STAT 407

Solutions to Assignment #1

Chapter 2

(1) $P\{X=0\} = \left[\begin{matrix}7\\2\end{matrix}\right] \Big/ \left[\begin{matrix}10\\2\end{matrix}\right] = \frac{14}{30}.$

(2) $-n, -n+2, -n+4, \ldots, n-2, n.$

(3) $P\{X=-2\} = \frac{1}{4} = P\{X=2\}$

$P\{X=0\} = \frac{1}{2}.$

(8) $p(0) = \frac{1}{2}, \quad p(1) = \frac{1}{2}.$

(9) $p(0) = \frac{1}{2}, \quad p(1) = \frac{1}{10}, \quad p(2) = \frac{1}{5},$

$p(3) = \frac{1}{10}, \quad p(3.5) = \frac{1}{10}.$

(10) $1 - \left[\begin{matrix}3\\2\end{matrix}\right] \left[\frac{1}{6}\right]^2 \left[\frac{5}{6}\right] - \left[\begin{matrix}3\\3\end{matrix}\right] \left[\frac{1}{6}\right]^3 = \frac{200}{216}.$

(25) A total of 7 games will be played if the first 6 result in 3 wins and 3 losses. Thus,

$$P\{7 \text{ games}\} = \binom{6}{3} p^3(1-p)^3.$$

Differentiation yields that

$$\begin{aligned}\frac{d}{dp}P\{7\} &= 20\left[3p^2(1-p)^3 - p^3 3(1-p)^2\right] \\ &= 60p^2(1-p)^2\,[1 - 2p].\end{aligned}$$

Thus, the derivative is zero when $p = 1/2$. Taking the second derivative shows that the maximum is attained at this value.

(26) Let X denote the number of games played.

(a) $P\{X = 2\} = p^2 + (1-p)^2$
$P\{X = 3\} = 2p(1-p)$
$E[X] = 2\left\{p^2 + (1-p)^2\right\} + 6p(1-p)$
$= 2 + 2p(1-p).$
Since $p(1-p)$ is maximized when $p = 1/2$, we see that $E[X]$ is maximized at that value of p.

(35) $P\{X > 20\} = \int_{20}^{\infty} \frac{10}{x^2} dx = \frac{1}{2}.$

(36) $$\begin{aligned}P\{D \le x\} &= \frac{\text{area of disk of radius } x}{\text{area of disk of radius } 1} \\ &= \frac{\pi x^2}{\pi} = x^2\end{aligned}$$

(37.) $$\begin{aligned}P\{M \le x\} &= P\{\max(X_1, \ldots, X_n) \le x\} \\ &= P\{X_1 \le x, \ldots, X_n \le x\} \\ &= \prod_{i=1}^{n} P\{X_i \le x\} \\ &= x^n.\end{aligned}$$

$$f_M(x) = \frac{d}{dx}P\{M \le x\} = nx^{n-1}.$$

Chapter 3

1. $\sum_x p_{X|Y}(x|y) = \frac{\sum_x p(x,y)}{p_Y(y)} = \frac{p_Y(y)}{p_Y(y)} = 1.$

3. $E[X|Y=1]=2$

$E[X|Y=2]=\frac{5}{3}$

$E[X|Y=3]=\frac{12}{5}.$

4. No.

5. (a) $P\{X=i|Y=3\} = P\{i$ white balls selected when choosing 3 balls from 3 white and 6 red$\}$

$$= \frac{\begin{bmatrix}3\\i\end{bmatrix}\begin{bmatrix}6\\3-i\end{bmatrix}}{\begin{bmatrix}9\\3\end{bmatrix}}, \quad i=0,1,2,3.$$

(b) By same reasoning as in (a), if $Y=1$, then X has the same distribution as the number of white balls chosen when 5 balls are chosen from 3 white and 6 red. Hence,

$$E[X|Y] = 5\frac{3}{9} = \frac{5}{3}.$$

11. $E[X|Y=y] = C\int_{-y}^{y} x(y^2-x^2)dx = 0$

12. $$f_{X|Y}(x|y) = \frac{\frac{1}{y}\exp^{-x/y}\exp^{-y}}{\exp^{-y}\int \frac{1}{y}\exp^{-x/y}dx} = \frac{1}{y}\exp^{-x/y}$$

Hence, given $Y=y$, X is exponential with mean y.

15. $$f_{X|Y=y}(x|y) = \frac{\frac{1}{y}\exp^{-y}}{f_y(y)} = \frac{\frac{1}{y}\exp^{-y}}{\int_0^y \frac{1}{y}\exp^{-y}\,dx}$$
$$= \frac{1}{y}, \quad 0 < x < y$$

$$E[X^2|Y=y] = \frac{1}{y}\int_0^y x^2 dx = \frac{y^2}{3}.$$

24. In all parts, let X denote the random variable whose expectation is desired, and start by conditioning on the result of the first flip. Also, h stands for heads and t for tails.

(a) $$E[X] = E[X|h]p + E[X|t](1-p)$$
$$= \left(1 + \frac{1}{1-p}\right)p + \left(1 + \frac{1}{p}\right)(1-p)$$
$$= 1 + p/(1-p) + (1-p)/p$$

(b) $$E[X] = (1 + E[\text{number of heads before first tail}])p + 1(1-p)$$
$$= 1 + p(1/(1-p) - 1)$$
$$= 1 + p/(1-p) - p$$

(c) Interchanging p and $1-p$ in (b) gives result: $1 + (1-p)/p - (1-p)$

(d) $$E[X] = (1 + \text{answer from } (a))p + (1 + 2/p)(1-p)$$
$$= (2 + p/(1-p) + (1-p)/p)p + (1 + 2/p)(1-p)$$

25. Let W denote the number of wins.

(a) $$E[W] = E[E[W|X]] = E[X + Xp] = (1+p)E[X] = (1+p)np$$

(b) $$E[W] = E[E[W|Y]] = E[1 + Yp] = 1 + p/p = 2$$
since Y is geometric with mean $1/p$.

(26) Let N_A and N_B denote the number of games needed given that you start with A and given that you start with B. Conditioning on the outcome of the first game gives

$$E[N_A] = E[N_A|w]p_A + E[N_A|l](1 - p_A)$$

Conditioning on the outcome of the next game gives

$$\begin{aligned} E[N_A|w] &= E[N_A|ww]p_B + E[N_A|wl](1 - p_B) \\ &= 2p_B + (2 + E[N_A])(1 - p_B) \\ &= 2 + (1 - p_B)E[N_A] \end{aligned}$$

As, $E[N_A|l] = 1 + E[N_B]$, we obtain that

$$\begin{aligned} E[N_A] &= (2 + (1 - p_B)E[N_A])p_A \\ &\quad + (1 + E[N_B])(1 - p_A) \\ &= 1 + p_A + p_A(1 - p_B)E[N_A] \\ &\quad + (1 - p_A)E[N_B] \end{aligned}$$

Similarly,

$$\begin{aligned} E[N_B] &= 1 + p_B + p_B(1 - p_A)E[N_B] \\ &\quad + (1 - p_B)E[N_A] \end{aligned}$$

Subtracting gives

$$\begin{aligned} &E[N_A] - E[N_B] \\ &= p_A - p_B + (p_A - 1)(1 - p_B)E[N_A] \\ &\quad + (1 - p_B)(1 - p_A)E[N_B] \end{aligned}$$

or

$$[1 + (1 - p_A)(1 - p_B)](E[N_A] - E[N_B]) = p_A - p_B$$

Hence, if $p_B > p_A$ then $E[N_A] - E[N_B] < 0$, showing that playing A first is better.

(40.) Let X denote the number of door chosen, and let N be the total number of days spent in jail.

(a) Conditioning on X, we get

$$E[N] = \sum_{i=1}^{3} E\{N|X = i\}P\{X = 1\}.$$

(continued on next page)

40 (continued)

The process restarts each time the prisoner returns to his cell. Therefore,

$$E(N|X=1) = 2 + E(N)$$
$$E(N|X=2) = 3 + E(N)$$
$$E(N|X=3) = 0.$$

and

$$E(N) = (.5)(2 + E(N)) + (.3)(3 + E(N)) + (.2)(0),$$

or

$$E(N) = 9.5 \text{ days}.$$

(b) Let N_i denote the number of additional days the prisoner spends after having initially chosen cell i.

$$E[N] = \frac{1}{3}(2 + E[N_1]) + \frac{1}{3}(3 + E[N_2]) + \frac{1}{3}(0) = \frac{5}{3} + \frac{1}{3}(E[N_1] + E[N_2]).$$

Now,

$$E[N_1] = \frac{1}{2}(3) + \frac{1}{2}(0) = \frac{3}{2}$$
$$E[N_2] = \frac{1}{2}(2) + \frac{1}{2}(0) = 1$$

and so,

$$E[N] = \frac{5}{3} + \frac{1}{3}\frac{5}{2} = \frac{5}{2}.$$

41. Let N denote the number of minutes in the maze. If L is the event the rat chooses its left, and R the event it chooses its right, we have by conditioning on the first direction chosen:

$$\begin{aligned} E(N) &= \frac{1}{2}E(N|L) + \frac{1}{2}E(N|R) \\ &= \frac{1}{2}\left[\frac{1}{3}(2) + \frac{2}{3}(5 + E(N))\right] + \frac{1}{2}[3 + E(N)] \\ &= \frac{5}{6}E(N) + \frac{21}{6} \\ &= 21. \end{aligned}$$

50. $$P\{N = n\} = \frac{1}{3}\left[\begin{bmatrix}10\\n\end{bmatrix}(.3)^n(.7)^{10-n} + \begin{bmatrix}10\\n\end{bmatrix}(.5)^n(.5)^{10-n} + \begin{bmatrix}10\\n\end{bmatrix}(.7)^n(.3)^{10-n}\right].$$

N is not binomial.

$$E[N] = 3\left[\frac{1}{3}\right] + 5\left[\frac{1}{3}\right] + 7\left[\frac{1}{3}\right] = 5.$$

62. Let W and L stand for the events that player A wins a game and loses a game, respectively. Let $P(A)$ be the probability that A wins, and let $P(C)$ be the probability that C wins, and note that this is equal to the conditional probability that a player about to compete against the person who won the last round is the overall winner.

$$\begin{aligned}P(A) &= (1/2)P(A|W) + (1/2)P(A|L)\\ &= (1/2)[1/2 + (1/2)P(A|WL)]\\ &\quad + (1/2)(1/2)P(C)\\ &= 1/4 + (1/4)(1/2)P(C)\\ &\quad + (1/4)P(C) = 1/4 + (3/8)P(C)\end{aligned}$$

Also,

$$P(C) = (1/2)P(A|W) = 1/4 + (1/8)P(C)$$

and so

$$P(C) = 2/7, \quad P(A) = 5/14,$$
$$P(B) = P(A) = 5/14$$

64. (a) $P(A) = 5/36 + (31/36)(5/6)P(A)$

$\rightarrow P(A) = 30/61$

(b) $E[X] = 5/36 + (31/36)[1 + 1/6 + (5/6)$

$(1 + E[X])] \rightarrow E[X] = 402/61$

(c) Let Y equal 1 if A wins on her first attempt, let it equal 2 if B wins on his first attempt, and let it equal 3 otherwise. Then

$$Var(X|Y = 1) = 0, \quad Var(X|Y = 2) = 0,$$
$$Var(X|Y = 3) = Var(X)$$

Hence,

$$E[Var(X|Y)] = (155/216)Var(X)$$

Also,

$$E[X|Y = 1] = 1, \quad E[X|Y = 2] = 2,$$
$$E[X|Y = 3] = 2 + E[X] = 524/61$$

and so

$$Var(E[X|Y]) = 1^2(5/36) + 2^2(31/216) + (524/61)^2(155/216) - (402/61)^2 \approx 10.2345$$

Hence, from the conditional variance formula we see that

$$Var(X) \approx z(155/216)Var(X) + 10.2345$$
$$\rightarrow Var(X) \approx 36.24$$

Chapter 4

5. Cubing the transition probability matrix, we obtain P^3:

$$\begin{bmatrix} 13/36 & 11/54 & 47/108 \\ 4/9 & 4/27 & 11/27 \\ 5/12 & 2/9 & 13/36 \end{bmatrix}$$

Thus,

$$E[X_3] = \frac{1}{4}47/108 + \frac{1}{4}11/27 + \frac{1}{2}13/36$$

6. It is immediate for $n = 1$, so assume for n. Now use induction.

8. Let the state on any day be the number of the coin that is flipped on that day.

$$\underline{P} = \begin{bmatrix} .7 & .3 \\ .6 & .4 \end{bmatrix}$$

and so,

$$\underline{P}^2 = \begin{bmatrix} .67 & .33 \\ .66 & .34 \end{bmatrix}$$

and

$$\underline{P}^3 = \begin{bmatrix} .667 & .333 \\ .666 & .334 \end{bmatrix}$$

Hence,

$$\frac{1}{2}\left[P_{11}^3 + P_{21}^3\right] \equiv .6665.$$

10. The answer is $1 - P_{0,2}^3$ for the Markov chain with transition probability matrix

$$\begin{bmatrix} .5 & .4 & .1 \\ .3 & .4 & .3 \\ 0 & 0 & 1 \end{bmatrix}$$

11. The answer is $\dfrac{P_{2,2}^4}{1 - P_{2,0}^4}$ for the Markov chain with transition probability matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ .3 & .4 & .3 \\ .2 & .3 & .5 \end{bmatrix}$$

14. (i) $\{0,1,2\}$ recurrent.

(ii) $\{0,1,2,3\}$ recurrent.

(iii) $\{0,2\}$ recurrent, $\{1\}$ transient, $\{3,4\}$ recurrent.

(iv) $\{0,1\}$ recurrent, $\{2\}$ recurrent, $\{3\}$ transient, $\{4\}$ transient.

18. If the state at time n is the n^{th} coin to be flipped then sequence of consecutive states constitute a two state Markov chain with transition probabilities

$$P_{1,1} = .6 = 1 - P_{1,2}, \quad P_{2,1} = .5 = P_{2,2}$$

(a) The stationary probabilities satisfy

$$\pi_1 = .6\pi_1 + .5\pi_2$$
$$\pi_1 + \pi_2 = 1$$

Solving yields that $\pi_1 = 5/9, \pi_2 = 4/9$. So the proportion of flips that use coin 1 is 5/9.

(b) $P^4_{1,2} = .44440$

23. Let the state be 0 if the last two trials were both successes. 1 if the last trial was a success and the one before it a failure. 2 if the last trial was a failure. The transition probability matrix of this Markov chain is

$$P = \begin{bmatrix} .8 & 0 & .2 \\ .5 & 0 & .5 \\ 0 & .5 & .5 \end{bmatrix}$$

This gives $\pi_0 = 5/11, \pi_1 = 2/11, \pi_2 = 4/11$. Consequently, the proportion of trials that are successes is $.8\pi_0 + .5(1 - \pi_0) = 7/11$.

24. Let the state be the color of the last ball selected, call it 0 if that color was red, 1 if white, and 2 if blue. The transition probability matrix of this Markov chain is

$$P = \begin{bmatrix} 1/5 & 0 & 4/5 \\ 2/7 & 3/7 & 2/7 \\ 3/9 & 4/9 & 2/9 \end{bmatrix}$$

Solve for the stationary probabilities to obtain the solution.

(28) There are 4 states: 1 = success on last 2 trials;
2 = success on last, failure on next to last;
3 = failure on last, success on next to last;
4 = failure on last 2 trials.

Transition probabilities are:

$$P_{1,1} = \frac{3}{4}, \quad P_{1,3} = \frac{1}{4}$$
$$P_{2,1} = \frac{2}{3}, \quad P_{2,3} = \frac{1}{3}$$
$$P_{3,2} = \frac{2}{3}, \quad P_{3,4} = \frac{1}{3}$$
$$P_{4,2} = \frac{1}{2}, \quad P_{4,4} = \frac{1}{2}.$$

Limiting probabilities are given by

$$\prod_1 = \frac{3}{4}\prod_1 + \frac{2}{3}\prod_2$$
$$\prod_2 = \frac{2}{3}\prod_3 + \frac{1}{2}\prod_4$$
$$\prod_3 = \frac{1}{4}\prod_1 + \frac{1}{3}\prod_2$$
$$\prod_1 + \prod_2 + \prod_3 + \prod_4 = 1,$$

and the solution is $\prod_1 = 1/2, \prod_2 = 3/16, \prod_3 = 3/16, \prod_4 = 1/8$. Hence, the desired answer is $\prod_1 + \prod_2 = 11/16$.

(29) Each employee moves according to a Markov chain whose limiting probabilities are the solution of

$$\prod_1 = .7\prod_1 + .2\prod_2 + .1\prod_3$$
$$\prod_2 = .2\prod_1 + .6\prod_2 + .4\prod_3$$
$$\prod_1 + \prod_2 + \prod_3 = 1.$$

Solving yields $\prod_1 = 6/17, \prod_2 = 7/17, \prod_3 = 4/17$. Hence, if N is large, it follows from the law of large numbers that approximately 6, 7, and 4 of each 17 employees are in categories 1, 2, and 3.

(33.) Consider the Markov chain whose state at time n is the type of exam number n. The transition probabilities of this Markov chain are obtained by conditioning on the performance of the class. This gives the following.

$$P_{11} = .3(1/3) + .7(1) = .8$$
$$P_{12} = P_{13} = .3(1/3) = .1$$
$$P_{21} = .6(1/3) + .4(1) = .6$$
$$P_{22} = P_{23} = .6(1/3) = .2$$
$$P_{31} = .9(1/3) + .1(1) = .4$$
$$P_{32} = P_{33} = .9(1/3) = .3$$

Let r_i denote the proportion of exams that are type $i, i = 1, 2, 3$. The r_i are the solutions of the following set of linear equations.

$$r_1 = .8\, r_1 + .6\, r_2 + .4\, r_3$$
$$r_2 = .1\, r_1 + .2\, r_2 + .3\, r_3$$
$$r_1 + r_2 + r_3 = 1$$

Since $P_{i2} = P_{i3}$ for all states i, it follows that $r_2 = r_3$. Solving the equations gives the solution

$$r_1 = 5/7, \quad r_2 = r_3 = 1/7.$$

(46) (i) Let the state be the number of umbrellas he has at his present location. The transition probabilities are

$$P_{0,x} = 1, P_{i,r-i} = 1 - p, P_{i,r-i+1} = p, \quad i = 1, \ldots, r.$$

(ii) We must show that $\prod_j = \sum_1 r_i P_{ij}$ is satisfied by the given solution. These equations reduce to

$$r_r = r_0 + r_1 p$$
$$r_j = r_{r-j}(1-p) + r_{r-j+1}p, \quad j = 1, \ldots, r-1$$
$$r_0 = r_r(1-p),$$

and it is easily verified that they are satisfied.

(iii) $pr_0 = \dfrac{pq}{r+q}$.

(iv)
$$\frac{d}{dp}\left[\frac{p(1-p)}{4-p}\right] = \frac{(4-p)(1-2p) + p(1-p)}{(4-p)^2} = \frac{p^2 - 8p + 4}{(4-p)^2}$$

$$p^2 - 8p + 4 = 0 \Rightarrow p = \frac{8 - \sqrt{48}}{2} = .55.$$