STAT 407

Solutions to Assignment 2

Chapter 4

56. This is just the probability that a gambler starting with m reaches her goal of $n+m$ before going broke, and is thus equal to $\frac{1-(q/p)^m}{1-(q/p)^{n+m}}$, where $q = 1 - p$.

57. Let A be the event that all states have been visited by time T. Then, conditioning on the direction of the first step gives

$$P(A) = P(A|\text{clockwise})p$$
$$+P(A|\text{counterclockwise})q$$
$$= p\frac{1-q/p}{1-(q/p)^n} + q\frac{1-p/q}{1-(p/q)^n}$$

The conditional probabilities in the preceding follow by noting that they are equal to the probability in the gambler's ruin problem that a gambler that starts with 1 will reach n before going broke when the gambler's win probabilities are p and q.

58. Using the hint, we see that the desired probability is

$$P\{X_{n+1} = i+1|X_n = i\}$$
$$\frac{P\{lim\, X_m = N|X_n = i, X_n + 1 = i+1\}}{P\{lim\, X_m = N|X_n = 1\}}$$
$$= \frac{p^P i + 1}{P_i}$$

and the result follows from Equation (4.74).

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59. Condition on the outcome of the initial play.

61. With $P_0 = 0,\ P_N = 1$

$$P_i = \alpha_i P_{i+1} + (1-\alpha_i)P_{i-1}, \quad i = 1, \ldots, N-1$$

These latter equations can be rewritten as

$$P_{i+1} - P_i = \beta_i(P_i - P_{i-1})$$

where $\beta_i = (1-\alpha_i)/\alpha_i$. These equations can now be solved exactly as in the original gambler's ruin problem. They give the solution

$$P_i = \frac{1 + \sum_{j=1}^{i-1} C_j}{1 + \sum_{j=1}^{N-1} C_j}, \quad i = 1, \ldots, N-1$$

where

$$C_j = \prod_{i=1}^{j} \beta_i$$

(c) P_{N-i}, where $\alpha_i = (N-i)/N$

Chapter 5

4. (a) 0, (b) $\frac{1}{27}$, (c) $\frac{1}{4}$.

5. e^{-1} by lack of memory.

6. Condition on which server initially finishes first. Now,

$$P\{\text{Smith is last}|\text{server 1 finishes first}\}$$
$$= P\{\text{server 1 finishes before server 2}\}$$
by lack of memory
$$= \frac{\lambda_1}{\lambda_1 + \lambda_2}.$$

Similarly,

$$P\{\text{Smith is last}|\text{server 2 finished first}\} = \frac{\lambda_2}{\lambda_1 + \lambda_2}$$

and thus

$$P\{\text{Smith is last}\} = \left[\frac{\lambda_1}{\lambda_1 + \lambda_2}\right]^2 + \left[\frac{\lambda_2}{\lambda_1 + \lambda_2}\right]^2.$$

9. Condition on whether machine 1 is still working at time t, to obtain the answer,

$$1 - e^{-\lambda_1 t} + e^{-\lambda_1 t}\frac{\lambda_1}{\lambda_1 + \lambda_2}$$

14. (a) The conditional density of X gives that $X < c$ is

$$f(x|X < c) = \frac{f(x)}{P\{x < c\}} = \frac{\lambda e^{-\lambda x}}{1 - e^{-\lambda c}}, \ 0 < x < c$$

Hence,

$$E[X|X < c] = \int_0^c x\lambda e^{-\lambda x} dx/(1 - e^{-\lambda c}).$$

Integration by parts yields that

$$\int_0^c x\lambda e^{-\lambda x}\, dx = -xe^{-\lambda x}\Big|_0^c + \int_0^c e^{-\lambda x} dx$$
$$= -ce^{-\lambda c} + (1 - e^{-\lambda c})/\lambda.$$

Hence,

$$E[X|X < c] = 1/\lambda - ce^{-\lambda c}/(1 - e^{-\lambda c}).$$

(b) $1/\lambda = E[X|X < c](1 - e^{-\lambda c}) + (c + 1/\lambda)e^{-\lambda c}$
This simplifies to the same answer as given in part (a).

15. Let T_i denote the time between the $(i-1)^{th}$ and the i^{th} failure. Then the T_i are independent with T_i being exponential with rate $(101 - i)/200$. Thus,

$$E[T] = \sum_{i=1}^{5} E[T_i] = \sum_{i=1}^{5} \frac{200}{101 - i}$$

$$Var(T) = \sum_{i=1}^{5} Var(T_i) = \sum_{i=1}^{5} \frac{(200)^2}{(101 - i)^2}$$

16. Letting T_i denote the time between departure $i-1$ and departure i, we have

$$E[T] = E[T_1] + E[T_2] + E[T_3]$$

The random variables T_1 and T_2 are both exponential with rate $\lambda_1 + \lambda_2$, and so have mean $1/(\lambda_1 + \lambda_2)$. To determine $E[T_3]$ consider the time at which the first customer has departed and condition on which server completes the next service. This gives:

$$\begin{aligned} E[T_3] &= E[T_3|\text{server } 1]\ [\lambda_1/(\lambda_1+\lambda_2)] \\ &= E[T_3|\text{server } 2][\lambda_2/(\lambda_1+\lambda_2)] \\ &= (1/\lambda_2)[\lambda_1/(\lambda_1+\lambda_2)] \\ &\quad + (1/\lambda_1)[\lambda_2/(\lambda_1+\lambda_2)]. \end{aligned}$$

Therefore,

$$E[\text{Time}] = 2/(\lambda_1+\lambda_2) + (1/\lambda_2)[\lambda_1/(\lambda_1+\lambda_2)] + (1/\lambda_1)[\lambda_2/(\lambda_1+\lambda_2)].$$

20. (a) $P_A = \dfrac{\mu_1}{\mu_1+\mu_2}$

(b) $P_B = 1 - \left(\dfrac{\mu_2}{\mu_1+\mu_2}\right)^2$

(c) $E[T] = 1/\mu_1 + 1/\mu_2 + P_A/\mu_2 + P_B/\mu_2$

21. $E[\text{time}] = E[\text{time waiting at } 1] + 1/\mu_1 + E[\text{time waiting at } 2] + 1/\mu_2.$

Now

$E[\text{time waiting at } 1] = 1/\mu_1,$

$E[\text{time waiting at } 2] = (1/\mu_2)\dfrac{\mu_1}{\mu_1+\mu_2}.$

The last equation follows by conditioning on whether or not the customer waits for server 2. Therefore,

$$E[\text{time}] = 2/\mu_1 + (1/\mu_2)[1 + \mu_1/(\mu_1+\mu_2)].$$

22. $E[\text{time}] = E[\text{time waiting for server 1}] + 1/\mu_1$
$+ E[\text{time waiting for server 2}] + 1/\mu_2.$

Now, the time spent waiting for server 1 is the remaining service time of the customer with server 1 plus any additional time due to that customer blocking your entrance. If server 1 finishes before server 2 this additional time will equal the additional service time of the customer with server 2. Therefore,

$$E[\text{time waiting for server 1}]$$
$$= 1/\mu_1 + E[\text{Additional}]$$
$$= 1/\mu_1 + (1/\mu_2)[\mu_1/(\mu_1 + \mu_2)].$$

Since when you enter service with server 1 the customer preceding you will be catering service with server 2, it follows that you will have to wait for server 2 if you finish service first. Therefore, conditioning on whether or not you finish first

$$E[\text{time waiting for server 2}]$$
$$= (1/\mu_2)[\mu_1/(\mu_1 + \mu_2)].$$

Thus,

$$E[\text{time}] = 2/\mu_1 + (2/\mu_2)[\mu_1/(\mu_1 + \mu_2)] + 1/\mu_2.$$

25. Parts (a) and (b) follow upon integration. For part (c), condition on which of X or Y is larger and use the lack of memory property to conclude that the amount by which it is larger is exponential rate λ. For instance, for $x < 0$,

$$fx - y(x)dx$$
$$= P\{X < Y\}P\{-x < Y - X < -x + dx | Y > X\}$$
$$= \frac{1}{2}\lambda e^{\lambda x} dx$$

For (d) and (e), condition on I.

26. (a) $$\frac{1}{\mu_1 + \mu_2 + \mu_3} + \sum_{i=1}^{3} \frac{\mu_i}{\mu_1 + \mu_2 + \mu_3} \frac{1}{\mu_i}$$
$$= \frac{4}{\mu_1 + mu_2 + \mu_3}$$

(b) $$\frac{1}{\mu_1 + \mu_2 + \mu_3} + (a) = \frac{5}{\mu_1 + \mu_2 + \mu_3}$$

30. Condition on which animal died to obtain

E[additional life]

$= E$[additional life | dog died]

$\dfrac{\lambda_d}{\lambda_c + \lambda_d} + E$[additional life | cat died] $\dfrac{\lambda_c}{\lambda_c + \lambda_d}$

$$= \frac{1}{\lambda_c}\frac{\lambda_d}{\lambda_c + \lambda_d} + \frac{1}{\lambda_d}\frac{\lambda_c}{\lambda_c + \lambda_d}$$

31. Condition on whether the 1 PM appointment is still with the doctor at 1:30, and use the fact that if she or he is then the remaining time spent is exponential with mean 30. This gives

E[time spent in office]

$$= 30(1 - e^{-30/30}) + (30 + 30)e^{-30/30}$$
$$= 30 + 30e^{-1}$$

34. (a) $\dfrac{\lambda}{\lambda + \mu_A}$

(b) $\dfrac{\lambda + \mu_A}{\lambda + \mu_A + \mu_B} \cdot \dfrac{\lambda}{\lambda + \mu_B}$

41. $\lambda_1/(\lambda_1 + \lambda_2)$.

42. (a) $E[S_4] = 4/\lambda$.

(b) $E[S_4|N(1) = 2]$
$= 1 + E$[time for 2 more events] $= 1 + 2/\lambda$.

(c) $E[N(4) - N(2)|N(1) = 3] = E[N(4) - N(2)]$
$= 2\lambda$.

The first equality used the independent increments property.

43. Let S_i denote the service time at server $i, i = 1, 2$ and let X denote the time until the next arrival. Then, with p denoting the proportion of customers that are served by both servers, we have

$$p = P\{X > S_1 + S_2\}$$
$$= P\{X > S_1\}PX > S_1 + S_2|X > S_1\}$$
$$= \frac{\mu_1}{\mu_1 + \lambda}\frac{\mu_2}{\mu_2 + \lambda}$$

44. (a) $e^{-\lambda T}$

(b) Let W denote the waiting time and let X denote the time until the first car. Then

$$E[W] = \int_0^\infty E[W|X = x]\lambda e^{-\lambda x}dx$$
$$= \int_0^T E[W|X = x]\lambda e^{-\lambda x}dx$$
$$+ \int_T^\infty E[W|X = x]\lambda e^{-\lambda x}dx$$
$$= \int_0^T (x + E[W])\lambda e^{-\lambda x}dx + Te^{-\lambda T}$$

Hence,

$$E[W] = T + e^{\lambda T}\int_0^T x\lambda e^{-\lambda x}dx$$

47. (a) $1/(2\mu) + 1/\lambda$

(b) Let T_i denote the time until both servers are busy when you start with i busy servers $i =$ 0.1. Then,

$$E[T_0] = 1/\lambda + E[T_1]$$

Now, starting with 1 server busy, let T be the time until the first event (arrival or departure); let $X = 1$ if the first event is an arrival and let it be 0 if it is a departure; let Y be the additional time after the first event until both servers are busy.

$$E[T_1] = E[T] + E[Y]$$
$$= \frac{1}{\lambda+\mu} + E[Y|X=1]\frac{\lambda}{\lambda+\mu}$$
$$+ E[Y|X=0]\frac{\mu}{\lambda+\mu}$$
$$= \frac{1}{\lambda+\mu} + E[T_0]\frac{\mu}{\lambda+\mu}$$

Thus,

$$E[T_0] - \frac{1}{\lambda} = \frac{1}{\lambda+\mu} + E[T_0]\,\frac{\mu}{\lambda+\mu}$$

or

$$E[T_0] = \frac{2\lambda+\mu}{\lambda^2}$$

Also,

$$E[T_1] = \frac{\lambda+\mu}{\lambda^2}$$

(continued)

47 (continued)

(c) Let L_i denote the time until a customer is lost when you start with i busy servers. Then, reasoning as in part (b) gives that

$$E[L_2] = \frac{1}{\lambda+\mu} + E[L_1]\frac{\mu}{\lambda+\mu}$$
$$= \frac{1}{\lambda+\mu} + (E[T_1] + E[L_2])\frac{\mu}{\lambda+\mu}$$
$$= \frac{1}{\lambda+\mu} + \frac{\mu}{\lambda^2} + E[L_2]\frac{\mu}{\lambda+\mu}$$

Thus,

$$E[L_2] = \frac{1}{\lambda} + \frac{\mu(\lambda+\mu)}{\lambda^3}$$

50. Let T denote the time until the next train arrives; and so T is uniform on $(0, 1)$. Note that, conditional on T, X is Poisson with mean $7T$.

(a) $E[X] = E[E[X|T]] = E[7T] = 7/2$.

(b) $E[X|T] = 7T$, $Var(X|T) = 7T$. By the conditional variance formula $Var(X) = 7E[T] + 49Var[T] = 7/2 + 49/12 = 91/12$.

59. The unconditional probability that the claim is type 1 is 10/11. Therefore,

$$P(1|4000) = \frac{P(4000|1)P(1)}{P(4000|1)P(1) + P(4000|2)P(2)}$$
$$= \frac{e^{-4}10/11}{e^{-4}10/11 + .2e^{-.8}1/11}$$

61. (a) Poisson with mean $cG(t)$.

(b) Poisson with mean $c[1 - G(t)]$.

(c) Independent.

66. The number of unreported claims is distributed as the number of customers in the system for the infinite server Poisson queue.

(a) $e^{-a(t)}(a(t))^n/n!$, where $a(t) = \lambda \int_0^t \bar{G}(y)dy$

(b) $a(t)\mu_F$, where μ_F is the mean of the distribution F.

70. (a) Let A be the event that the first to arrive is the first to depart, let S be the first service time, and let $X(t)$ denote the number of departures by time t.

$$
\begin{aligned}
P(A) &= \int P(A|S=t)g(t)dt \\
&= \int P\{X(t)=0\}g(t)dt \\
&= \int e^{-\lambda \int_0^t G(y)dy} g(t)dt
\end{aligned}
$$

(b) Given $N(t)$, the number of arrivals by t, the arrival times are iid uniform $(0,t)$. Thus, given $N(t)$, the contribution of each arrival to the total remaining service times are independent with the same distribution, which does not depend on $N(t)$.

(c) and (d) If, conditional on $N(t)$, X is the contribution of an arrival, then

$$E[X] = \frac{1}{t}\int_0^t \int_{t-s}^{\infty} (s+y-t)g(y)dyds$$

$$E[X^2] = \frac{1}{t}\int_0^t \int_{t-s}^{\infty} (s+y-t)^2 g(y)dyds$$

$$E[S(t)] = \lambda t E[X] \quad Var(S(t)) = \lambda t E[X^2]$$

77 Answer is $e^{-11}(11)^n/n!$

Chapter 6

1. Let us assume that the state is (n,m). Male i mates at a rate λ with female j, and therefore it mates at a rate λm. Since there are n males, matings occur at a rate λnm. Therefore,

$$v_{(n,m)} = \lambda nm.$$

Since any mating is equally likely to result in a female as in a male, we have

$$P_{(n,m);(n+1,m)} = P_{(n,m)(n,m+1)} = \frac{1}{2}.$$

3. This is not a birth and death process since we need more information than just the number working. We also must know which machine is working. We can analyze it by letting the states be

b : both machines are working
1 : 1 is working, 2 is down
2 : 2 is working, 1 is down
0_1: both are down, 1 is being serviced
0_2: both are down, 2 is being serviced.

$$v_b = \mu_1 + \mu_2,\ v_1 = \mu_1 + \mu,\ v_2 = \mu_2 + \mu,$$

$$v_{0_1} = v_{0_2} = \mu$$

$$P_{b,1} = \frac{\mu_2}{\mu_2 + \mu_1} = 1 - P_{b,2}, \quad P_{1,b} = \frac{\mu}{\mu + \mu_1}$$
$$= 1 - P_{1,0_2}$$

$$P_{2,b} = \frac{\mu}{\mu + \mu_2} = 1 - P_{2,0_1}, \quad P_{0_1,1} = P_{0_2,2} = 1.$$

5. (a) Yes.
(b) It is a pure birth process.
(c) If there are i infected individuals then since a contact will involve an infected and an uninfected individual with probability $i(n-i)/\binom{n}{2}$, it follows that the birth rates are $\lambda_i = \lambda i(n-i)/\binom{n}{2}$, $i = 1, \ldots, n$. Hence,

$$E[\text{time all infected}] = \frac{n(n-1)}{2\lambda} \sum_{i=1}^{n} 1/[i(n-i)].$$

6. Starting with $E[T_0] = \dfrac{1}{\lambda_0} = \dfrac{1}{\lambda}$, employ the identity

$$E[T_i] = \frac{1}{\lambda_i} + \frac{\mu_i}{\lambda_i} E[T_{i-1}]$$

to successively compute $E[T_i]$ for $i = 1, 2, 3, 4$.

(a) $E[T_0] + \cdots + E[T_3]$.
(b) $E[T_2] + E[T_3] + E[T_4]$.