

STAT 721 — Midterm Exam No. 2 — December 2, 2005

NAME _____

Solutions

1. Let X_1, \dots, X_n be a random sample from a $\text{Poisson}(\lambda)$ distribution, where λ is unknown ($\lambda > 0$).

- (a) Find the MLE of $\lambda^2 e^{-\lambda}$.

$$L(\lambda) = \prod_{i=1}^n \left[e^{-\lambda} \frac{\lambda^{x_i}}{x_i!} \right] = e^{-n\lambda} \cdot \frac{\lambda^{\sum x_i}}{\prod x_i!}$$

$$\log L(\lambda) = -n\lambda + (\sum x_i) \log \lambda - \log(\prod x_i!)$$

$\frac{d}{d\lambda} \log L(\lambda) = -n + \frac{\sum x_i}{\lambda} = 0$ has unique soln $\hat{\lambda} = \frac{\sum x_i}{n} = \bar{x}$. Since $\frac{d^2}{d\lambda^2} \log L(\lambda) = -\sum x_i / \lambda^2 < 0$, the unique extremum must be a global max.

Since \bar{x} is MLE of λ , by an invariance theorem for MLE's, we have that $\bar{x}^2 e^{-\bar{x}}$ is the MLE of $\lambda^2 e^{-\lambda}$.

- (b) Find the best (i.e., uniformly minimum variance) unbiased estimator of $\lambda^2 e^{-\lambda}$.

$T = \sum_{i=1}^n X_i$ is a complete, sufficient for λ by the Factorization theorem and the one-dim. exponential family representation.

There are two possible solutions, both using the Lehmann-Scheffe theorem:

① An unbiased est of $\lambda^2 e^{-\lambda}$ is $2 \mathbb{E}[I_{\{X_1=2\}}]$, since $\mathbb{E}[I_{\{X_1=2\}}] = P[X_1=2] = \frac{\lambda^2}{2} e^{-\lambda}$.

$$\text{For } t \geq 2, \mathbb{E}[2 \mathbb{E}[I_{\{X_1=2\}}] \mid T=t] = 2 \mathbb{P}[X_1=2 \mid T=t] = 2 \mathbb{P}[X_1=2] \mathbb{P}\left[\sum_{i=2}^n X_i = t-2\right]$$

$$= \left[2e^{-\lambda} \frac{\lambda^2}{2} e^{-(n-1)\lambda} \frac{[(n-1)\lambda]^{t-2}}{(t-2)!} \right] / \left[e^{-\lambda} \frac{(n\lambda)^t}{t!} \right]$$

$$= \frac{(n-1)^{t-2}}{(t-2)!} \frac{t!}{n^t} = \frac{t(t-1)(n-1)^{t-2}}{n^t}$$

So the UMVU est is $\frac{T(T-1)(n-1)}{n^T}$

(2)

① (continued)

② (Another solution)

The UMVU est. must be a function of the complete sufficient statistic $T = \sum_{i=1}^n X_i$, so we try to solve

$$E_\lambda[\delta(T)] = \lambda^2 e^{-\lambda}$$

or $\sum_{t=0}^{\infty} \delta(t) \frac{e^{-n\lambda} (n\lambda)^t}{t!} = \lambda^2 e^{-\lambda}$

or $\sum_{t=0}^{\infty} \delta(t) \frac{n^t \lambda^t}{t!} = \lambda^2 e^{-(n-1)\lambda} = \lambda^2 \sum_{t=0}^{\infty} \frac{(n-1)^t \lambda^t}{t!}$
 $= \sum_{t=0}^{\infty} \frac{(n-1)^t \lambda^{t+2}}{t!} = \sum_{t=2}^{\infty} \frac{(n-1)^{t-2} \lambda^t}{(t-2)!}$,

In order for these two power series to agree for all $\lambda > 0$, we must

have $\delta(0) = 0$, $\delta(1) = 0$; and for $t \geq 2$,

$$\delta(t) \frac{n^t}{t!} = \frac{(n-1)^{t-2}}{(t-2)!}, \quad \delta(t) = \frac{t(t-1)(n-1)^{t-2}}{n^t}$$

So $\delta(T) = \frac{T(T-1)(n-1)^{T-2}}{n^T}$ is the UMVU est of $\lambda^2 e^{-\lambda}$.

(3)

2. Let X_1, \dots, X_n be a random sample from a Bernoulli(p) distribution with unknown p , $0 < p < 1$.

(a) Compute the Cramér-Rao lower bound on the variance of an unbiased estimator of p^3 .

$$f(x|p) = p^x (1-p)^{1-x}, \quad x=0,1$$

$$\log f(x|p) = x \log p + (1-x) \log (1-p),$$

$$\frac{\partial}{\partial p} \log f(x|p) = \frac{x}{p} - \frac{1-x}{1-p} = \frac{x-p}{p(1-p)}$$

$$I(p) = E_p \left\{ \left[\frac{\partial}{\partial p} \log f(x|p) \right]^2 \right\} = E_p \left[\frac{(x-p)^2}{p^2(1-p)^2} \right] = \frac{p(1-p)}{p^2(1-p)^2} = \frac{1}{p(1-p)}$$

$\tau(p) = p^3$, $\tau'(p) = 3p^2$, so the C-R lower bound on the var. of an UE of p^3 is $\frac{[\tau'(p)]^2}{n I(p)} = \frac{(3p^2)^2}{n(1/p(1-p))} = \frac{9p^5(1-p)}{n}$.

(b) Find the best (i.e., uniformly minimum variance) unbiased estimator of p^3 .

$$\text{Let } \delta = \begin{cases} 1 & \text{if } X_1 = X_2 = X_3 = 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$\text{Then } E_p [\delta(X_1, \dots, X_n)] = P_p [X_1 = X_2 = X_3 = 1] = p^3, \text{ so } \delta \text{ is an UMVUE of } p^3.$$

By the usual exponential family representation, $T = \sum_{i=1}^n X_i$ is complete suff-stat for p .

So the UMVUE of p^3 is $E[\delta | T=t]$.

$$\begin{aligned} \text{Fix } t \geq 3. \text{ Then } E_p [\delta | T=t] &= \frac{P[X_1=1, X_2=1, X_3=1] P\left[\sum_{i=4}^n X_i = t-3\right]}{P\left[\sum_{i=1}^n X_i = t-3\right]} \\ &= \frac{p^3 \binom{n-3}{t-3} p^{t-3} (1-p)^{(n-3)-(t-3)}}{\binom{n}{t} p^t (1-p)^{n-t}} = \frac{\binom{n-3}{t-3}}{\binom{n}{t}} = \frac{t(t-1)(t-2)}{n(n-1)(n-2)}. \end{aligned}$$

So $\frac{T(T-1)(T-2)}{n(n-1)(n-2)}$ is the UMVUE of p^3 , where $T = \sum_{i=1}^n X_i$.

(4)

3. Let X_1, \dots, X_n be a random sample from the distribution with pdf

$$f(x|\beta, \theta) = \begin{cases} \frac{1}{\beta} e^{-(x-\theta)/\beta}, & \text{if } \theta \leq x < \infty, \\ 0 & \text{elsewhere,} \end{cases}$$

where β ($\beta > 0$) and θ ($-\infty < \theta < \infty$) are both unknown parameters.

(a) Find a complete sufficient statistic for (θ, β) .

$$\prod_{i=1}^n f(x_i|\beta, \theta) = \frac{1}{\beta^n} e^{-\sum_{i=1}^n (x_i - \theta)/\beta} I_{\{X_{(1)} \geq \theta\}},$$

so by the Factorization Theorem, $(X_{(1)}, \sum_{i=1}^n x_i)$ is sufficient for (θ, β) . Another equivalent statistic which is also sufficient (by way of a 1-1 mapping) is $(X_{(1)}, \sum_{i=1}^n (x_i - X_{(1)}))$.

See appendix A.1 to A.7 for proof of completeness of this sufficient statistic. (I didn't expect anyone to get it on the test! It took me hours to do it.)

(b) Find the best (i.e., uniformly minimum variance) unbiased estimator of β .

We will find (by inspection) a function of the complete suff. stat which satisfy $E_{\theta, \beta}(\delta) = \beta$ for all $\theta \in \mathbb{R}$ and β .

Since $X_{(1)} - \theta$ is exponential, mean β/n , we have $E(X_{(1)}) = \theta + \beta/n$.

Since $\sum_{i=1}^n (x_i - \theta)$ is exponential, mean $n\beta$, $E(\sum_{i=1}^n x_i) = n\theta + n\beta$,

or $E(\bar{X}) = \theta + \beta$. So $E[\bar{X} - X_{(1)}] = \theta + \beta - (\theta + \frac{\beta}{n}) = \beta(\frac{n-1}{n})$.

So $E\left\{\frac{n}{n-1} [\bar{X} - X_{(1)}]\right\} = \beta$.

Since $\frac{n}{n-1} (\bar{X} - X_{(1)})$ is a function of the complete suff. stat ~~$(X_{(1)}, \sum_{i=1}^n x_i)$~~ , which is an unbiased est of β , it follows from Lehmann-Scheffe ₃ that it is VMVUest of β .

APPENDIX

Proof of "completeness" in problem 3:

WLOG, we can replace the sufficient statistic $(X_{(1)}, \sum_{i=1}^n X_i)$ with the equivalent sufficient statistic $(X_{(1)}, \sum_{i=1}^n (X_i - X_{(1)}))$.

First note that the joint pdf of the order statistics $(X_{(1)}, X_{(2)}, \dots, X_{(n)})$ is

$$f(x_{(1)}, x_{(2)}, \dots, x_{(n)}) = \frac{1}{n!} \frac{1}{B^n} e^{-\sum_{i=1}^n (x_{(i)} - \theta)/\beta}$$

if $\theta < x_{(1)} < x_{(2)} < \dots < x_{(n)}$

$$= 0 \quad \text{otherwise.}$$

Also it is easy to see that the marginal pdf of $X_{(1)}$

$$\text{is: } f(x_{(1)}) = \frac{n}{B} e^{-n(x_{(1)} - \theta)/\beta}$$

if $x_{(1)} > \theta$

$$= 0 \quad \text{otherwise.}$$

Hence the conditional density of $(X_{(2)}, X_{(3)}, \dots, X_{(n)})$ given

that $X_{(1)} = x_{(1)} (> \theta)$ is

$$f(x_{(2)}, \dots, x_{(n)} | X_{(1)} = x_{(1)}) = \frac{f(x_{(1)}, x_{(2)}, \dots, x_{(n)})}{f(x_{(1)})}$$

$$= \begin{cases} \frac{1}{(n-1)!} \frac{1}{B^{n-1}} e^{-\sum_{i=2}^n (x_{(i)} - x_{(1)})} & \text{if } \theta < x_{(1)} < x_{(2)} < \dots < x_{(n)} \\ 0 & \text{otherwise} \end{cases}$$

after some algebra

Hence the cond. distribution of $(X_{(2)}, \dots, X_{(n)})$, given $X_{(1)} = x_{(1)}$,

is the same as the unconditional distribution of

$$(x_{(1)} + Y_{(1)}, x_{(1)} + Y_{(2)}, \dots, x_{(1)} + Y_{(n-1)}),$$

where Y_1, Y_2, \dots, Y_{n-1} denotes a random sample of size $n-1$

from an exponential dist., with mean β .

So the conditional dist. of $(X_{(2)} - X_{(1)}, X_{(3)} - X_{(1)}, \dots, X_{(n)} - X_{(1)})$

given $X_{(1)} = x_{(1)}$ is the same as the uncond. dist. of $(Y_{(1)}, Y_{(2)}, \dots, Y_{(n-1)})$.

Thus the cond. dist. of $\sum_{i=2}^n (X_{(i)} - X_{(1)})$ given $X_{(1)} = x_{(1)}$ is the

same as the uncond. dist. of $\sum_{i=1}^{n-1} Y_i$. Equivalently, the

conditional dist. of ~~$\sum_{i=1}^{n-1} Y_i$~~ $\sum_{i=1}^n (X_i - X_{(1)})$

given $X_{(1)} = x_{(1)}$ is the same as the unconditional of $\sum_{i=1}^{n-1} Y_i$,

which is Gamma $(\beta, n-1)$ since the Y_i 's are iid exponential with mean β .

Since this is true for all values of $x_{(1)} (\geq \beta)$, we have shown now

that ~~the~~ the sufficient statistic $X_{(1)}$ and $T \stackrel{\text{def}}{=} \sum_{i=1}^n (X_i - X_{(1)})$

are independent. [Note: I worked too hard to get this. If we fix the value of β , the independence follows directly from Basu's Theorem.]

(A.3)

Furthermore, we have shown that the joint pdf of the sufficient statistics $X_{(1)}$ and T is:

$$f(x, t) = \frac{n}{\beta} e^{-n(x-\theta)/\beta} \frac{1}{\Gamma(n-i) \beta^{n-i}} t^{n-2} e^{-t/\beta}, \text{ if } 0 < x < \infty \text{ and } 0 < t < \infty, \\ = 0, \text{ otherwise.}$$

Now suppose that $g(x, t)$ is a function such that

$$(1) E_{\theta, \beta} g(X_{(1)}, T) = 0 \text{ for all } \theta \in \mathbb{R} \text{ and all } \beta > 0.$$

To complete the proof that $(X_{(1)}, T)$ is ~~sufficiently~~ complete, it suffices to show that (1) implies

$$(2) g(x, t) = 0 \text{ for all } x \in \mathbb{R} \text{ and } t > 0$$

(except perhaps on a set of prob. 0).

Now (2) can be written as

$$(3) \int_{x=\theta}^{\infty} \int_{t=0}^{\infty} g(x, t) \frac{n}{\beta} e^{-n(x-\theta)/\beta} \frac{1}{\Gamma(n-i)} t^{n-2} e^{-t/\beta} dt dx = 0 \text{ for all } \theta \in \mathbb{R} \text{ and } \beta > 0.$$

or

$$(4) e^{n\theta/\beta} \int_{x=0}^{\infty} \left[\int_{t=0}^{\infty} g(x, t) \frac{1}{\Gamma(n-i)} t^{n-2} e^{-t/\beta} dt \right] \frac{n}{\beta} e^{-nx/\beta} dx = 0 \text{ for all } \theta \text{ and all } \beta > 0.$$

~~Since $e^{n\theta/\beta} > 0$ for all real θ ,~~

~~we must have~~

it follows that (4) is equivalent to

$$(5) \quad \int_{x=0}^{\infty} H(x, \beta) e^{-\theta x/\beta} dx = 0 \quad \forall \theta \text{ and } \forall \beta > 0,$$

where $H(x, \beta) \stackrel{\text{def}}{=} \int_{t=0}^{\infty} g(x, t) \frac{1}{\Gamma(n-1)} t^{n-2} e^{-t/\beta} dt$.

Next, fix $\beta > 0$, so that the integrand in (5) is a function of x alone. By the Fund-Thm of Calculus, we can differentiate (5) with respect to θ to obtain

$$(6) \quad -H(\theta, \beta) e^{-\theta/\beta} = 0 \quad \forall \theta \in \mathbb{R}, \text{ for fixed } \beta > 0.$$

It follows that ~~$H(x, \beta)$~~ $H(x, \beta) = 0 \quad \forall x \in \mathbb{R} \text{ and } \forall \beta > 0$,

that is, that

$$(7) \quad \int_{t=0}^{\infty} g(x, t) \frac{1}{\Gamma(n-1)} t^{n-2} e^{-t/\beta} dt = 0 \quad \text{for all } x \in \mathbb{R} \text{ and for all } \beta > 0.$$

Now fix $x \in \mathbb{R}$, so that $g(x, t)$ in the integrand is a function of t alone. Since (7) holds for all $\beta > 0$, it follows from the completeness of the one-dimensional Gamma $(n-1, \beta)$ family that

$$g(x, t) = 0 \quad \text{for fixed } x \in \mathbb{R} \text{ and for all } t > 0.$$

Since this is true for every $x \in \mathbb{R}$, it follows that $g(x, t) = 0$ for all $x \in \mathbb{R}$ and all $t > 0$. \square