significant implies not that there is NO difference in the population, but only the sample does NOT provide enough evidence to rule out the possibility of NO population difference.

Independent-samples t tests of the difference between Two means

Two samples are independent if the selection of observational units for one sample has no bearing on which observational units are in the other sample.
The null hypothesis for the t-test based on two independent samples is, as usual, an assumption of no difference in the population. $H_0: \mu_1 = \mu_2$; $H_A: \mu_1 \neq \mu_2$
or $H_0: \mu_1 - \mu_2 = 0$, $H_A: \mu_1 - \mu_2 \neq 0$

In effect, the difference $\mu_1 - \mu_2$ is the parameter that we wish to estimate on the basis of our two independent random samples. If the two sample means are $\bar{x}_1$ and $\bar{x}_2$, the sample difference $\bar{x}_1 - \bar{x}_2$ is an unbiased estimate of the population difference $\mu_1 - \mu_2$ and the sampling distribution of the difference $\bar{x}_1 - \bar{x}_2$ is normal if the sampling distribution of $\bar{x}_1$ and $\bar{x}_2$ are both normal.

$\sigma^2(X_1 - X_2) = \sigma^2(X_1) + \sigma^2(X_2)$ if X_1 & X_2 independent.

$$\sigma^2(\bar{X}_1 - \bar{X}_2) = \sigma^2(\bar{X}_1) + \sigma^2(\bar{X}_2)$$
$$= \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$$ for two independent random samples

If we can assume that the two populations being sampled have equal variance, then

$$\sigma^2(\bar{X}_1 - \bar{X}_2) = \sigma^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right).$$

Taking square roots, we obtain the standard error of the difference between the means of two independent samples from populations with equal variances:

$$\sigma(\bar{x}_1-\bar{x}_2)=\sqrt{\sigma^2\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}.$$

$\sigma^2(\bar{x}_1-\bar{x}_2)$ can be estimated by

$$S^2_{pooled}=\frac{SS_1+SS_2}{df_1+df_2}=\frac{\Sigma(X_1-\bar{X}_1)^2+\Sigma(X_2-\bar{X}_2)^2}{(n_1-1)+(n_2-1)}$$

$$=\frac{(n_1-1)S_1^2+(n_2-1)S_2^2}{n_1+n_2-2}.$$

$$\therefore\ S_{\bar{X}_1-\bar{X}_2}=\sqrt{S^2_{pooled}\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}$$

test statistic $t_{obs}=\dfrac{(\bar{x}_1-\bar{x}_2)}{S_{\bar{X}_1-\bar{X}_2}}$

follows the t-distribution with n_1+n_2-2 d.f.

$|t|\geq t_{\alpha/2}(n_1+n_2-2)$ reject Ho.

If <u>NOT</u> assuming $\sigma_1^2=\sigma_2^2$, then

$$t_{obs}=\frac{\bar{X}_1-\bar{X}_2}{\sqrt{\frac{S_1^2}{n_1}+\frac{S_2^2}{n_2}}}$$

d.f. for t_{obs} : $$\frac{\left[\frac{S_1^2}{n_1}+\frac{S_2^2}{n_2}\right]^2}{\left(\frac{S_1^2}{n_1}\right)^2+\left(\frac{S_2^2}{n_2}\right)^2}$$

6.23a

```
get file='d:\stat601.14\polit\p133.sav'.
do if (gp=1).
compute diff=abs(pulse-95).
else.
compute diff=abs(pulse-105).
end if.
oneway diff by gp.
```

Oneway:p133.spo

ANOVA

diff

	Sum of Squares	df	Mean Square	F	Sig.
Between Groups	5.000	1	5.000	.134	.719
Within Groups	672.800	18	37.378		
Total	677.800	19			

```
t-test groups=gp(1,2)/variables=pulse.
```

T-Test

Group Statistics

	gp	N	Mean	Std. Deviation	Std. Error Mean
pulse pulse	1.00 experimental	10	95.0000	12.42757	3.92994
	2.00 control	10	105.0000	11.77568	3.72380

Independent Samples Test

		Levene's Test for Equality of Variances	
		F	Sig.
pulse pulse	Equal variances assumed	.134	.719
	Equal variances not assumed		

6.23b

Independent Samples Test

		t-test for Equality of Means			
		t	df	Sig. (2-tailed)	Mean Difference
pulse pulse	Equal variances assumed	-1.847	18	.081	-10.00000
	Equal variances not assumed	-1.847	17.948	.081	-10.00000

Independent Samples Test

		t-test for Equality of Means		
		Std. Error Difference	95% Confidence Interval of the Difference	
			Lower	Upper
pulse pulse	Equal variances assumed	5.41397	-21.37434	1.37434
	Equal variances not assumed	5.41397	-21.37670	1.37670

Pairwise t-test or correlated T-Test:

When the observations from two populations occur in pairs or are related, then independent samples t-test is inappropriate

e.g.

patient	BP at time 1	BP at time 2
1	–	–
2	–	–
3	–	–

$H_0: \mu_{time1} = \mu_{time2}$

$$t = \frac{\bar{D}}{\sqrt{\frac{S_D^2}{n}}}$$, if $|t| > t_{\alpha/2}(n-1)$ reject H_0

$|t| < t_{\alpha/2}(n-1)$ DO NOT reject H_0

e.g. Polit's book (p137)

Direct (X_1)	Indirect (X_2)	$D = X_1 - X_2$
130	128	2
102	100	2
154	155	-1
113	110	3
139	140	-1
125	120	5
156	155	1
108	105	3
161	160	1
105	107	-2
$\bar{X}_1 = 129.3$	$\bar{X}_2 = 128.0$	$\Sigma D = 13$, $\bar{D} = 1.3$

$S_D^2 = 4.678$, $\frac{S_D^2}{n} = 0.4678$, $t_{obs} = \frac{1.3}{\sqrt{.4678}} = 1.90$

$t_{.025}(9) = 2.26$, $t_{obs} = 1.9 < 2.26$ DO NOT reject H_0

6.25

```
get file='d:\stat601.14\polit\p137.sav'.
compute diff=x1-x2.
t-test pairs=x1 with x2.
```

T-Test:p137.spo

Paired Samples Statistics

		Mean	N	Std. Deviation	Std. Error Mean
Pair 1	x1	129.3000	10	22.35099	7.06800
	x2	128.0000	10	23.01690	7.27858

Paired Samples Correlations

		N	Correlation	Sig.
Pair 1	x1 & x2	10	.996	.000

Paired Samples Test

		Paired Differences					
					95% Confidence Interval of the Difference		
		Mean	Std. Deviation	Std. Error Mean	Lower	Upper	t
Pair 1	x1 - x2	1.30000	2.16282	.68394	-.24719	2.84719	1.901

Paired Samples Test

		df	Sig. (2-tailed)
Pair 1	x1 - x2	9	.090

```
t-test testvals=0/variables=diff.
```

T-Test

One-Sample Statistics

	N	Mean	Std. Deviation	Std. Error Mean
diff	10	1.3000	2.16282	.68394

One-Sample Test

	Test Value = 0					
					95% Confidence Interval of the Difference	
	t	df	Sig. (2-tailed)	Mean Difference	Lower	Upper
diff	1.901	9	.090	1.30000	-.2472	2.8472

Power of a test

Decision	Actual Situation: H_0 True	Actual Situation: H_0 False
H_0: Not rejected	Correct Decision	Type II error
H_0: rejected	Type I error	Correct Decision

$\alpha = P\{\text{Type I error}\} = P(\text{reject } H_0 \mid H_0 \text{ True})$

$\beta = P\{\text{Type II error}\} = P(H_0 \text{ \underline{NOT} rejected} \mid H_0: \text{False})$
$= P\{\text{test statistic in acceptance region} \mid H_0: \text{False}$
$= P\{\text{test statistic in acceptance region} \mid H_A: \text{True}$

$\text{Power} = 1 - \beta = P\{\text{test statistic in rejection region} \mid H_A: \text{true}\}$

(A) Suppose we have a random sample of n observations from a normal population with a known variance σ^2. Suppose we want to test the null hypothesis:

$$H_0: \mu = \mu_0$$
$$H_A: \mu > \mu_0 \quad \text{with the level of significance of}$$

$$\alpha = P\{\bar{x} > \bar{x}^* \mid H_0: \text{True}\} = P\left\{\frac{\bar{x} - \mu_0}{(\sigma/\sqrt{n})} > \frac{\bar{x}^* - \mu_0}{(\sigma/\sqrt{n})} \,\middle|\, H_0: \text{True}\right.$$

$$= P\{z > z_\alpha \mid H_0: \text{true}\}$$

$$\therefore \frac{\bar{x}^* - \mu_0}{(\sigma/\sqrt{n})} = z_\alpha \Rightarrow \bar{x}^* = \mu_0 + z_\alpha\left(\frac{\sigma}{\sqrt{n}}\right)$$

$\bar{x} > \bar{x}^*$ reject H_0

$\bar{x} \leq \bar{x}^*$ Do <u>NOT</u> reject H_0

Suppose the *true* mean is μ_1 rather than the hypothesized value μ_0. The power of the test is

$$\text{Power} = 1-\beta = P\{\bar{X} > \bar{x}^*\} = P\{\bar{X} > \mu_0 + \frac{\sigma}{\sqrt{n}} z_\alpha\}$$

where $\bar{X} \sim N(\mu_1, \frac{\sigma^2}{n})$

$$\therefore \text{Power} = P\left\{ \frac{\bar{X}-\mu_1}{(\sigma/\sqrt{n})} > \frac{\mu_0 + \frac{\sigma}{\sqrt{n}} z_\alpha - \mu_1}{(\sigma/\sqrt{n})} \right\}$$

$$= P\left\{ Z > \frac{\mu_0 - \mu_1 + \frac{\sigma}{\sqrt{n}} z_\alpha}{(\sigma/\sqrt{n})} \right\}$$

e.g. Let μ be the mean annual salary of teacher

$H_0: \mu \le \$25K$

$H_A: \mu > \$25K$

Assume that it is normally distributed with standard deviation $\sigma = \$4000$. Sample size $n = 100$, $\alpha = 0.01$

Find power of the test if the *true* mean is 26K

Sol/n

$\alpha = 0.01 \Rightarrow z_\alpha = 2.33$

$$\alpha = P\{\bar{X} > \bar{x}^*\} = P\left\{ \frac{\bar{X}-\mu_0}{(\sigma/\sqrt{n})} > \frac{\bar{x}^* - \mu_0}{(\sigma/\sqrt{n})} \right\}$$

$$= P\left\{ Z > \frac{\bar{x}^* - 25000}{400} \right\} = P\{Z > 2.33\}$$

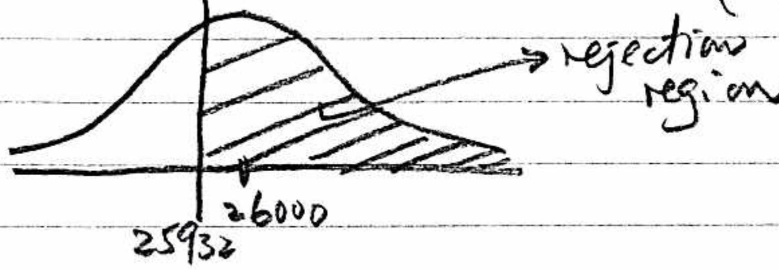
$$\therefore \frac{\bar{x}^* - 25000}{400} = 2.33 \Rightarrow \bar{x}^* = 25000 + 2.33 \times 400$$

$$= 25932$$

$$\text{Power} = P\{\bar{x} > 25932 \mid H_A : \text{true}\} = P\{\bar{x} > 25932 \mid \mu_1 = 26000\}$$

$$= P\left\{\frac{\bar{x} - 26000}{\left(\frac{4000}{\sqrt{100}}\right)} > \frac{25932 - 26000}{\left(\frac{400}{\sqrt{100}}\right)}\right\} = P\{Z > -0.17\}$$

$$= 0.5675$$



The rejection region consists of all values $\bar{x}$ exceeding 25932 when $\mu = 26000$. The shaded area shows the power of the test represents the probability that we will correct reject H_0 when in fact $\mu = \$26000$.
Since the power of the test is $1-\beta = 0.5675$, it follows that $\beta = 0.4325$. Thus, for $\mu = \$26000$, the probability is 0.5675 that we will reject H_0 in favour of H_A. Consequently the probability is 0.4325 that we will incorrectly accept H_0.

(B) Suppose we have a random sample of n observations from a normal population with a known variance σ^2

$H_0: \mu = \mu_0$, $H_A: \mu < \mu_0$, with level of sig. α

$$\alpha = P\{\bar{x} < \bar{x}^* \mid H_0 : \text{true}\} = P\left\{\frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} < \left(\frac{\bar{x}^* - \mu_0}{\frac{\sigma}{\sqrt{n}}}\right)\right\}$$

$$= P\{Z < -Z_\alpha\} \Rightarrow \frac{\bar{x}^* - \mu_0}{\sigma/\sqrt{n}} = -Z_\alpha \Rightarrow \bar{x}^* = \mu_0 - \left(\frac{\sigma}{\sqrt{n}}\right)\cdot$$

Suppose the mean is μ_1 rather than the hypothesized value μ_0, i.e. $\bar{X} \sim N\left(\mu_1, \frac{\sigma^2}{n}\right)$

$$\text{Power} = P\left\{\bar{X} < \bar{X}^*\right\} = P\left\{\frac{\bar{X}-\mu_1}{\left(\frac{\sigma}{\sqrt{n}}\right)} < \frac{\bar{X}^*-\mu_1}{\left(\frac{\sigma}{\sqrt{n}}\right)}\right\}$$

where $\bar{X}^* = \mu_0 - \frac{\sigma}{\sqrt{n}} z_\alpha$

e.g. $H_0: \mu = 32$
$H_A: \mu < 32$

It was assumed that the population was normal with $\sigma^2 = 16$. The sample size $n = 25$ and level of significance was 0.05. Find the power of the test, in fact $\mu = 29.5$

Soln: $\alpha = 0.05$ $\quad z_\alpha = 1.645$

$$0.05 = P\left\{\bar{X} < \bar{X}^*\right\} = P\left\{\frac{\bar{X}-\mu_0}{\left(\frac{\sigma}{\sqrt{n}}\right)} < \frac{\bar{X}^*-\mu_0}{\left(\frac{\sigma}{\sqrt{n}}\right)}\right\}$$

$$= P\left\{z < \frac{\bar{X}^*-32}{(4/5)}\right\} = P\left\{z < -1.645\right\}$$

$$\therefore \frac{\bar{X}^*-32}{(4/5)} = -1.645 \Rightarrow \bar{X}^* = 32 - 1.645 \times \frac{4}{5}$$
$$= 30.684$$

$$\text{Power} = P\left\{\bar{X} < \bar{X}^* \mid \mu_1 = 29.5\right\} = P\left\{\frac{\bar{X}-29.5}{(4/5)} < \frac{30.684-29}{(4/5)}\right.$$

$$= P\left\{z < 1.48\right\} = 0.9306$$

ⓒ Two sided test

$$H_0: \mu = \mu_0$$
$$H_A: \mu \neq \mu_0$$

The power of the test is the probability that the sample mean $\bar{X}$ is less than $\bar{X}_1^*$ OR greater than $\bar{X}_2^*$ when $\mu = \mu_1$ where $\bar{X}_1^*$ and $\bar{X}_2^*$ are the critical values given by

$$\bar{X}_1^* = \mu_0 - Z_{\alpha/2} \frac{\sigma}{\sqrt{n}}, \quad \bar{X}_2^* = \mu_0 + Z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

i.e. $\text{Power} = P\{\bar{X} < \bar{X}_1^*\} + P\{\bar{X} > \bar{X}_2^*\}$
where $\bar{X} \sim N(\mu_1, \frac{\sigma^2}{n})$

$$\text{Power} = P\left\{Z < \frac{\bar{X}_1^* - \mu_1}{(\sigma/\sqrt{n})}\right\} + P\left\{Z > \frac{\bar{X}_2^* - \mu_1}{(\sigma/\sqrt{n})}\right\}$$

e.g. $H_0: \mu = \$300$
$H_A: \mu \neq \$300$

The population is normally distributed with s.d. $\sigma = 50$, sample size $n = 25$, $\alpha = 0.05$.
Find power of the test if in fact $\mu = 270$

Soln: $\alpha = 0.05$, $Z_{\alpha/2} = 1.96$

$$\bar{X}_1^* = 300 - \frac{50}{5} \times 1.96 = 280.40$$

$$\bar{X}_2^* = 300 + \frac{50}{5} \times 1.96 = 319.60$$

$$\text{Power} = P\{\bar{X} < \bar{X}_1^*\} + P\{\bar{X} > \bar{X}_2^*\}$$
$$= P\{\bar{X} < 280.40\} + P\{\bar{X} > 319.60\}, \quad \bar{X} \sim N(270, \frac{2500}{25}$$

$$= P\left\{Z < \frac{280.40-270}{10}\right\} + P\left\{Z > \frac{319.60-270}{10}\right\}$$

$$= P\{Z < 1.04\} + P\{Z > 4.96\} = 0.850$$

$\beta = 1 - \text{Power} = 1 - 0.850 = 0.150$

<u>Choice of sample size for testing mean</u>

(a) $H_0: \mu = \mu_0$ $\quad n = (z_\alpha + z_\beta)^2 \dfrac{\sigma^2}{\delta^2}$ where $\delta = \mu_1 - \mu_0$
$H_A: \mu > \mu_0$

(b) $H_0: \mu = \mu_0$ $\quad n = (z_\alpha + z_\beta)^2 \dfrac{\sigma^2}{\delta^2}$ where $\delta = \mu_1 - \mu_0$
$H_A: \mu < \mu_0$

(c) $H_0: \mu = \mu_0$ $\quad n = (z_{\alpha/2} + z_\beta)^2 \dfrac{\sigma^2}{\delta^2}$ where $\delta = \mu_1 - \mu_0$
$H_A: \mu \neq \mu_0$

e.g. $H_0: \mu = 68$ kg
$H_A: \mu > 68$ kg for male students at a college $\alpha = 0.05$, $\sigma = 5$

Find the sample size required if the power of our test is 0.95 when the true mean is 69 kg.

Soln: $\alpha = 0.05$, power $= 0.95 \Rightarrow \beta = 0.05$

$z_\alpha = 1.645$, $\quad z_\beta = 1.645$

$\mu_1 = 69$, $\therefore \delta = 69 - 68 = 1$

$$n = (1.645 + 1.645)^2 \frac{5^2}{1} = 270.6$$

270 observations is required if the test to reject H_0 95% of the time when in fact, μ as large as 69 kg.

e.g. Suppose we want to test the null hypothesis $\mu = \$300$ against the alternative hypothesis $\mu \neq \$300$ for a population where standard deviation is $\sigma = \$12$, How large a sample will we need, if the probability of a Type I error to be 0.05 and the probability of a Type II error is to be 0.05 for $\mu = \$305$.

Soln: $\alpha = 0.05$, $Z_{\alpha/2} = 1.96$, $\sigma = 12$

$\beta = 0.05$, $Z_\beta = 1.645$, $\delta = 300 - 305 = -5$

$$n = (1.96 + 1.645)^2 \frac{144}{(-5)^2}$$

$$= 74.86$$

$\therefore n = 75$

For further reading:

Introduction to Sample Size Determination & Power Analysis for Clinical Trials. John M. Lachin, Clinical Trials 2, page 93–113 (1981).

Analysis of variance

Data: interval or ratio

When testing for differences in the mean of more than two populations, we usually do NOT proceed by considering all combinations of two populations at a time and testing for differences in each pair. First, such an approach would require several tests rather than just one. Second, if each individual test were using a level of significance of, say $\alpha = 0.05$ then the overall level of significance would be higher than 0.05

e.g. $H_0: \mu_1 = \mu_2 = \mu_3$ were true and if the three null hypothesis $\mu_1 = \mu_2$, $\mu_1 = \mu_3$, $\mu_2 = \mu_3$ were each tested using a 0.05 level of significance, then P(accept each single H_0) $= 0.95$

P(accept all 3 H_0) will be at least $(0.95)^3 = 0.857$ (If 3 tests are independent, probability will be exactly 0.857). Consequently, the probability of rejecting H_0 when it was true could be as large as $1 - 0.857 = 0.143$ which is substantially greater than 0.05

Thus we want to test simultaneously for difference among the means of all the populations and we want the joint level of significance of the test to be alpha. To perform this test we may use of F distribution and use a method called Analysis of Variance (ANOVA)

6.34

To test

$H_0: \mu_1 = \mu_2 = \cdots = \mu_k$

H_A: At least two of the means differ.

Logic:

Population

1	2	...	k
x_{11}	x_{12}		x_{1k}
x_{21}	x_{22}		x_{2k}
$\vdots$			$\vdots$
$x_{n_1 1}$	$x_{n_2 2}$		$x_{n_k k}$
$\bar{x}_1$	$\bar{x}_2$		$\bar{x}_k$

$n = n_1 + n_2 + \cdots + n_k$

$$\bar{x} = \frac{\sum_{j=1}^{k}\sum_{i=1}^{n_j} x_{ij}}{n} = \frac{\sum_{j=1}^{k} n_j \bar{x}_j}{n}$$

$$SST = \sum_{j=1}^{k}\sum_{i=1}^{n_j}(x_{ij}-\bar{x})^2$$

SST: Total sum of squares

$$= \sum_{j=1}^{k}\sum_{i=1}^{n_j}(x_{ij}-\bar{x}_j+\bar{x}_j-\bar{x})^2$$

$$= \sum_{j=1}^{k}\sum_{i=1}^{n_j}(x_{ij}-\bar{x}_j)^2 + \sum_{j=1}^{k}\sum_{i=1}^{n_j}(\bar{x}_j-\bar{x})^2$$

$$= \underbrace{\sum_{j=1}^{k}\sum_{i=1}^{n_j}(x_{ij}-\bar{x}_j)^2}_{} + \underbrace{\sum_{j=1}^{k} n_j(\bar{x}_j-\bar{x})^2}_{}$$

$$= SSW \qquad\qquad SSB$$

SSW: sum of squares within (or error sum of squares)

SSB: sum of squares between

Def^n: Mean Square between $MSB = \frac{SSB}{K-1}$

Def^n: Mean Square within (or Mean Square Error) $MSW = \frac{SSW}{n-K}$
sometimes it is denoted by MSE.

In one way ANOVA, the appropriate test statistic is
$F = \frac{MSB}{MSW}$ with d.f. $(K-1, n-K)$

When H_0 is true, SSB is about the same as SSW because all variations simply from random fluctuations. But when the groups are systematically different from one another, SSB tends to be large, relative to variation within the groups. The larger the between group variation, the greater is the likelihood that the samples do <u>NOT</u> come from populations with equal means.

e.g.

GP=1	GP=2	GP=3
0	1	5
6	4	6
2	3	10
4	2	8
3	0	6
$\bar{x}_1 = 3.0$	$\bar{x}_2 = 2.0$	$\bar{x}_3 = 7.0$

$\bar{x} = 4.0$

$SSW = (0-3)^2 + (6-3)^2 + (2-3)^2 + (4-3)^2 + (3-3)^2 + (1-2)^2 + (4-2)^2 + (3-2)^2$
$+ (2-2)^2 + (0-2)^2 + (7-5)^2 + (7-6)^2 + (10-7)^2 + (8-7)^2 + (6-7)^2$
$= 9 + 9 + 1 + 1 + 0 + 1 + 4 + 1 + 0 + 4 + 4 + 1 + 9 + 1 + 1 = 46$

$SSB = 5 \times (3-4)^2 + 5 \times (2-4)^2 + 5 \times (7-4)^2 = 5 + 20 + 45 = 70$

$F = \frac{MSB}{MSW} = \frac{(70/2)}{(46/12)} = \frac{35}{3.834} = 9.1289$ d.f. $(2, 12)$