STAT 407

Solutions to Assignment #3

Chapter 6

12. (a) If the state is the number of individuals at time t, we get a birth and death process with

$$\lambda_n = n\lambda + \theta \qquad n < N$$

$$\lambda_n = n\lambda \qquad n \geq N$$

$$\mu_n = n\mu.$$

(b) Let P_i be the long-run probability that the system is in state i. Since this is also the proportion of time the system is in state i, we are looking for $\sum_{i=3}^{\infty} P_i$.

We have $\lambda_k P_k = \mu_{k+1} P_{k+1}$.

This yields

$$P_1 = \frac{\theta}{\mu} P_0$$

$$P_2 = \frac{\lambda + \theta}{2\mu} P_1 = \frac{\theta(\lambda + \theta)}{2\mu^2} P_0$$

$$P_3 = \frac{2\lambda + \theta}{2\mu} P_2 = \frac{\theta(\lambda + \theta)(2\lambda + \theta)}{6\mu^3} P_0.$$

For $k \geq 4$, we get

$$P_k = \frac{(k-1)\lambda}{k\mu} P_{k-1},$$

which implies

$$P_k = \frac{(k-1)(k-2)\cdots(3)}{(k)(k-1)\cdots(4)} \left[\frac{\lambda}{\mu}\right]^{k-3}$$

$$P_3 = \frac{3}{k} \left[\frac{\lambda}{\mu}\right]^{k-3} P_3;$$

therefore $\sum_{k=3}^{\infty} P_k = 3 \left[\frac{\mu}{\lambda}\right]^3 P_3 \sum_{k=3}^{\infty} \frac{1}{k} \left[\frac{\lambda}{\mu}\right]^k,$

(continued)

12(b) (continued)

but $$\sum_{k=1}^{\infty}\frac{1}{k}\left[\frac{\lambda}{\mu}\right]^k = \log\left[\frac{1}{1-\frac{\lambda}{\mu}}\right]$$

$$= \log\left[\frac{\mu}{\mu-\lambda}\right] \text{ if } \frac{\lambda}{\mu} < 1.$$

So $$\sum_{k=3}^{\infty} P_k = 3\left[\frac{\mu}{\lambda}\right]^3 P_3\left[\log\left[\frac{\mu}{\mu-\lambda}\right] - \frac{\lambda}{\mu} - \frac{1}{2}\left[\frac{\lambda}{\mu}\right]^2\right]$$

$$\sum_{k=3}^{\infty} P_k = 3\left[\frac{\mu}{\lambda}\right]^3\left[\log\left[\frac{\mu}{\mu-\lambda}\right] - \frac{\lambda}{\mu} - \frac{1}{2}\left[\frac{\lambda}{\mu}\right]^2\right] \frac{\theta(\lambda+\theta)(2\lambda+\theta)}{6\mu^3} P_0.$$

Now $\sum_0^{\infty} P_i = 1$ implies

$$P_0 = \left[1 + \frac{\theta}{\mu} + \frac{\theta(\lambda+\theta)}{2\mu^2} + \frac{1}{2\lambda^3}\theta(\lambda+\theta)(2\lambda+\theta) \times \left[\log\left[\frac{\mu}{\mu-\lambda}\right] - \frac{\lambda}{\mu} - \frac{1}{2}\left[\frac{\lambda}{\mu}\right]^2\right]\right]^{-1}.$$

And finally,

$$\sum_{k=3}^{\infty} P_k = \left[\left[\frac{1}{2\lambda^3}\right]\left[\log\left[\frac{\mu}{\mu-\lambda}\right] - \frac{\lambda}{\mu} - \frac{1}{2}\left[\frac{\lambda}{\mu}\right]^2\right] \theta(\lambda+\theta)(2\lambda+\theta)\right] \Big/ \left[1 + \frac{\theta}{\mu} + \frac{\theta(\lambda+\theta)}{2\mu^2} + \frac{\theta(\lambda+\theta)(2\lambda+\theta)}{2\lambda^3} \times \left[\log\left[\frac{\mu}{\mu-\lambda}\right] - \frac{\lambda}{\mu} - \frac{1}{2}\left[\frac{\lambda}{\mu}\right]^2\right]\right].$$

13. With the number of customers in the shop as the state, we get a birth and death process with

$\lambda_0 = \lambda_1 = 3 \quad \mu_1 = \mu_2 = 4.$

Therefore

$$P_1 = \frac{3}{4}P_0 \quad P_2 = \frac{3}{4} \quad P_1 = \left[\frac{3}{4}\right]^2 P_0.$$

And since $\sum_0^2 P_i = 1$, we get

$$P_0 = \left[1 + \frac{3}{4} + \left[\frac{3}{4}\right]^2\right]^{-1} = \frac{16}{37}.$$

(a) The average number of customers in the shop is

$$P_1 + 2P_2 = \left[\frac{3}{4} + 2\left[\frac{3}{4}\right]^2\right] P_0$$
$$= \frac{30}{16}\left[1 + \frac{3}{4} + \left[\frac{3}{4}\right]^2\right]^{-1} = \frac{30}{37}.$$

(b) The proportion of customers that enter the shop is

$$\frac{\lambda(1 - P_2)}{\lambda} = 1 - P_2 = 1 - \frac{9}{16} \cdot \frac{16}{37} = \frac{28}{37}.$$

(c) Now $\mu = 8$, and so

$$P_0 = \left[1 + \frac{3}{8} + \left[\frac{3}{8}\right]^2\right]^{-1} = \frac{64}{97}.$$

So the proportion of customers who now enter the shop is

$$1 - P_2 = 1 - \left[\frac{3}{8}\right]^2 \frac{264}{97} = 1 - \frac{9}{97} = \frac{88}{97}.$$

The rate of added customers is therefore

$$\lambda\left[\frac{88}{97}\right] - \lambda\left[\frac{28}{37}\right] = 3\left[\frac{88}{97} - \frac{28}{37}\right] = 0.45.$$

The business he does would improve by 0.45 customers per hour.

14. Letting the number of cars in the station be the state, we have a birth and death process with

$$\lambda_0 = \lambda_1 = \lambda_2 = 20, \quad \lambda_i = 0,\ i > 2,$$

$$\mu_1 = \mu_2 = 12.$$

Hence,

$$P_1 = \frac{5}{3}P_0,\ P_2 = \frac{5}{3}P_1 = \left[\frac{5}{3}\right]^2 P_0,$$
$$P_3 = \frac{5}{3}P_2 = \left[\frac{5}{3}\right]^3 P_0$$

and as $\sum_0^3 P_i = 1$, we have

$$P_0 = \left[1 + \frac{5}{3} + \left[\frac{5}{3}\right]^2 + \left[\frac{5}{3}\right]^2\right]^{-1} = \frac{27}{272}.$$

(a) The fraction of the attendant's time spent servicing cars is equal to the fraction of time there are cars in the system and is therefore $1 - P_0 = 245/272$.

(b) The fraction of potential customers that are lost is equal to the fraction of customers that arrive when there are three cars in the station and is therefore

$$P_3 = \left[\frac{5}{3}\right]^3 P_0 = 125/272.$$

15. With the number of customers in the system as the state, we get a birth and death process with

$$\lambda_0 = \lambda_1 = \lambda_2 = 3\ \lambda_i = 0, \quad i \geq 4,$$
$$\mu_1 = 2\ \mu_2 = \mu_3 = 4.$$

Therefore, the balance equations reduce to

$$P_1 = \frac{3}{2}P_0\ P_2 = \frac{3}{4}P_1 = \frac{9}{8}P_0\ P_3 = \frac{3}{4}P_2 = \frac{27}{32}P_0.$$

And therefore,

$$P_0 = \left[1 + \frac{3}{2} + \frac{9}{8} + \frac{27}{32}\right]^{-1} = \frac{32}{143}.$$

(a) The fraction of potential customers that enter the system is

$$\frac{\lambda(1 - P_3)}{\lambda} = 1 - P_3 = 1 - \frac{27}{32} \times \frac{32}{143} = \frac{116}{143}.$$

(b) With a server working twice as fast we would get

$$P_1 = \frac{3}{4}P_0\ P_2 = \frac{3}{4}P_1 = \left[\frac{3}{4}\right]^2 P_0\ P_3 = \left[\frac{3}{4}\right]^3 P_0,$$

and $P_0 = \left[1 + \frac{3}{4} + \left[\frac{3}{4}\right]^2 + \left[\frac{3}{4}\right]^3\right]^{-1} = \frac{64}{175}.$

So that now

$$1 - P_3 = 1 - \frac{27}{64} = 1 - \frac{64}{175} = \frac{148}{175}.$$

22. The number in the system is a birth and death process with parameters

$$\lambda_n = \lambda/(n+1), \quad n \geq 0$$
$$\mu_n = \mu, \quad n \geq 1.$$

From Equation (5.3),

$$1/P_0 = 1 + \sum_{n=1}^{\infty} (\lambda/\mu)^n/n! = e^{\lambda/\mu}$$

and

$$P_n = P_0(\lambda/\mu)^n/n! = e^{-\lambda/\mu}(\lambda/\mu)^n/n!, \quad n \geq 0.$$

23. Let the state denote the number of machines that are down. This yields a birth and death process with

$$\lambda_0 = \frac{3}{10}, \ \lambda_1 = \frac{2}{10}, \ \lambda_2 = \frac{1}{10}, \ \lambda_i = 0, \quad i \geq 3$$
$$\mu_1 = \frac{1}{8}, \ \mu_2 = \frac{2}{8}, \ \mu_3 = \frac{2}{8}.$$

The balance equations reduce to

$$P_1 = \frac{3/10}{1/8}P_0 = \frac{12}{5}P_0$$
$$P_2 = \frac{2/10}{2/8}P_1 = \frac{4}{5}P_1 = \frac{48}{25}P_0$$
$$P_3 = \frac{1/10}{2/8}P_2 = \frac{4}{10}P_3 = \frac{192}{250}P_0.$$

Hence, using $\sum_0^3 P_i = 1$, yields

$$P_0 = \left[1 + \frac{12}{5} + \frac{48}{25} + \frac{192}{250}\right]^{-1} = \frac{250}{1522}.$$

(a) Average number not in use

$$= P_1 + 2P_2 + 3P_3 = \frac{2136}{1522} = \frac{1068}{761}.$$

(b) Proportion of time both repairmen are busy

$$= P_2 + P_3 = \frac{672}{1522} = \frac{336}{761}.$$

Chapter 7

1. (a) Yes, (b) no, (c) no.

2. (a) S_n is Poisson with mean $n\mu$.

   (b) $P\{N(t) = n\}$

$$= P\{N(t) \geq n\} - P\{N(t) \geq n+1\}$$
$$= P\{S_n \leq t\} - P\{S_{n+1} \leq t\}$$
$$= \sum_{k=0}^{[t]} e^{-n\mu}(n\mu)^k/k!$$
$$- \sum_{k=0}^{[t]} e^{-(n+1)\mu}[(n+1)\mu]^k/k!,$$

   where $[t]$ is the largest integer not exceeding t.

3. By the one-to-one correspondence of $m(t)$ and F, it follows that $\{N(t), t \geq 0\}$ is a Poisson process with rate 1/2. Hence,

$$P\{N(5) = 0) = e^{-5/2}.$$

4. (a) No! Suppose, for instance, that the interarrival times of the first renewal process are identically equal to 1. Let the second be a Poisson process. If the first interarrival time of the process $\{N(t), t \geq 0\}$ is equal to 3/4, then we can be certain that the next one is less than or equal to 1/4.

   (b) No! Use the same processes as in (a) for a counter example. For instance, the first interarrival will equal 1 with probability $e^{-\lambda}$, where λ is the rate of the Poisson process. The probability will be different for the next interarrival.

   (c) No, because of (a) or (b).

5. The random variable N is equal to $N(I) + 1$ where $\{N(t)\}$ is the renewal process whose interarrival distribution is uniform on (0, 1). By the results of Example 2c,

$$E[N] = a(1) + 1 = e.$$

[illegible] Once every five months.

Chapter 10

1. $X(s) + X(t) = 2X(s) + X(t) - X(s)$.

   Now $2X(s)$ is normal with mean 0 and variance $4s$ and $X(t) - X(s)$ is normal with mean 0 and variance $t - s$. As $X(s)$ and $X(t) - X(s)$ are independent, it follows that $X(s) + X(t)$ is normal with mean 0 and variance $4s + t - s = 3s + t$.

2. The conditional distribution $X(s) - A$ given that $X(t_1) = A$ and $X(t_2) = B$ is the same as the conditional distribution of $X(s - t_1)$ given that $X(0) = 0$ and $X(t_2 - t_1) = B - A$, which by (1.4) is normal with mean $\frac{s - t_1}{t_2 - t_1}(B - A)$ and variance $\frac{(s - t_1)}{t_2 - t_1}(t_2 - s)$. Hence the desired conditional distribution is normal with mean $A + \frac{(s - t_1)(B - A)}{t_2 - t_1}$ and variance $\frac{(s - t_1)(t_2 - s)}{t_2 - t_1}$.

3. $E[X(t_1)X(t_2)X(t_3)]$

$$
\begin{aligned}
&= E[E[X(t_1)X(t_2)X(t_3) \mid X(t_1), X(t_2)]] \\
&= E[X(t_1)X(t_2)E[X(t_3) \mid X(t_1), X(t_2)]] \\
&= E[X(t_1)X(t_2)X(t_2)] \\
&= E[E[X(t_1)E[X^2(t_2) \mid X(t_1)]] \\
&= E[X(t_1)E[X^2(t_2) \mid X(t_1)]] \qquad (*) \\
&= E[X(t_1)\{(t_2 - t_1) + X^2(t_1)\}] \\
&= E[X^3(t_1)] + (t_2 - t_1)E[X(t_1)] \\
&= 0
\end{aligned}
$$

   where the equality $(*)$ follows since given $X(t_1)$, $X(t_2)$ is normal with mean $X(t_1)$ and variance $t_2 - t_1$. Also, $E[X^3(t)] = 0$ since $X(t)$ is normal with mean 0.

4. (a) $P\{T_a < \infty\} = \lim_{t \to \infty} P\{T_a \le t\}$

$$
\begin{aligned}
&= \frac{2}{\sqrt{2r}} \int_0^\infty e^{-y^2/2} dy \quad \text{by (10.6)} \\
&= 2P\{N(0,1) > 0\} = 1.
\end{aligned}
$$

   Part (b) can be proven by using

$$E[T_a] = \int_0^\infty P\{T_a > t\} dt$$

   in conjunction with (2.3).

5. $P\{T_1 < T_{-1} < T_2\} = P\{\text{hit 1 before } -1 \text{ before } 2\}$

$$
\begin{aligned}
&= P\{\text{hit 1 before } -1\} \\
&\quad \times P\{\text{hit } -1 \text{ before } 2 \mid \text{hit 1 before } -1\} \\
&= \frac{1}{2} P\{\text{down 2 before up 1}\} \\
&= \frac{1}{2}\frac{1}{3} = \frac{1}{6}.
\end{aligned}
$$

   The next to last equality follows by looking at the Brownian motion when it first hits 1.

6. The probability of recovering your purchase price is the probability that a Brownian motion goes up c by time t. Hence the desired probability is

$$1 - P\{\max_{0 \le s \le t} X(s) \ge c\} = 1 - \frac{2}{\sqrt{2\pi t}} \int_{c/\sqrt{t}}^\infty e^{-y^2/2} dy$$

7. Let $M = \{\max_{t_1 \le s \le t_2} X(s) > x\}$. Condition on $X(t_1)$ to obtain

$$P(M) = \int_{-\infty}^\infty P(M \mid X(t_1) = y) \frac{1}{\sqrt{2\pi t_1}} e^{-y^2/2t_1} dy$$

   Now, use that

$$P(M \mid X(t_1) = y) = 1 \quad y \ge x$$

   and, for $y < x$

$$
\begin{aligned}
P(M \mid X(t_1) = y) &= P\{\max_{0 < s < t_2 - t_1} X(s) > x - y\} \\
&= 2P\{X(t_2 - t_1) > x - y\}
\end{aligned}
$$