right)e^{-20\lambda_2}$$

3 (c) find the probability that the first transistor that fails is of Type B.

$$\frac{3\lambda_2}{3\lambda_2+5\lambda_1}$$

16 3. A customer entering a shoeshine shop first receives service from Server A, whose service time is exponentially distributed with rate μ_1. The customer then receives service from Server B, whose service time is exponentially distributed with rate μ_2.

Suppose that a customer arrives at the shop to find that Server A is idle, but that another customer is being served by Server B. What is the expected total amount of time that the entering customer is in the shop?

[Hint: Note that when the entering customer finishes with Server A, then it is possible that Server B is still busy with the previous customer, in which case the entering customer will have to wait for Server B to finish with the previous customer.]

exp. time at Server A

prob that A finishes before B

expected time to wait until B finishes with other customer

$$\underbrace{\frac{1}{\mu_1}}_{3} + \underbrace{\frac{\mu_1}{\mu_1+\mu_2}\,\frac{1}{\mu_2}}_{4} + \underbrace{\frac{1}{\mu_2}}_{3}$$

expected time for service by B

10 4. Events occur according to a non-homogeneous Poisson process with intensity function

$$\lambda(t) = \begin{cases} 1+t & \text{if } 0 \le t \le 3 \\ 10-2t & \text{if } 3 < t \le 5 \end{cases}$$

4 (a) Find the mean value function of this process.

For $0 \le t \le 3$, $m(t) = \int_0^t (1+y)\,dy = y + \frac{y^2}{2}\Big]_0^t = t + t^2/2.$

For $3 < t \le 5$, $m(t) = m(3) + \int_3^t (10-2y)\,dy = 7.5 + 10y - y^2\Big]_3^t$

$= 7.5 + 10t - t^2 - (30-9) = -13.5 + 10t - t^2$

So $m(t) = t + \frac{t^2}{2}$, $0 \le t \le 3$ ← 2

$= 10t - t^2 - 13.5$, $3 \le t \le 5$ ← 2

The wrong answer $m(5) = 11.5$ is worth 1 mark.

3 (b) Find the probability that no events occur between time $t = 2$ and time $t = 4$.

Expected # of events between $t=2$ and $t=4$ is $m(4) - m(2) =$

$= -13.5 + 40 - 16 - [2 + \frac{4}{2}] = 10.5 - 4 = \underline{\underline{6.5}}$

So $P[0 \text{ events}] = e^{-6.5}$ $(\approx .001503)$

(1 mark for calculating with wrong expected value)

3 (c) Find the probability that at least two events occur between time $t = 2$ and time $t = 4$.

$P[\#\text{ events} \ge 2] = 1 - P[0 \text{ events}] - P[1 \text{ event}]$

$= 1 - e^{-6.5} - 6.5\,e^{-6.5}$ (≈ 0.9887)