

①

STAT 721 Solutions to Assignment #3

7.1

| x | MLE at x |
|-----|------------|
| 0 | 1 |
| 1 | 1 |
| 2 | 2 |
| 3 | 3 |
| 4 | 3 |

(since $\max(\frac{1}{3}, \frac{1}{4}, 0) = \frac{1}{3}$, etc.)

Another MLE is:

| x | MLE at x |
|-----|------------|
| 0 | 1 |
| 1 | 1 |
| 2 | 3 |
| 3 | 3 |
| 4 | 3 |

← note $\max\{f(x_1), f(x_2), f(x_3)\}$ is attained at both $x=2$ and $x=3$

This example shows that a MLE need not be unique.

7.6 (a) $\prod_{i=1}^n f(x_i; \theta) = \theta^n (\prod x_i)^{-2} I_{\{x_{(1)} \geq \theta\}}$, so $X_{(1)} = \min(X_1, \dots, X_n)$ is sufficient.

(b) $\log L(\theta) = n \log \theta - 2 \sum_{i=1}^n \log x_i \quad \bar{x} \quad \text{if } X_{(1)} \geq \theta \quad (\text{i.e. if } 0 < \theta \leq X_{(1)})$
 $= -\infty \quad \text{otherwise}$

Since $\log L(\theta)$ is increasing on $(0, X_{(1)}]$, max is attained at $\theta = X_{(1)}$.
Hence MLE $\hat{\theta} = X_{(1)}$.

(c) $E X = \theta \int_{\theta}^{\infty} x \cdot \frac{1}{x^2} dx = \theta \int_{\theta}^{\infty} \frac{dx}{x} = \theta [\ln(\infty) - \ln(\theta)] = \infty$

Since $EX = \infty$, ~~method of moments~~ fails,

i.e., there is no method of moments estimator of θ .

(2)

$$7.9 \quad EX = \int x f(x|\theta) dx = \int_0^\theta \frac{x}{\theta^2} dx = \left[\frac{x^2}{2\theta^2} \right]_0^\theta = \frac{1}{\theta^2} \cdot \frac{\theta^2}{2} = \frac{\theta}{2}$$

Method of moments = equate $\bar{X} = m_1 = \frac{\theta}{2}$.

Solution is $\theta = 2\bar{X}$. So method of moments est is $\hat{\theta}_{MM} = 2\bar{X}$

$$\text{MLE : } L(\theta) = \begin{cases} \frac{1}{\theta^n} & \text{if } \bar{X}_{(n)} \leq \theta \\ 0 & \text{elsewhere} \end{cases}$$

So $L(\theta)$ reaches max at $\hat{\theta} = \bar{X}_{(n)}$.

So the MLE is $\hat{\theta}_{ML} = \bar{X}_{(n)}$.

The mean of ~~$\hat{\theta}_{MM}$~~ is $E[2\bar{X}] = 2EX_1 = 2\frac{\theta}{2} = \theta$,

and the variance is $\text{Var}(2\bar{X}) = 4 \frac{\text{Var}(X_1)}{n} = \frac{4}{n} \frac{\theta^2}{12} = \frac{\theta^2}{3n}$.

$$\text{For the MLE, } E(\bar{X}_{(n)}) = \int_0^\theta x \frac{nx^{n-1}}{\theta^n} dx = \frac{n}{\theta^n} \frac{\theta^{n+1}}{n+1} = \frac{n}{n+1} \theta,$$

$$\text{Also } E(\bar{X}_{(n)}^2) = \int_0^\theta x^2 \frac{nx^{n-1}}{\theta^n} dx = \frac{n}{\theta^n} \frac{\theta^{n+2}}{n+2} = \frac{n}{n+2} \theta^2, \text{ so}$$

$$\text{Var}(\bar{X}_{(n)}) = \frac{n}{n+2} \theta^2 - \frac{n^2}{(n+1)^2} \theta^2 = \frac{n}{(n+2)(n+1)^2} \theta^2.$$

Comparison. $\hat{\theta}_{MM}$ is unbiased $\hat{\theta}_{MLE}$ is not, so the method of moments estimator wins in terms of bias. In all other respects the MME loses to the MLE: the MME is not a function of the sufficient stat $X_{(n)}$ and so could be improved. The MME has much higher variance than the MLE — even when $n=1$, the variances are $\theta^3/3$ vs $\theta^2/12$. On balance, I prefer the MLE

(3)

7.11 (a) $\log L(\theta) = \log [\theta^n (\prod x_i)^{\theta-1}] = n \log \theta - (\theta-1) \sum \log x_i$

$\frac{d}{d\theta} \log L(\theta) = \frac{n}{\theta} - \sum \log x_i$. Equate this to 0 and

solve to obtain $\hat{\theta} = \frac{n}{-\sum \log x_i}$. To see that this yields

a global maximum, note that $\frac{d^2}{d\theta^2} \log L(\theta) = -\frac{n}{\theta^2} < 0$.

So the MLE is $\hat{\theta} = \frac{n}{-\sum_{i=1}^n \log x_i}$.

[The minus sign looks screwy until one notes that $0 < x < 1$ implies $-\log x_i > 0$.]

To compute the variance of the MLE, we first need to figure out its distribution.

Let $Y_i = -\log x_i$. Then the pdf of Y_i is

$$g(y) = f(e^{-y}) |e^{-y}| = \theta e^{-y(\theta-1)} e^{-y} = \theta e^{-y/\theta}, 0 < y < \infty \\ = 0 \quad \text{elsewhere}$$

So Y_1, Y_2, \dots, Y_n are iid, each with exponential

distribution $f(y) = \frac{1}{\beta} e^{-y/\beta}, y > 0$, with $\beta = 1/\theta$,

Hence (using reg. mgf's), $\sum_{i=1}^n Y_i = -\sum_{i=1}^n \log x_i$

has Gamma distribution with $\alpha = n, \beta = 1/\theta$

(as listed on page 629 of text).

(7.11) (a) (continued)

If we let $V = \frac{1}{\sum_{i=1}^n \log x_i}$. Then V has an inverted gamma distribution with parameters n and $\theta = V\beta$. Its pdf, worked out at the ~~start~~ top of p.52 of the text, is $\frac{1}{(n-1)! \beta^n} \left(\frac{1}{y}\right)^{n+1} e^{-1/(By)}$, $0 < y < \infty$, where $\beta = 1/\theta$.

Then we can work out the first two moments of $\hat{\theta} = nV$ from the first two moments of V , which we can evaluate using the trick that the pdf of the inverted gamma ~~function~~ integrates to 1, no matter what the integer n is. So

$$\begin{aligned} E[V] &= \int_0^\infty \frac{1}{(n-1)! \beta^n} \left(\frac{1}{y}\right)^{n+1} \left(\frac{1}{y}\right)^{-1} e^{-1/(By)} dy \\ &= \int_0^\infty \frac{1}{(n-1)! \beta^{n-1+1}} \left(\frac{1}{y}\right)^{(n-1)+1} e^{-1/(By)} dy \\ &= \frac{(n-2)!}{(n-1)! \beta} \int_0^\infty \frac{1}{[(n-1)-1]!} \frac{1}{\beta^{n-1}} \left(\frac{1}{y}\right)^{(n-1)+1} e^{-1/(By)} dy \\ &= \frac{(n-2)!}{(n-1)! \beta} \cdot 1 = \frac{\theta}{n-1}. \text{ So } E[nV] = \frac{n\theta}{n-1} \end{aligned}$$

A similar calculation shows that $E(V^2) = \frac{(n-3)!}{(n-1)! \beta^2} = \frac{\theta^2}{(n-1)(n-2)}$

(5)

$$\text{So } E[nV]^2 = n^2 EU^2 = \frac{n^2 \theta^2}{(n-1)(n-2)}.$$

$$\begin{aligned} \text{So } \text{Var}(\hat{\theta}) &= E[(nV)^2] - [E(nV)]^2 = \frac{n^2 \theta^2}{(n-1)(n-2)} - \frac{n^2 \theta^2}{(n-1)^2} \\ &= \frac{n^2}{(n-1)^2(n-2)} \theta^2, \text{ which is easily seen to } \rightarrow 0 \\ &\quad \text{as } n \rightarrow \infty. \end{aligned}$$

$$(b) EX_1 = \int_0^1 \theta x^{\theta-1} x dx = \int_0^1 \theta x^\theta dx = \theta \left[\frac{x^{\theta+1}}{\theta+1} \right]_0^1 = \frac{\theta}{\theta+1}$$

So equate $\bar{X} = \frac{\theta}{\theta+1}$ and solve for θ .

$$\bar{X}\theta + \bar{X} = \theta, \quad \theta(1-\bar{X}) = \bar{X}, \quad \theta = \frac{\bar{X}}{1-\bar{X}}$$

So ~~MLE~~ method of moments estimator is $\underline{\frac{\bar{X}}{1-\bar{X}}}$

(Makes sense, because $\bar{X} \in (0,1)$ with prob 1
and so $\frac{\bar{X}}{1-\bar{X}}$ stays in $(0,\infty)$ with prob 1).

7.37 Let $Y_i = |X_i|, i=1 \dots n$. The model is equivalent to the model that Y_1, \dots, Y_n are iid $\text{Unif}[0, \theta]$.

We know that $\max(Y_1, \dots, Y_n)$ is complete suff stat for θ and that $E[\max(Y_1, \dots, Y_n)] = \frac{n}{n+1} \theta$. Thus $\frac{n+1}{n} \max Y_i$ is an unbias est of θ which is a function of a complete sufficient statistic, hence is a best unbiased est. in terms of the original X_i 's,

$\frac{n+1}{n} \max\{|X_1|, |X_2|\}$ is best VE of θ .

7.38 To work this question, we need to use Cor 7.3.15 in text (necc & suff conditions for attainment of C-R lower bound). [Note: my apologies for skipping this topic in my lectures]. In each case, we write down $a(\theta)[W(x) - g(\theta)] = \frac{d}{d\theta} \log L(\theta|x)$; and check whether it holds for some $g(\theta)$ and some U.E. $W(x)$ of $g(\theta)$.

(a) When $f(x|\theta) = \theta x^{\theta-1}$, $0 < x < 1$, $\theta > 0$

$$\log L(\theta|x) = \log [\theta^n (\pi x_i)^{\theta-1}] = n \log \theta + (\theta-1) \sum_{i=1}^n \log x_i$$

$$\text{So } \frac{d}{d\theta} \log L(\theta|x) = \frac{n}{\theta} + \sum_{i=1}^n \log x_i.$$

$$\text{Write } a(\theta)[W(x) - g(\theta)] = \sum_{i=1}^n \log x_i + \frac{n}{\theta}.$$

For the above identity to hold, $g(\theta)$ must be some non-zero multiple of $\frac{1}{\theta}$. Take, e.g. $g(\theta) = 1/\theta$. This forces $a(\theta) = -n$ and $W(x) = -\frac{\sum_{i=1}^n \log x_i}{n}$. Then, by the Cor.,

$W(x) = -\frac{\sum \log x_i}{n}$ attains the C-R lower bound, provided that $W(x)$ is an unbiased est of $\frac{1}{\theta}$. (continued)

7

7.38(a) (continued)

$E W(X) = -E \log(X_1) = \frac{1}{\theta}$, since we showed in an earlier problem that $\log X_1 \sim \text{exponential with } \beta = \frac{1}{\theta}$.

$$\underline{7.38(b)}: \quad L(\theta|x) = \left(\frac{\log \theta}{\theta-1} \right)^n \theta^{\sum x_i}$$

$$\log L(\theta|x) = n \log(\log \theta) - n \log(\theta-1) + \sum x_i \log \theta$$

$$\frac{\partial}{\partial \theta} \log L(\theta|x) = \frac{n}{\log \theta} - \frac{1}{\theta} - \frac{n}{\theta-1} + \frac{\sum x_i}{\theta}$$

Try to find $g(\theta)$, $W(x)$, $a(\theta)$ for which

$$a(\theta) [W(x) - g(\theta)] = \frac{n}{\theta} \left\{ \frac{\sum x_i}{n} - \left[\frac{\theta}{\theta-1} - \frac{1}{\log \theta} \right] \right\}$$

Unique solution (up to a non-zero multiplicative constant) is.

$$g(\theta) = \frac{\theta}{\theta-1} - \frac{1}{\log \theta}, \quad W(x) = \bar{x}, \quad a(\theta) = \frac{n}{\theta}.$$

\bar{x} attains the C-R lower bound for estimating $g(\theta)$ provided that $E_\theta(\bar{x}) = g(\theta)$.

$$E_\theta(\bar{x}) = E_\theta(x_1) = \frac{1}{\theta-1} \int_0^1 x \theta^x \log \theta dx = \frac{1}{\theta-1} \int_0^1 x d\theta^x$$

$$= \frac{1}{\theta-1} \left[x \theta^x \Big|_0^1 - \int_0^1 \theta^x dx \right] = \frac{1}{\theta-1} \left[\theta - \frac{\theta^x}{\log \theta} \Big|_0^1 \right]$$

$$= \frac{1}{\theta-1} \left[\theta - \frac{\theta-1}{\log(\theta)} \right] = \frac{\theta}{\theta-1} - \frac{1}{\log(\theta)} = g(\theta). \quad \checkmark$$

7.44 $X_1, \dots, X_n \stackrel{iid}{\sim} N(\theta, 1)$.

We have seen before that \bar{X} is a complete sufficient statistic for θ . If we can show that $E_\theta(\bar{X}^2 - \frac{1}{n}) = \theta^2$, then (by Lehmann-Scheffe Thm) it follows that $\bar{X}^2 - \frac{1}{n}$ is the best VT of θ^2 .

$$\begin{aligned} E(\bar{X}^2) &= \frac{1}{n^2} E\left[\left(\sum_{i=1}^n X_i\right)^2\right] = \frac{1}{n^2} \left[nEX_1^2 + n(n-1)E(X_1 X_2) \right] \\ &= \frac{1}{n^2} \left[n[\text{Var } X_1 + (EX)^2] + n(n-1)(EX_1)^2 \right] \\ &= \frac{1}{n^2} [n(1+\theta^2) + n(n-1)\theta^2] = \frac{1}{n} [1+\theta^2 + (n-1)\theta^2] = \frac{1}{n}[n\theta^2 + \theta^2] \\ &= \theta^2 + \frac{1}{n}, \text{ so } E\left[\bar{X}^2 - \frac{1}{n}\right] = \theta^2. \quad \checkmark \end{aligned}$$

There are many ways to compute $\text{Var}(\bar{X}^2)$, ranging from crude to elegant (e.g. Stein's identity). I will give a crude derivation relying on algebra and the fact that if $Z \sim N(0, 1)$, then $EZ=0$, $EZ^2=1$, $EZ^3=0$ and $EZ^4=3$ (if you haven't seen $EZ^4=3$, show it using mgf or by integration by parts $\int x^4 \varphi(x) dx = - \int x^3 d\varphi(x) = \text{etc.}$)

$$\bar{X} - \theta \sim N(0, \frac{1}{n}), \text{ so } \sqrt{n}(\bar{X} - \theta) \sim N(0, 1).$$

We have already that $E\bar{X}=\theta$ and $E\bar{X}^2=\theta^2+\frac{1}{n}$.

Since $E[n^{3/2}(\bar{X}-\theta)^3]=0$, we have $E[\bar{X}^3-3\theta\bar{X}^2+3\theta^2\bar{X}-\theta^3]=$

$$\begin{aligned} \text{so } E(\bar{X}^3) &= 3\theta E(\bar{X}^2) - 3\theta^2 E\bar{X} + \theta^3 = 3\theta(\theta^2 + \frac{1}{n}) - 3\theta^3 + \theta^3 \\ &= \theta^3 + \frac{3\theta}{n} \end{aligned}$$

(continued)

7.44 (continued)

$$\text{also } E[n^2(\bar{X} - \theta)^4] = 3, \text{ so}$$

$$E[\bar{X}^4 - 4\theta\bar{X}^3 + 6\theta^2\bar{X}^2 - 4\theta^3\bar{X} + \theta^4] = \frac{3}{n^2}$$

$$\text{so } E(\bar{X}^4) = \frac{3}{n^2} + 4\theta E(\bar{X}^3) - 6\theta^2 E(\bar{X}^2) + 4\theta^3 E(\bar{X}) - \theta^4$$

$$= \frac{3}{n^2} + 4\theta \left[\theta^3 + \frac{3\theta}{n} \right] - 6\theta^2 \left(\theta^2 + \frac{1}{n} \right) + 4\theta^4 - \theta^4$$

$$= \theta^4 + \frac{3}{n^2} + \frac{12\theta^2}{n} - \frac{6\theta^2}{n} = \theta^4 + \frac{6\theta^2}{n} + \frac{3}{n^2}$$

$$\text{Thus } \text{Var}(\bar{X}^2 - \frac{1}{n}) = \text{Var}(\bar{X}^2) = E(\bar{X}^4) - [E(\bar{X}^2)]^2$$

$$= \theta^4 + \frac{6\theta^2}{n} + \frac{3}{n^2} - (\theta^2 + \frac{1}{n})^2$$

$$= \theta^4 + \frac{6\theta^2}{n} + \frac{3}{n^2} - \theta^4 - \frac{2\theta^2}{n} - \frac{1}{n^2} = \underline{\underline{\frac{4\theta^2}{n} + \frac{2}{n^2}}}$$

Cramer-Rao Lower bound is $\frac{\left[\frac{d}{d\theta} \theta^2 \right]^2}{n E_\theta \left[\left(\frac{\partial}{\partial \theta} \log f(x, \theta) \right)^2 \right]}$

$$\text{Numerator is } (2\theta)^2 = 4\theta^2.$$

$$\log f(x|\theta) = \log \frac{1}{\sqrt{2\pi}} - \frac{1}{2}(x-\theta)^2, \quad \frac{\partial}{\partial \theta} \log f(x|\theta) = -\frac{1}{2} 2(x-\theta) - 1 \\ = x - \theta$$

$$\text{So denominator is } n E_\theta [(x-\theta)^2] = n \text{Var}(X_1) = n.$$

So C-R lower bd on variance of VE of θ^2 is $\frac{4\theta^2}{n} \left(< \frac{4\theta^2}{n} + \frac{2}{n^2} \right)$

7.46 $X_1, X_2, X_3 \stackrel{iid}{\sim} U[\theta, 2\theta]$

(a) $E X_i = \frac{\theta + 2\theta}{2} = \frac{3}{2}\theta$ - Estimate $\bar{X} = \frac{3}{2}\hat{\theta}_{MM}$

and solve to get $\hat{\theta}_{MM} = \frac{2}{3}\bar{X}$.

(b) First note that $0 < \theta < a < b < 2\theta \Rightarrow \frac{b}{a} < 2 \Rightarrow \frac{b}{2} < a$,

hence $P_\theta\left[\frac{X_{(3)}}{2} > X_{(1)}\right] = 0$ for all $\theta > 0$. So we will

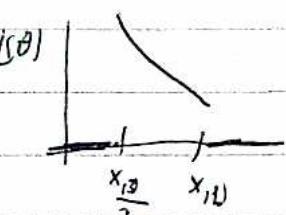
assume that $\frac{X_{(3)}}{2} \leq X_{(1)}$ (and define the MLE to be any estimator $f(x_1, \dots, x_n)$ that one want when $\frac{X_{(3)}}{2} > X_{(1)}$).).

$$\text{Given } \frac{X_{(3)}}{2} \leq X_{(1)}, L(\theta | x_1, \dots, x_n) = \begin{cases} \frac{1}{\theta^3} & \text{if } \theta \leq x_i \leq 2\theta, \\ 0 & \text{elsewhere} \end{cases} \quad i=1, 2, 3.$$

The condition $\{\theta \leq x_i \leq 2\theta, i=1, 2, 3\}$ is equivalent to $\{X_{(1)} \geq \theta \text{ and } X_{(3)} \leq 2\theta\}$

which is equivalent to $\frac{X_{(3)}}{2} \leq \theta \leq X_{(1)}$ (when $\frac{X_{(3)}}{2} \leq X_{(1)}$).

$$\text{So } L(\theta | x_1, x_2, x_3) = \begin{cases} \frac{1}{\theta^3} & \text{when } \frac{X_{(3)}}{2} \leq \theta \leq X_{(1)} \\ 0 & \text{elsewhere} \end{cases}$$



so MLE is $\hat{\theta} = \frac{X_{(3)}}{2}$.

So $\hat{\theta} = \frac{X_{(3)}}{2}$ is a MLE of θ .

$$E(X_{(3)}) = \theta + \frac{3}{4}\theta = \frac{7}{4}\theta \text{ so } E \hat{\theta} = E\left[\frac{X_{(3)}}{2}\right] = \frac{7}{8}\theta$$

Take $k = \frac{8}{7}$ then $\frac{8}{7}\hat{\theta} = \frac{4}{7}X_{(3)}$ is an unb. est of θ .

(continued)

(7.46) (c) The sufficient statistic is $(X_{(1)}, X_{(3)})$. Since $\frac{4}{7} X_{(3)}$ is an unbiased est. of θ which is already a function of the sufficient statistic, it cannot be improved by 'Rao-Blackwellization'; that is, $E\left[\frac{4}{7} X_{(3)} \mid (X_{(1)}, X_{(3)})\right] = \frac{4}{7} X_{(3)}$ (the same estimator).

The moment estimator $\frac{2}{3} \bar{X} = \frac{2}{9} (X_1 + X_2 + X_3) = \frac{2}{9} (X_{(1)} + X_{(2)} + X_{(3)})$

~~is not~~ is an VE of θ which is not a function of the sufficient statistic $(X_{(1)}, X_{(3)})$. So it can be improved. Let

$X_{(1)}, X_{(3)}$ be two possible values of $X_{(1)}$ and $X_{(3)}$ (i.e., $\frac{X_{(3)}}{2} \leq X_{(1)}$)

$$\begin{aligned} \text{Then } E\left[\frac{2}{9} (X_{(1)} + X_{(2)} + X_{(3)}) \mid X_{(1)} = x_{(1)}, X_{(3)} = x_{(3)}\right] &= \\ &= \frac{2}{9} \left(x_{(1)} + x_{(3)} + E[X_{(2)} \mid X_{(1)} = x_{(1)}, X_{(3)} = x_{(3)}] \right) \\ &= \frac{2}{9} \left[x_{(1)} + x_{(3)} + \frac{1}{2} (x_{(1)} + x_{(3)}) \right] = \frac{2}{9} \frac{3}{2} (x_{(1)} + x_{(3)}) = \frac{1}{3} (x_{(1)} + x_{(3)}). \end{aligned}$$

Here we have used the fact that the conditional dist. of $X_{(2)}$ given $X_{(1)} = x_{(1)}$ and $X_{(3)} = x_{(3)}$ is uniform on $[x_{(1)}, x_{(3)}]$, and so the conditional expectation is $(x_{(1)} + x_{(3)})/2$.

So, by the Rao-Blackwell Theorem, $\frac{1}{3} (X_{(1)} + X_{(3)})$ is an unbiased estimator of θ with strictly smaller variance than that of $\frac{2}{3} \bar{X}$.

$$7.46(d) \quad X_1 = 1.29, X_2 = 0.86, X_3 = 1.33$$

$\hat{\theta}_{MM} = \frac{2}{9}(1.29 + 0.86 + 1.33) = 0.7733$ ← In this case, the estimate is reasonable because all 3 X_i 's lie between $\hat{\theta} = 0.7733$ and $2\hat{\theta} = 1.5467$.

B

$X_{(1)} = 0.86, X_{(2)} = 1.29, X_{(3)} = 1.33$. Note that $\frac{X_{(3)}}{2} = 0.665 < X_{(1)} = 0.86$

so the ~~ML~~ est $\hat{\theta} = \frac{X_{(3)}}{2} = 0.665$ makes sense in this case. Note that all 3 observations lie between $\hat{\theta} = 0.665$ and $2\hat{\theta} = 1.33$.

7.48 (a) C-R lower bound on variance of an VE of p is

$$\frac{1}{n E_p \left\{ \left[\frac{\partial}{\partial p} \log f(x_i | p) \right]^2 \right\}}$$

$$\begin{aligned} \log f(x | p) &= \log [p^x (1-p)^{1-x}] \\ &= x \log p + (1-x) \log (1-p) \end{aligned}$$

$$\frac{\partial}{\partial p} \log f(x | p) = \frac{x}{p} + \frac{1-x}{1-p} (-1) = \frac{x}{p} - \frac{1-x}{1-p} = \frac{(1-p)x - p(1-x)}{p(1-p)}$$

$$= \frac{x-p}{p(1-p)}$$

$$E_p \left[\left(\frac{x-p}{p(1-p)} \right)^2 \right] = \frac{E[(x-p)^2]}{p^2(1-p)^2} = \frac{\text{Var } X_1}{p^2(1-p)} = \frac{1-p}{p^2(1-p)^2} = \frac{1}{p(1-p)}$$

So C-R lower bound is $\frac{1}{n \frac{1}{p(1-p)}} = \frac{p(1-p)}{n}$, (contains)

7.48(a) (continued)

The MLE of p : $L(p|x_1, \dots, x_n) = \prod_{i=1}^n [p^{x_i} (1-p)^{1-x_i}]$
 $= p^{\sum x_i} (1-p)^{n-\sum x_i}$, so $\log L(p|x_1, \dots, x_n) = (\sum x_i) \log p + (n - \sum x_i) \log(1-p)$

$$\frac{d}{dp} \left[(\sum x_i) \log p + (n - \sum x_i) \log(1-p) \right] = \frac{\sum x_i}{p} - \frac{n - \sum x_i}{1-p}.$$

$\frac{d}{dp} [\cdot] = 0$ when $\frac{\sum x_i}{p} = \frac{n - \sum x_i}{1-p}$ or $\sum x_i - p \sum x_i = np - (n - \sum x_i)p$
or $p = \frac{\sum x_i}{n}$. One can easily check that this ~~is~~ $\neq 0$.

corresponds to a unique global max of $\log L(p)$, so $\hat{p} = \frac{\sum x_i}{n}$

is the MLE.

Since $\sum_{i=1}^n x_i \sim \text{Binom}(n, p)$, $\text{Var}(\sum x_i) = np(1-p)$

$$\therefore \text{Var}(\hat{p}) = \frac{\text{Var}(\sum x_i)}{n^2} = \frac{np(1-p)}{n^2} = \frac{p(1-p)}{n}.$$

\hat{p} agrees with the C-R lower bound

(b) $X_1 X_2 X_3 X_4 = 1 \Leftrightarrow X_1 = 1, X_2 = 1, X_3 = 1, X_4 = 1$
(since each $X_i = 0$ or 1),

$$\therefore E_p(X_1 X_2 X_3 X_4) = 1 \cdot P_p[X_1 = 1, X_2 = 1, X_3 = 1, X_4 = 1]$$

$$= \prod_{i=1}^4 P_p[X_i = 1] = p \cdot p \cdot p \cdot p = p^4.$$

indep of
 X_1, X_2, X_3, X_4

$T = \sum_{i=1}^n X_i$ is a complete sufficient statistic

for θ . (continued).

Hence $E\{X_1 X_2 X_3 X_4 \mid \sum_{i=1}^n X_i = k\}$ is UMVUE of p^4 , by the Lehmann-Scheffe Theorem.

$$\begin{aligned}
 E\{X_1 X_2 X_3 X_4 \mid \sum_{i=1}^n X_i = k\} &= P[X_1=1, X_2=1, X_3=1, X_4=1 \mid \sum_{i=1}^n X_i = k] \\
 &= \frac{P[X_1=1, X_2=1, X_3=1, \cancel{X_4=1}, \sum_{i=1}^n X_i = k-4]}{P[\sum_{i=1}^n X_i = k]} \\
 &= \frac{p^4 \cdot \binom{n-4}{k-4}}{\binom{n}{k} p^k (1-p)^{n-k}} = \frac{\binom{n-4}{k-4}}{\binom{n}{k}} \\
 &= \frac{(n-4)!}{(k-4)! (n-k)!} = \frac{k(k-1)(k-2)(k-3)}{n(n-1)(n-2)(n-3)}, \text{ when } k=4, 5, \dots, n
 \end{aligned}$$

Clearly, when $k=0, 1, 2, 3$, $E\{X_1 X_2 X_3 X_4 \mid \sum_{i=1}^n X_i = k\} = 0$.

Hence, with $T = \sum_{i=1}^n X_i$, the best unbiased est. of p^4 is $\frac{T(T-1)(T-2)(T-3)}{n(n-1)(n-2)(n-3)}$.

7.52 (a) $X_1, \dots, X_n \stackrel{iid}{\sim} \text{Poisson}(\lambda)$

$$\prod_{i=1}^n f(x_i | \lambda) = \prod_{i=1}^n \left[e^{-\lambda} \frac{\lambda^{x_i}}{x_i!} \right] = e^{-n\lambda} \frac{\lambda^{\sum x_i}}{\prod x_i!}.$$

By factorization, $\sum x_i$ is sufficient for λ . We also need to show that the family of distributions of $T = \sum_{i=1}^n X_i$ is complete. We can do that either by quoting a theorem about completeness in exponential families — or we can do the following direct verification.

Suppose that $E_\lambda[g(T)] = 0$ for all $\lambda > 0$. To show completeness we need to deduce that $g(t) = 0$ for $t = 0, 1, 2, 3, \dots$

$T \sim \text{Poisson}(n\lambda)$, so $E_\lambda[g(T)] = 0$ implies

$$0 = \sum_{t=0}^{\infty} g(t) e^{-n\lambda} \frac{(n\lambda)^t}{t!} = e^{-n\lambda} \sum_{t=0}^{\infty} \frac{n^t g(t)}{t!} \lambda^t \quad \text{for all } \lambda > 0.$$

This is possible only if $\frac{n^t g(t)}{t!} = 0$ for $t = 0, 1, 2, \dots$

that is, if $g(t) = 0$ for $t = 0, 1, 2, \dots$. So T is complete.

Now

$$E_\lambda(\bar{X}) = E_\lambda\left[\frac{\sum X_i}{n}\right] = \frac{n E X_1}{n} = E X_1 = \lambda. \quad \text{So } \bar{X} \text{ is a function}$$

of the complete sufficient statistic $T = \sum X_i$, which is an BE of λ .

It follows from the Lehmann-Scheffé Theorem that \bar{X} is the