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STAT 721Assignment #1Solutions

2.2(a) $y = x^2$ is 1-1 from $0 < x < 1$ to $0 < y < 1$.

inverse map is $x = \sqrt{y}$, $0 < y < 1$

$$\begin{aligned} \underline{f_Y(y)} &= f_X(\sqrt{y}) \left| \frac{d\sqrt{y}}{dy} \right|, \quad 0 < y < 1 \\ &= 1 \cdot \left| \frac{1}{2} y^{-1/2} \right| \\ &= \underline{\underline{\frac{1}{2\sqrt{y}}}}, \quad 0 < y < 1 \end{aligned}$$

(b) $y = -\log x$ is a 1-1 map from $(0, 1)$ onto $(0, \infty)$

inverse map is $x = e^{-y}$. also $\left| \frac{d}{dy} e^{-y} \right| = |-e^{-y}| = e^{-y}$,

$$\begin{aligned} \text{so } f_Y(y) &= f_X(e^{-y}) \cdot e^{-y}, \quad 0 < y < \infty \\ &= \frac{(n+m+1)!}{n!m!} e^{-yn} (1-e^{-y})^m e^{-y} \\ &= \frac{(n+m+1)!}{n!m!} e^{-(n+1)y} (1-e^{-y})^m, \quad 0 < y < \infty. \end{aligned}$$

(c) $y = e^x$ is a 1-1 map of $(0, \infty)$ onto $(1, \infty)$.

inverse map is $x = \log y$, and $\left| \frac{d(\log y)}{dy} \right| = \frac{1}{y}$, so

$$f_Y(y) = \frac{1}{e^2} \log y e^{-(\log y/e)^2/2} \cdot \frac{1}{y}, \quad 1 < y < \infty.$$

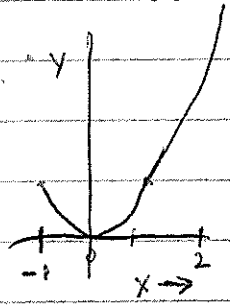
2.3 $Y = \frac{X}{X+1}$ is a 1-1 map of $\{0, 1, 2, 3, \dots\}$ onto $\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\}$

Given that y is a possible value of Y , the corresponding x is obtained by solving $y = \frac{x}{x+1}$ for x , which yields $x = \frac{y}{1-y}$

So $f_Y(y) = P[Y=y] = P[X = \frac{y}{1-y}] = \frac{1}{3} \left(\frac{2}{3}\right)^{\frac{y}{1-y}}$ for $y \in \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\}$

2.7(a) Note that $y = x^2$ is not a 1-1 map of $[-1, 2]$ onto its range $[0, 4]$, so we cannot use the standard

change of variable formula. One way to proceed is to calculate $F_Y(y) = P[Y \leq y] = P[X^2 \leq y]$ for all possible y , then differentiate.



Case 1 $0 \leq y \leq 1$. Then $P[Y \leq y] = P[X^2 \leq y] = P[-\sqrt{y} < X < \sqrt{y}]$

$= \int_{-\sqrt{y}}^{\sqrt{y}} \frac{2}{9} (x+1) dx$. Note $P[Y \leq 1] = \int_{-1}^1 \frac{2}{9} (x+1) dx$

$= \frac{2}{9} \left[\frac{x^2}{2} + x \right]_{-1}^1 = \frac{2}{9} \left(\frac{1}{2} + 1 - \left(\frac{1}{2} - 1 \right) \right) = \frac{4}{9}$

Case 2 $1 \leq y \leq 4$. Then

$P[Y \leq y] = P[Y \leq 1] + P[1 < Y \leq y] = \frac{4}{9} + P[1 < X^2 \leq y]$

$= \frac{4}{9} + P[1 < X \leq \sqrt{y}]$ (continued)

2.7(a) (continued)

$$\begin{aligned} \text{So } F_Y(y) &= F_X(\sqrt{y}) - F_X(-\sqrt{y}) & 0 < y < 1 \\ &= \frac{4}{9} + F_X(\sqrt{y}) - F_X(1) & 1 < y < 4. \end{aligned}$$

$$\begin{aligned} \text{So, when } 0 < y < 1, \quad f_Y(y) &= \frac{d}{dy} [F_X(\sqrt{y}) - F_X(-\sqrt{y})] \\ &= f_X(\sqrt{y}) \cdot \frac{1}{2} y^{-1/2} - f_X(-\sqrt{y}) \cdot (-\frac{1}{2} y^{-1/2}) \\ &= \frac{2}{9} (\sqrt{y} + 1) \cdot \frac{1}{2} \frac{1}{\sqrt{y}} + \frac{2}{9} (-\sqrt{y} + 1) \cdot \frac{1}{2} \frac{1}{\sqrt{y}} \\ &= \frac{2}{9} \frac{1}{\sqrt{y}}. \end{aligned}$$

$$\text{And when } 1 < y < 4, \quad f_Y(y) = f_X(\sqrt{y}) \cdot \frac{1}{2} y^{-1/2} = \frac{2}{9} (\sqrt{y} + 1) \cdot \frac{1}{2} \frac{1}{\sqrt{y}} = \frac{1}{9} \left(1 + \frac{1}{\sqrt{y}}\right)$$

$$\begin{aligned} \text{So } f_Y(y) &= \frac{2}{9} \frac{1}{\sqrt{y}}, \quad 0 < y < 1 \\ &= \frac{1}{9} \left(1 + \frac{1}{\sqrt{y}}\right), \quad 1 < y < 4. \end{aligned}$$

$$\begin{aligned} \text{(2.23) (a) For } 0 < y < 1, \quad P[Y \leq y] &= P[X^2 \leq y] = P[-\sqrt{y} < X < \sqrt{y}] \\ &= F_X(\sqrt{y}) - F_X(-\sqrt{y}), \end{aligned}$$

$$\begin{aligned} \text{so } f_Y(y) &= \frac{d}{dy} [F_X(\sqrt{y}) - F_X(-\sqrt{y})] = f_X(\sqrt{y}) \cdot \frac{1}{2} y^{-1/2} - f_X(-\sqrt{y}) \cdot (-\frac{1}{2} y^{-1/2}) \\ &= \frac{1}{2} y^{-1/2} [f_X(\sqrt{y}) + f_X(-\sqrt{y})] = \frac{1}{2\sqrt{y}} \left[\frac{1}{2} (1 + \sqrt{y}) + \frac{1}{2} (1 - \sqrt{y}) \right] \\ &= \frac{1}{2\sqrt{y}} \end{aligned}$$

$$\text{So } f_Y(y) = \begin{cases} \frac{1}{2\sqrt{y}} & 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases} \quad (\text{continued})$$

2.23 (continued)

$$(b) EY = \int_0^1 y \frac{1}{2\sqrt{y}} dy = \frac{1}{2} \int_0^1 y^{1/2} dy = \frac{1}{2} \left[\frac{2}{3} y^{3/2} \right]_0^1 = \frac{1}{3}$$

$$EY^2 = \int_0^1 y^2 \frac{1}{2} y^{-1/2} dy = \frac{1}{2} \int_0^1 y^{3/2} dy = \frac{1}{2} \left[\frac{2}{5} y^{5/2} \right]_0^1 = \frac{1}{5}$$

$$\therefore \text{Var}(Y) = E(Y^2) - (EY)^2 = \frac{1}{5} - \frac{1}{9} = \frac{4}{45}$$

2.30 (a) $M_x(t) = E e^{tx}$

When $t=0$, $M_x(t) = 1$

When $t \neq 0$,

$$M_x(t) = E e^{tx} = \int_0^c \frac{1}{c} e^{tx} dx = \frac{1}{c} \left[\frac{1}{t} e^{tx} \right]_0^c = \frac{1}{ct} (e^{tc} - 1)$$

(b) When $t \neq 0$,

$$M_x(t) = E e^{tx} = \frac{2}{c^2} \int_0^c x e^{tx} dx = \frac{2}{c^2} \int_0^c x d\left(\frac{1}{t} e^{tx}\right)$$

$$= \frac{2}{c^2} \left\{ \left[x \frac{1}{t} e^{tx} \right]_0^c - \int_0^c \frac{1}{t} e^{tx} dx \right\}$$

$$= \frac{2}{c^2} \left\{ \frac{c}{t} e^{tc} - \frac{1}{t^2} \int_0^c d e^{tx} \right\} = \frac{2}{c^2} \left\{ \frac{c}{t} e^{tc} - \frac{1}{t^2} (e^{tc} - 1) \right\}$$

(c) ~~$E e^{tx} = \frac{1}{2\beta} \left[\int_{-\infty}^{\alpha} e^{tx} e^{-\frac{(x-\alpha)}{\beta}} dx + \int_{\alpha}^{\infty} e^{tx} e^{-\frac{(\alpha-x)}{\beta}} dx \right]$~~

$$E e^{tx} = \frac{1}{2\beta} \int_{-\infty}^{\alpha} e^{tx} e^{-\frac{(x-\alpha)}{\beta}} dx + \int_{\alpha}^{\infty} e^{tx} e^{-\frac{(\alpha-x)}{\beta}} dx$$

Simplify this by making change of variables $y = \frac{\alpha-x}{\beta}$

in the first integral, $z = \frac{x-\alpha}{\beta}$ in the second integral,

combine into a single integral over $(0, \infty)$, then evaluate.

Details and answer omitted. This is a double exponent distribution (see bottom of page 623).

$$\begin{aligned}
 \textcircled{2.30} \quad (d) \quad E e^{tX} &= \sum_{x=0}^{\infty} e^{tx} P[X=x] \\
 &= \sum_{x=0}^{\infty} \binom{r+x-1}{x} p^r [(1-p)e^t]^x \\
 &= \sum_{x=0}^{\infty} \binom{r+x-1}{x} \left\{ 1 - (1-p)e^t \right\}^r \left[(1-p)e^t \right]^x \frac{p^r}{\left[1 - (1-p)e^t \right]^r} \\
 &= \left[\frac{p}{1 - (1-p)e^t} \right]^r \sum_{x=0}^{\infty} \binom{r+x-1}{x} Q^r (1-Q)^x \\
 &\quad (\text{where } Q = 1 - (1-p)e^t)
 \end{aligned}$$

$$= \left[\frac{p}{1 - (1-p)e^t} \right]^r, \quad \text{The series converges (and the calculation makes sense) provided that } (1-p)e^t < 1 \quad \text{or } t < \log(1/(1-p))$$

$$\textcircled{2.32} \quad \left. \frac{d}{dt} S(t) \right|_{t=0} = \left. \frac{d}{dt} \log(M_x(t)) \right|_{t=0} = \left. \frac{M'_x(t)}{M_x(t)} \right|_{t=0} = \frac{M'_x(0)}{M_x(0)} = \frac{EX}{1} = EX$$

$$\left. \frac{d^2}{dt^2} S(t) \right|_{t=0} = \left. \frac{d}{dt} \left[\frac{M'_x(t)}{M_x(t)} \right] \right|_{t=0} = \frac{M_x(0)M_x''(0) - [M'_x(0)]^2}{[M_x(0)]^2} = \frac{1 \cdot EX^2 - (EX)^2}{1^2}$$

$$= EX^2 - (EX)^2 = \text{Var}(X).$$

~~2.32~~

(6)

$$(3.22) (a) E[X(X-1)] = \sum_{x=0}^{\infty} x(x-1) P[X=x]$$

$$= \sum_{x=2}^{\infty} x(x-1) e^{-\lambda} \frac{\lambda^x}{x!} = \sum_{x=2}^{\infty} e^{-\lambda} \frac{\lambda^2 \lambda^{x-2}}{(x-2)!}$$

$$= e^{-\lambda} \lambda^2 \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{-\lambda} \lambda^2 e^{\lambda} = \lambda^2$$

$$\therefore EX^2 - EX = \lambda^2 \quad \text{since } EX = \lambda, \quad E(X^2) - \lambda = \lambda^2$$

$$\therefore EX^2 = \lambda^2 + \lambda \quad \text{and } \text{Var}(X) = E(X^2) - (EX)^2 = \lambda^2 + \lambda - \lambda^2 = \lambda.$$

(b) One way to compute $\text{Var}(X) = E(X^2) - (EX)^2$ is to

$$\text{take } M_x(t) = \left(\frac{\mu}{1 - (1-\mu)e^t} \right)^r \quad (\text{derived in an earlier}$$

$$\text{problem}) \text{ and calculate } E(X^2) = \left. \frac{d^2}{dt^2} M_x(t) \right|_{t=0}.$$

For (c) and (d), use the usual trick of writing the

integrand in $\int x f(x) dx$ and $\int x^2 f(x) dx$ as a constant

times ~~it~~ a gamma (α, β) [beta (α, β)]

density (with different parameter values).

(c) If $X \sim DE(\mu, \sigma)$, then (by defn) $Y = \frac{X-\mu}{\sigma}$ is $DE(0, 1)$

$$\therefore EX = \mu + \sigma EY \quad \text{and} \quad \text{Var}(X) = \sigma^2 \text{Var} Y.$$

$$EY = \int_{-\infty}^0 \frac{1}{2} y e^y dy + \int_0^{\infty} \frac{1}{2} y e^{-y} dy = 0, \quad \text{so } EX = \mu + \sigma \cdot 0 = \mu$$

$$\text{Var}(Y) = EY^2 = \int_{-\infty}^0 \frac{1}{2} y^2 e^y dy + \int_0^{\infty} \frac{1}{2} y^2 e^{-y} dy = 2 \cdot \frac{1}{2} \int_0^{\infty} y^2 e^{-y} dy = -y^2 e^{-y} \Big|_0^{\infty} + \int_0^{\infty} 2y e^{-y} dy$$

3.22 (e) (continued)

$$= 2 \int_0^{\infty} y e^{-y} dy = 2, \text{ so } \text{Var}(X) = (\text{Var } Y) \sigma^2 = 2\sigma^2.$$

3.23

(a) $f(x)$ is ≥ 0 , and $\int_{-\infty}^{\infty} f(x) dx = \int_0^{\infty} \frac{\beta \alpha^\beta}{x^{\beta+1}} dx$

$$= \beta \alpha^\beta \int_{\alpha}^{\infty} x^{-\beta-1} dx = \beta \alpha^\beta \int_{\alpha}^{\infty} d\left[-\frac{1}{\beta} x^{-\beta}\right]$$

$$= \beta \alpha^\beta \left[-\frac{1}{\beta} x^{-\beta}\right]_{\alpha}^{\infty} = \beta \alpha^\beta \left[0 + \frac{1}{\beta} \alpha^{-\beta}\right] = 1, \checkmark$$

(b) $EX = \int_{\alpha}^{\infty} \frac{\beta \alpha^\beta x}{x^{\beta+1}} dx = \int_{\alpha}^{\infty} \frac{\beta \alpha^\beta}{x^{(\beta-1)+1}} dx$

$$= \frac{\beta \alpha}{\beta-1} \int_{\alpha}^{\infty} \frac{(\beta-1) \alpha^{\beta-1}}{x^{(\beta-1)+1}} dx = \frac{\beta \alpha}{\beta-1}$$

Note that the integrand here is a p.d.f., integrating to 1,

if and only if $\beta-1 > 0$ (or $\beta > 1$). If $\beta \leq 1$, then

$EX = \infty$. This is because $\int_{\alpha}^{\infty} \frac{1}{x^v} dx < \infty$ if $v > 1$
 $= \infty$ if $v \leq 1$.

Similarly, $E(X^2) = \frac{\beta \alpha^2}{\beta-2} \int_{\alpha}^{\infty} \frac{(\beta-2) \alpha^{\beta-2}}{x^{(\beta-2)+1}} dx = \frac{\beta \alpha^2}{\beta-2}$ if $\beta > 2$
 $= \infty$ if $\beta \leq 2$.

(continued)

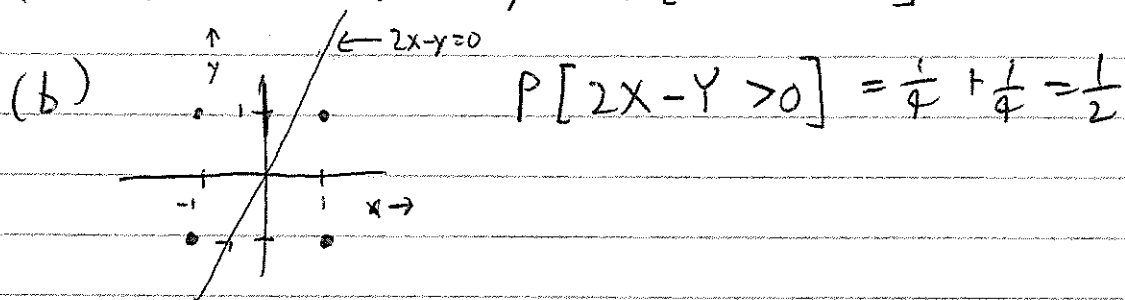
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3.23 (b) (continued)

$$\begin{aligned} \text{So } \text{Var}(X) &= E(X^2) - (EX)^2 = \frac{\beta \alpha^2}{\beta - 2} - \frac{\beta^2 \alpha^2}{(\beta - 1)^2} \quad \text{if } \beta > 2 \\ &= \infty \quad \text{if } 1 < \beta \leq 2 \\ &= \text{undefined} \quad \text{if } \beta \leq 1 \end{aligned}$$

(c) Already shown above in part (b). The authors of the textbook take "variance does not exist" to mean "variance = ∞ " or "variance is undefined."

4.1 (a) $P(X^2 + Y^2 = 2) = 1$, so $P[X^2 + Y^2 < 1] = 0$.



(c)

x	y	x+y
1	1	2
1	-1	0
-1	1	0
-1	-1	2

$P[|X + Y| < 2] = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

4.4 (a) $1 = C \int_{y=0}^1 \int_{x=0}^2 (x+2y) dx dy = C \int_0^1 \left[\frac{x^2}{2} + 2xy \right]_0^2 dy = C \int_0^1 [2+4y] dy$
 $= C [2y + 2y^2]_0^1 = 4C$, so $C = \frac{1}{4}$.

(b) For $0 < x < 2$, $f(x) = \int \frac{1}{4} (x+2y) dy = \frac{1}{4} (xy + y^2) \Big|_0^1 = \frac{1}{4} (x+1)$
 Check: $\int_0^2 \frac{1}{4} (x+1) dx = \frac{1}{4} \left[\frac{x^2}{2} + x \right]_0^2 = \frac{1}{4} (2+2) = 1$. ✓

4.4 (continued)

(c) For $0 \leq x_0 \leq 2$ and $0 \leq y_0 \leq 1$,

$$\begin{aligned}
 F_{x,y}(x_0, y_0) &= \int_{y=0}^{y_0} \int_{x=0}^{x_0} \frac{1}{4} (x+2y) dx dy = \frac{1}{4} \int_0^{y_0} \left[\frac{x^2}{2} + 2xy \right]_0^{x_0} dy \\
 &= \int_0^{y_0} \frac{1}{4} \left[\frac{x_0^2}{2} + 2x_0 y \right] dy = \frac{1}{4} \left[\frac{x_0^2 y}{2} + x_0 y^2 \right]_0^{y_0} \\
 &= \frac{1}{4} \left[\frac{x_0^2 y_0}{2} + x_0 y_0^2 \right] = \frac{1}{8} x_0 y_0 [x_0 + 2y_0].
 \end{aligned}$$

[check: when $x_0=2, y_0=1, F(2,1) = \frac{1}{8} \cdot 2 \cdot 1 \cdot 4 = 1$]

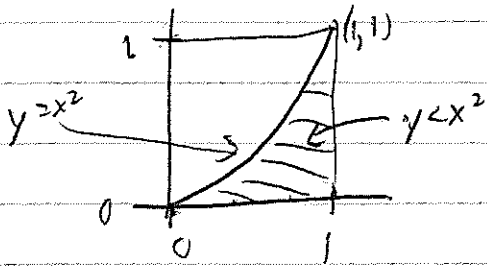
So $F_{x,y}(x_0, y_0) = \frac{1}{8} x_0 y_0 [x_0 + 2y_0]$ $0 \leq x_0 \leq 2, 0 \leq y_0 \leq 1$
 $= 0$ $x_0 \leq 0$ or $y_0 \leq 0$
 $= 1$ $x_0 > 2$ and $y_0 > 1$.

(d) $z = \frac{9}{(x+1)^2}$ is a $1-1$ ^{decreasing} map from $0 < x < 2$ to $1 < z < 9$

inverse map $(x+1)^2 = \frac{9}{z}$, $x+1 = \frac{3}{\sqrt{z}}$, $x = 3z^{-1/2} - 1$

So $f_z(z) = \frac{1}{4} (x+1) \left| \frac{dx}{dz} \right|$
 $= \frac{1}{4} \cdot 3z^{-1/2} \left| -\frac{3}{2} z^{-3/2} \right| = \frac{9}{8} z^{-2} = \frac{9}{8z^2}$, $1 < z < 9$
 $= 0$ otherwise

4.5 (a)



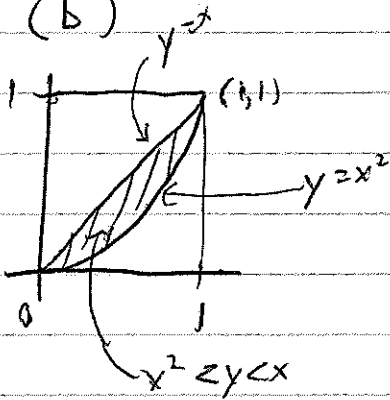
$$P[X > \sqrt{Y}] = P[Y < X^2]$$

$$= \int_{x=0}^1 \left[\int_{y=0}^{x^2} (x+y) dy \right] dx$$

$$= \int_0^1 \left[xy + \frac{y^2}{2} \right]_0^{x^2} dx = \int_0^1 \left(x^3 + \frac{x^4}{2} \right) dx$$

$$= \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{5} = \underline{\underline{\frac{7}{20}}}$$

(b)



$$P[X^2 < Y < X] = \int_{x=0}^1 \int_{y=x^2}^x 2x dy dx$$

$$= \int_{x=0}^1 [2xy]_{x^2}^x dx = \int_0^1 (2x^2 - 2x^3) dx$$

$$= \frac{2}{3} - \frac{2}{4} = \underline{\underline{\frac{1}{6}}}$$

4.20 The mapping $(x_1, x_2) \rightarrow (y_1, y_2)$ takes \mathbb{R}^2 onto $\mathbb{R}^+ \times [-1, 1]$

since $y_2^2 = \frac{x_1^2}{x_1^2 + x_2^2} \leq 1$

The inverse mapping is: $x_1 = y_2 \sqrt{y_1}$

and $x_2^2 = y_1 - x_1^2 = y_1 - y_1 y_2^2 = y_1(1 - y_2^2)$ so $x_2 = \pm \sqrt{y_1} \sqrt{1 - y_2^2}$

This inverse map is not 1-1, so we have to use a method explained in the text (which I did not cover in the lecture),

namely split the domain \mathbb{R}^2 into several disjoint pieces \mathcal{A}_1, \dots (continued)

(11)

such that the mapping $(x_1, x_2) \rightarrow (y_1, y_2)$ is 1-1 from each of the Q_i 's to its range, then combine using equation (4.36) at the bottom of page 161 of the text. In the problem at hand it is clear that the mapping from each of the 9 quadrants of the (x_1, x_2) -plane is 1-1. Consider the first quadrant

$Q_1 = \{(x_1, x_2); x_1 > 0, x_2 > 0\}$. Then $(x_1, x_2) \rightarrow (y_1, y_2)$ is 1-1 from

Q_1 onto $B_1 = \{y_1 > 0, 0 < y_2 < 1\}$, with inverse map $x_1 = y_2 y_1^{1/2}$,

$x_2 = y_1^{1/2} (1 - y_2)^{1/2}$. The Jacobian is

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} y_2 \frac{1}{2} y_1^{-1/2} & y_1^{1/2} \\ \frac{1}{2} y_1^{-1/2} (1 - y_2)^{1/2} & y_1^{1/2} \left(\frac{1}{2} (1 - y_2)^{-1/2} (-2y_2) \right) \end{vmatrix}$$

$$= -\frac{1}{2} y_2^2 (1 - y_2)^{-1/2} - \frac{1}{2} (1 - y_2)^{1/2} = -\frac{1}{2} (1 - y_2)^{-1/2} [y_2^2 + (1 - y_2)]$$

$$= -\frac{1}{2} (1 - y_2)^{-1/2}, \quad \text{so } |J| = \frac{1}{2} (1 - y_2)^{-1/2}.$$

Since $f_{(x_1, x_2)} = \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}(x_1^2 + x_2^2)}$,

the piece of the joint density of (Y_1, Y_2) corresponding to (x_1, x_2) in first quadrant is

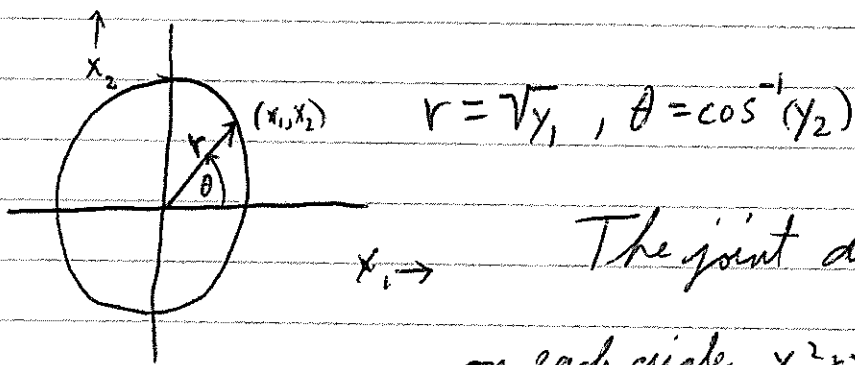
$$\frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2} y_1} |J| = \frac{1}{2\pi\sigma^2} e^{-\frac{y_1}{2\sigma^2}} \cdot \frac{1}{2} (1 - y_2)^{-1/2}$$

Combining similar calculations for the other 3 quadrants yields (via formula 4.3.6), the joint p.d.f. of Y_1 and Y_2 ,

$$f_{Y_1, Y_2}(y_1, y_2) = \frac{1}{\pi \sigma^2} e^{-y_1/(2\sigma^2)} (1-y_2^2)^{-1/2},$$

$y_1 > 0, -1 < y_2 < 1.$

(b) $f_{Y_1, Y_2}(y_1, y_2)$ factors as a (function of Y_1) \times (function of Y_2), hence Y_1 and Y_2 are independent.



The joint density of (X_1, X_2) is constant on each circle $x_1^2 + x_2^2 = r^2$. This makes it

intuitively clear that the radius $R = \sqrt{Y_1}$ should be independent of the angle $\theta = \cos^{-1}(Y_2)$ and hence that (Y_1, Y_2) are independent.

(4.27) The joint p.d.f. of X and Y is

$$f_{X, Y}(x, y) = \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2} [(x-\mu)^2 + (y-\nu)^2]}$$

$-\infty < x < \infty$
 $-\infty < y < \infty$

(continued)

4.27 (continued)

$$\begin{aligned} U &= X+Y \\ V &= X-Y \end{aligned} \quad , \quad \text{so } X = \frac{U+V}{2} \quad , \quad Y = \frac{U-V}{2}$$

so clearly the mapping $(x,y) \rightarrow (u,v)$ is 1-1 from \mathbb{R}^2 onto \mathbb{R}^2 ,

with Jacobian $J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$, so $|J| = \frac{1}{2}$.

So the joint pdf of U and V is

$$\begin{aligned} f_{U,V}(u,v) &= \frac{1}{4\pi\sigma^2} e^{-\frac{1}{2\sigma^2} \left[\left(\frac{u+v}{2} - \mu \right)^2 + \left(\frac{u-v}{2} - \gamma \right)^2 \right]} \quad \begin{matrix} -\infty < u < \infty \\ -\infty < v < \infty \end{matrix} \\ &= \frac{1}{4\pi\sigma^2} e^{-\frac{1}{8\sigma^2} \left[(u+v-2\mu)^2 + (u-v-2\gamma) \right]^2} \end{aligned}$$

Now we do some algebra with the exponent:

$$\begin{aligned} (u+v-2\mu)^2 + (u-v-2\gamma)^2 &= u^2 + 2uv + v^2 - 4\mu(u+v) + 4\mu^2 \\ &\quad + u^2 - 2uv + v^2 - 4\gamma(u-v) + 4\gamma^2 \end{aligned}$$

$$= 2u^2 + 2v^2 - 4u(\mu+\gamma) - 4v(\mu-\gamma) + 4\mu^2 + 4\gamma^2$$

$$= 2 \left\{ [u - (\mu+\gamma)]^2 + [v - (\mu-\gamma)]^2 \right\} \quad \begin{matrix} \text{(note that } (\mu+\gamma)^2 + (\mu-\gamma)^2 \\ = 2(\mu^2 + \gamma^2) \end{matrix}$$

Hence

$$\begin{aligned} f_{U,V}(u,v) &= \frac{1}{2\pi(2\sigma^2)} e^{-\frac{1}{2(2\sigma^2)} \left\{ [u - (\mu+\gamma)]^2 + [v - (\mu-\gamma)]^2 \right\}} \\ &= \frac{1}{\sqrt{2\pi(2\sigma^2)}} e^{-\frac{1}{2(2\sigma^2)} [u - (\mu+\gamma)]^2} \cdot \frac{1}{\sqrt{2\pi(2\sigma^2)}} e^{-\frac{1}{2(2\sigma^2)} [v - (\mu-\gamma)]^2} \end{aligned}$$

This factors into (a fn. of u) \times (a fn. of v). Hence U and V are indep.

Furthermore, we observe that $U \sim N(\mu+\gamma, 2\sigma^2)$, $V \sim N(\mu-\gamma, 2\sigma^2)$.

$$(4.30)(a) \quad Y|X=x \sim N(x, x^2), \text{ so } E[Y|X=x] = x, \text{ so } E[Y|X] = X$$

$$\text{so } E(Y) = E_x E[Y|X] = EX = \int_0^1 x dx = \underline{\underline{\frac{1}{2}}},$$

$$\text{Also } E[Y^2|X=x] = \mu_x^2 + \sigma_x^2 = x^2 + x^2 = 2x^2,$$

$$\text{so } E[Y^2|X] = 2X^2, \text{ so } E(Y^2) = E[E[Y^2|X]] = 2E(X^2) = 2 \int_0^1 x^2 dx = \underline{\underline{\frac{2}{3}}}$$

$$\text{Hence } \text{var}(Y) = E[Y^2] - (EY)^2 = \frac{2}{3} - \left(\frac{1}{2}\right)^2 = \underline{\underline{\frac{5}{12}}}.$$

$$E[XY|X=x] = x E[Y|X=x] = x \cdot x = x^2, \text{ so } E[XY|X] = X^2$$

$$\text{so } E[XY] = E[E[XY|X]] = EX^2 = \frac{1}{3},$$

$$\text{so } \text{Cov}(X, Y) = \underline{\underline{E[XY] - (EX)(EY)}} = \frac{1}{3} - \frac{1}{2} \cdot \frac{1}{2} = \underline{\underline{\frac{1}{12}}}$$

(b) The joint p.d.f of X and Y is

$$f_{X,Y}(x,y) = f(x)f(y|x) = 1 \cdot \frac{1}{\sqrt{2\pi x^2}} e^{-\frac{1}{2x^2}(y-x)^2} \quad \begin{matrix} 0 < x < 1 \\ -\infty < y < \infty \end{matrix}$$

~~Define~~ Define $Q = Y/X$ and $R = X$.

$$q = \frac{y}{x}, r = x \Rightarrow x = r, y = qr, \text{ so the change of}$$

variables $(x,y) \rightarrow (q,r)$ is 1-1 from $(0,1) \times \mathbb{R} \rightarrow \mathbb{R} \times (0,1)$

$$\text{The Jacobian is } J = \begin{vmatrix} \frac{\partial x}{\partial q} & \frac{\partial x}{\partial r} \\ \frac{\partial y}{\partial q} & \frac{\partial y}{\partial r} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ r & q \end{vmatrix} = -r, \text{ so } |J| = r.$$

(continued)

4.30 (b) (continued)

So the joint pdf of Q and R is

$$f_{Q,R}(q,r) = \frac{1}{\sqrt{2\pi}r^2} e^{-\frac{1}{2r^2}(qr-r)^2}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(q-1)^2}, \quad \begin{matrix} 0 < r < 1 \\ -\infty < q < \infty \end{matrix}$$

~~Since the marginal pdf of R is known to be $f_R(r) = 1$, $0 < r < 1$, \therefore~~

The marginal pdf of Q is

$$f_Q(q) = \int_0^1 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(q-1)^2} dr = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(q-1)^2}, \quad -\infty < q < \infty$$

Since the marginal pdf of R is known to be $f_R(r) = 1$, $0 < r < 1$,

we see that

$$f_{Q,R}(q,r) = f_Q(q) f_R(r) \quad \begin{matrix} \text{for all } r, 0 < r < 1 \\ \text{and all } q, -\infty < q < \infty \end{matrix}$$

\Rightarrow Hence $Q = \frac{Y}{X}$ and $R = X$ are independent.

(Furthermore $Y \sim N(1, 1)$).