UNIVERSITY OF CALGARY

DEPARTMENT OF MATHEMATICS AND STATISTICS

STAT 407 — Test no. 2 — Nov. 21, 2008 — TIME: 50 min.

NAME: ~~[illegible]~~ Solutions

Marks
10

1. Suppose that on each play of a game a gambler either wins \$5 with probability 2/3 or loses \$5 with probability 1/3. The gambler continues betting until she or he is either up \$30 or down \$15. What is the probablity that the gambler finishes up \$30?

1 unit = \$5

P[up 6 units before down 3 units | starting at 0]

Convert scale to ~~[illegible]~~ use Gambler's ruin formula as follows:

$+6 \longrightarrow N=9$

$+0 \longrightarrow i=3$

$-3 \longrightarrow 0$

$p=2/3 \quad N=9$
$q=1/3 \quad i=3$

$$P_i = \frac{1-\left(\frac{q}{p}\right)^i}{1-\left(\frac{q}{p}\right)^N}$$

$$= \frac{1-\left(\frac{1}{2}\right)^3}{1-\left(\frac{1}{2}\right)^9} = 0.87675$$

[10 marks for correct solution. Partial credit of 1 to 5 marks for various partially correct solutions.

ID number: ____________________

10

2. Ten items are simultaneously put on a life test. Suppose the lifetimes of the individual items are independent exponential random variables with common mean 50 hours. The test will end when there have been a total of three failures. If T is the time at which the test ends, find $E(T)$ and $Var(T)$.

Let T_i = time of ith failure, $i=1,2,3$

$\frac{1}{\lambda} = 50$ hrs, (mean of one item)

$T_1 \sim$ exponential, mean $\frac{50}{10}$

$T_2 \sim$ " , " $50/9$

$T_3 \sim$ " , " $50/8$

4

$$E(T) = E(T_1) + E(T_2) + E(T_3) = \frac{50}{10} + \frac{50}{9} + \frac{50}{8}$$

$$= 16.81 \text{ hr.}$$

2

T_1, T_2, T_3 indep,

$$\therefore \cancel{V(T_{1,2,3})}\ V(T) = V(T_1) + V(T_2) + V(T_3)$$

$$= \left(\frac{50}{10}\right)^2 + \left(\frac{50}{9}\right)^2 + \left(\frac{50}{8}\right)^2$$

4

$$= 94.93\ (\text{hr})^2$$

2

10 3. Consider a two-server system in which a customer is served first by server 1, then by server 2, and then departs. The service times at server i are exponential random variables with rates μ_i, $i = 1, 2$. When you arrive, you find server 1 free and two customers at server 2 — customer A in service and customer B waiting in line.

(a) Find P_A, the probability that A is still in service when you move over to server 2.

$$P_A = \frac{\mu_1}{\mu_1+\mu_2} \leftarrow 5$$

(b) Find P_B, the probability that B is still in the system when you move over to server 2.

$$P_B = 1 - P[\text{both } A \text{ and } B \text{ served before you}]$$

$$= 1 - \left(\frac{\mu_2}{\mu_1+\mu_2}\right)^2 \leftarrow 5$$

OR $$P_B = \frac{\mu_1}{\mu_1+\mu_2} + \frac{\mu_2}{\mu_1+\mu_2}\frac{\mu_1}{\mu_1+\mu_2}$$

(Both answers are logically right, and it is easy to see that the two answers are algebraically equivalent].

10 4. Events occur according to a non-homogeneous Poisson process with intensity function

$$\lambda(t) = \begin{cases} 5 - t^2 & \text{if } 0 \le t \le 2 \\ t - 1 & \text{if } 2 < t \le 7 \end{cases}$$

4 (a) Find the mean value function of this process.

For $0 \le t \le 2$, $m(t) = \int_0^t (5-v^2)\,dv = 5v - \frac{v^3}{3}\Big]_0^t = 5t - \frac{t^3}{3}$

For $2 < t \le 7$, $m(t) = m(2) + \int_2^t (v-1)\,dv = 10 - \frac{8}{3} + \frac{v^2}{2} - v\Big]_2^t = 10 - \frac{8}{3} + \frac{t^2}{2} - t + 2 -$

$= \frac{23}{3} + \frac{t^2}{2} - t$

So $m(t) = 5t - t^3/3, \quad 0 \le t \le 2$

$= \frac{22}{3} + \frac{t^2}{2} - t \quad 2 < t \le 7$

2

4 (b) Find the probability that at most one event occurs between time $t = 1$ and time $t = 4$.

$m(4) - m(1) = \frac{22}{3} + \frac{16}{2} - 4 - (5 - \frac{1}{3}) = \frac{23}{3} - 1 = \frac{20}{3} = 6.\bar{6}$ 2

$P[X=0] + [X=1] = e^{-20/3} + \frac{20}{3} e^{-20/3} = .009757$

2

2 (c) Find the probability that at least three events occur between time $t = 1$ and time $t = 4$.

$1 - P[X=0] - P[X=1 - P[X=2]$

$= 1 - e^{-20/3} - \frac{20}{3} e^{-20/3} - \underbrace{\frac{1}{2}\left(\frac{20}{3}\right)^2}_{22\frac{2}{9}} e^{-20/3} \quad = .9620$

10

5. Consider a birth and death process with birth rates

$$\lambda_n = n\lambda + \theta, \text{ for } n \geq 0$$

and death rates

$$\mu_n = n\mu, \text{ for } n \geq 1,$$

where $\lambda = 5$, $\theta = 2$ and $\mu = 3$. Calculate the expected time to go from state 0 to state 3.

$$E(T_0) = \frac{1}{\lambda_0} = \frac{1}{\theta} = \frac{1}{2}$$

$$E(T_1) = \frac{1}{\lambda_1} + \frac{\mu_1}{\lambda_1} E(T_0) = \frac{1}{\lambda+\theta} + \frac{\mu}{\lambda+\theta} E(T_0)$$

$$= \frac{1}{7} + \frac{3}{7} \cdot \frac{1}{2} = \frac{5}{14}$$

$$\frac{1}{12} + \frac{5}{28} = \frac{7+15}{84} = \frac{22}{84} = \frac{11}{42}$$

$$ET_2 = \frac{1}{\lambda_2} + \frac{\mu_2}{\lambda_2} ET_1 = \frac{1}{2\lambda+\theta} + \frac{2\mu}{2\lambda+\theta} E(T_1) = \frac{1}{12} + \frac{6}{12} \cdot \frac{5}{14} = \frac{11}{42}$$

$$E(T) = ET_0 + ET_1 + ET_2 = \frac{1}{2} + \frac{5}{14} + \frac{11}{42} = \frac{21+15+11}{42} = \frac{47}{42} \leftarrow 2$$

2

2 2 2

$$= 1.1190$$