STAT 721 — Midterm Exam — October 31, 2005

~~NAME~~ Solutions

1. Let X be a random variable uniformly distributed on (-2, 1) –that is, its pdf is $f(x) = 1/3$ for $x \in (-2, 1)$; $= 0$ elsewhere.

(a) Find the pdf of $Y = 3(X + 2)^2 - 5$.

$y = 3(x+2)^2 - 5$ is a 1-1 map of $(-2,1)$ onto $(-5,22)$

Inverse transf. is $x = -2 + \sqrt{\frac{y+5}{3}}$.

For $y \in (-5,22)$, $g(y) = f\left(-2+\sqrt{\frac{y+5}{3}}\right)\left|\frac{dx}{dy}\right|$

$$= \frac{1}{3}\,\frac{1}{2}\left(\frac{y+5}{3}\right)^{-1/2}\frac{1}{3} = \frac{1}{18}\left(\frac{y+5}{3}\right)^{-1/2}.$$

So $g(y) = \frac{1}{18}\left(\frac{y+5}{3}\right)^{-1/2}$, $-5 < y < 22$
$= 0$ elsewhere

(b) Find the pdf of $W = X^2$.

Case 1. $w \in [0,1]$. Then the cdf of W is

$$G(w) = P[W \le w] = P[X^2 \le w] = P[-\sqrt{w} < X < \sqrt{w}]$$
$$= \frac{1}{3}\,2\sqrt{w} = \frac{2}{3}\sqrt{w}$$

Case 2. $w \in [1,4]$. Then

$$G(w) = \frac{2}{3} + P[1 \le W \le w] = \frac{2}{3} + P[-\sqrt{w} < X < -1]$$
$$= \frac{2}{3} + \frac{1}{3}(-1+\sqrt{w})$$

So $\boxed{g(w)} = G'(w)$ $\boxed{\begin{aligned} &= \tfrac{1}{3}w^{-1/2}, \quad 0<w<1 \\ &= \tfrac{1}{6}w^{-1/2}, \quad 1<w<4 \end{aligned}}$

2. Suppose that the random variable X has pdf

$$f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x} & \text{if } 0 < x < \infty \\ 0 & \text{elsewhere,} \end{cases}$$

where $\alpha > 0$.

(a) Find the moment generating function of X. (Be sure to indicate the range of values of t for which $M_X(t)$ exists.)

$$M_X(t) = Ee^{tX} = \int_0^\infty \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x} e^{tx}\, dx$$

$$= \int_0^\infty \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x(1-t)}\, dx \quad (\leftarrow \text{converges only if } t<1)$$

change of variables $y = x(1-t)$ $\rightarrow$

$$= \int_0^\infty \frac{1}{\Gamma(\alpha)} \left(\frac{y}{1-t}\right)^{\alpha-1} e^{-y} \frac{dy}{1-t}$$

$$= \frac{1}{(1-t)^\alpha} \int_0^\infty \frac{1}{\Gamma(\alpha)} y^{\alpha-1} e^{-y}\, dy$$

$$= \frac{1}{(1-t)^\alpha}, \quad t < 1.$$

(b) Let X_1, X_2 be a random sample of size 2 from the distribution given above. Find the pdf of $Y = X_1/X_2$.

Change of variables $y_1 = \frac{x_1}{x_2}$, $y_2 = x_2$ — 1:1 from $R^+ \times R^+$ onto $R^+ \times R^+$.

Inverse transformation $\begin{cases} x_1 = y_1 y_2 \\ x_2 = y_2 \end{cases}$ with Jacobian $J = \begin{vmatrix} y_2 & y_1 \\ 0 & 1 \end{vmatrix} = y_2$.

So joint pdf of Y_1 and Y_2 is

$$g(y_1, y_2) = f(y_1 y_2, y_2) \cdot y_2 = \frac{1}{\Gamma(\alpha)} (y_1 y_2)^{\alpha-1} e^{-y_1 y_2} \frac{1}{\Gamma(\alpha)} y_2^{\alpha-1} e^{-y_2} y_2$$

$$= \frac{1}{\Gamma^2(\alpha)} y_1^{\alpha-1} y_2^{2\alpha-1} e^{-y_2(1+y_1)}$$

So, for fixed y_1, $0 < y_1 < \infty$,

$$g(y_1) = \int_0^\infty g(y_1, y_2)\, dy_2 = \frac{1}{[\Gamma(\alpha)]^2} y_1^{\alpha-1} \int_0^\infty y_2^{2\alpha-1} e^{-y_2(1+y_1)}\, dy_2$$

with $z = y_2(1+y_1)$ $\rightarrow$

$$= \frac{1}{[\Gamma(\alpha)]^2} y_1^{\alpha-1} \int_0^\infty \frac{z^{2\alpha-1}}{(1+y_1)^{2\alpha-1}} e^{-z} \frac{dz}{1+y_1} = \frac{1}{[\Gamma(\alpha)]^2} \frac{y_1^{\alpha-1}}{(1+y_1)^{2\alpha}} \int_0^\infty z^{2\alpha-1} e^{-z}\, d$$

$$= \frac{\Gamma(2\alpha)}{[\Gamma(\alpha)]^2} y_1^{\alpha-1} \quad , 0 < y_1 < \infty.$$

3. Let $X_1, \ldots, X_n$ be i.i.d. random variables, each with pdf

$$f(x) = \begin{cases} \frac{3x^2}{\theta^3} & \text{for } 0 < x < \theta \\ 0 & \text{elsewhere,} \end{cases}$$

where $\theta > 0$.

(a) Find a sufficient statistic for θ.

For $x_1 > 0, x_2 > 0, \ldots, x_n > 0$, the joint pdf is

$$\prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} \left\{ \left[\frac{3x_i}{\theta} \right] I_{\{x_i \leq \theta\}} \right\} = \frac{3^n \left(\prod_{i=1}^{n} x_i\right)^2}{\theta^{3n}} I_{\{x_{(n)} < \theta\}}$$

where $x_{(n)} = \max\{x_1, \ldots, x_n\}$.

So, by the Factorization Theorem, $X_{(n)}$ is sufficient for θ.

(b) Show that the sufficient statistic (found in part (a)) is complete.

First, we find the pdf of $X_{(n)}$:

$$F_{X_{(n)}}(x) = P[X_{(n)} < x] = [F_{X_1}(x)]^n = \frac{x^{3n}}{\theta^{3n}}, \quad 0 < x < \theta.$$

So pdf is $f_{X_{(n)}}(x) = 3n x^{3n-1} / \theta^{3n}, \quad 0 < x < \theta.$

Suppose that $E_\theta\, g(X_{(n)}) = 0$ for all $\theta > 0$.

Then $\int_0^\theta \frac{3n\, x^{3n-1}}{\theta^{3n}} g(x)\, dx = 0$ for all $\theta > 0$.

Thus $\int_0^\theta g(x) x^{3n-1}\, dx = 0$ for all $\theta > 0$

So $0 = \frac{d}{d\theta} \int_0^\theta g(x) x^{3n-1}\, dx = g(\theta)\theta^{3n-1}$ for all $\theta > 0$.

So $g(\theta) = 0$ for all $\theta > 0$, i.e. $g(x) = 0$, for all $x > 0$.

By the def'n of completeness, it follows that $X_{(n)}$ is complete

4. Let $X_1, \dots, X_n$ be i.i.d., each with pdf

$$f_\beta(x) = \begin{cases} \frac{1}{2\beta^3} x^2 e^{-x/\beta} & \text{for } 0 < x < \infty \\ 0 & \text{elsewhere,} \end{cases}$$

where $\beta > 0$.

(a) Find a complete sufficient statistic T for β.

$$f_\beta(x) = \frac{1}{2\beta^3} x^2 e^{-x \cdot 1/\beta}$$

$$= c(\beta)\, h(x)\, e^{T(x)\, w(\theta)} \quad \text{, with } T(x) = x. \quad \text{with } k=1,$$

Thus the distribution of the X_i's is an exponential family!

So $\sum_{i=1}^n T(X_i) = \sum_{i=1}^n X_i$ is sufficient for β.

Also, since the parameter space $\beta > 0$ contains a one-dim. open set, it follows that the family of distributions of $\sum_{i=1}^n X_i$ is complete.

(b) Let the statistic W be defined by $W(X_1, \dots, X_n) = X_{(1)}/X_{(n)}$. Is the statistic T (of part (a)) independent of the statistic W? Give a careful justification of your answer. You are allowed to make the additional assumption that the sufficient statistic of part (a) is minimal sufficient.

By Basu's Theorem, independence of T and W will follow if we show that W is ancillary (i.e., its distribution does not depend on β).

A change of variables shows that $\frac{X_i}{\beta}$ has pdf $x^2 e^{-x}$,

Thus β is a scale parameter; write $X_i = \beta Z_i$, $i = 1, \dots, n$, where $Z_i \overset{iid}{\sim}$ pdf $x^2 e^{-x}$, $x > 0$.

Then $\frac{X_{(1)}}{X_{(n)}}$ has the same distr. as $\frac{\beta Z_{(1)}}{\beta Z_{(n)}} = \frac{Z_{(1)}}{Z_{(n)}}$ which has a distribution not depending on β. So $\frac{X_{(1)}}{X_{(n)}}$ is ancillary.

Hence, by Basu's theorem, $\frac{X_{(1)}}{X_{(n)}}$ is indep of the complete suff stat $T = \sum_i X_i$.