STAT 721

Solutions to Assignment # 4

unknowns

(7.19) Y_i's are indep, $Y_i \sim N(\beta x_i, \sigma^2)$ $i=1,\dots,n$

fixed

(a)

$$\prod_{i=1}^{n} f(y_i \mid \beta, \sigma^2) = \prod_{i=1}^{n}\left[\frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{1}{2\sigma^2}(y_i-\beta x_i)^2}\right]$$

$$= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \exp\left[-\frac{1}{2\sigma^2}\sum (y_i-\beta x_i)^2\right]$$

$$\cancel{= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \exp\left[-\frac{1}{2\sigma^2}\sum y_i^2+\frac{\beta}{\sigma^2}\sum x_i y_i\right] \exp\left[\dots\right]}$$

$$= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \exp\left[-\frac{1}{2\sigma^2}\sum y_i^2+\frac{\beta}{\sigma^2}\sum x_i y_i - \frac{\beta^2}{2\sigma^2}\sum x_i^2\right]$$

$$= \underbrace{\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{-\frac{\beta^2}{2\sigma^2}\sum x_i^2}}_{g(\theta,\sigma^2)} \underbrace{\exp\left[-\frac{1}{2\sigma^2}\sum y_i^2+\frac{\beta}{2\sigma^2}\sum x_i y_i\right]}_{h[(\theta,\sigma^2),(\sum y_i^2,\ \sum x_i y_i)]}$$

(since $x_1,\dots,x_n$ are constants)

By the Factorization Thm, $\left(\sum_{i=1}^{n} x_i Y_i,\ \sum_{i=1}^{n} Y_i^2\right)$ is sufficient for (θ,σ^2)

(b) $\log L(\theta,\sigma^2) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\sum (y_i-\beta x_i)^2$

The maximizing values (assuming they are not on the boundary of the parameter space) must necessarily satisfy

$$\frac{\partial}{\partial\theta}\log L(\theta,\sigma^2)=0 \quad \text{and} \quad \frac{\partial}{\partial\sigma^2}\log L(\theta,\sigma^2)=0$$

(continued)

7.19 (b) (continued)

or $\quad -\frac{1}{2\sigma^2}\sum 2(y_i-\beta x_i)(-x_i) = 0 \quad (\Rightarrow \sum x_i y_i = \beta \sum x_i^2)$

and $\quad -\frac{n}{2\sigma^2} + \frac{1}{2(\sigma^2)^2}\sum (y_i-\beta x_i)^2 = 0 \quad (\Rightarrow \sigma^2 = \frac{\sum (y_i-\beta x_i)^2}{n})$.

The solutions are $\boxed{\hat\beta = \frac{\sum_{i=1}^n x_i Y_i}{\sum_{i=1}^n x_i^2}}$, $\boxed{\widehat{\sigma^2} = \frac{\sum (y_i-\hat\beta x_i)^2}{n}}$

These are the MLE's of β and σ^2.

To show that $\hat\beta$ is unbiased, calculate

$$E\hat\beta = E\left[\frac{\sum x_i Y_i}{\sum x_i^2}\right] = \frac{1}{\sum x_i^2}\sum x_i EY_i =$$

$$= \frac{1}{\sum x_i^2}\sum x_i(\beta x_i) = \frac{1}{\sum x_i^2}\beta\sum x_i^2 = \beta \ .. \checkmark$$

[Note: for the answers to (a) and (b) to make sense, we must assume that not all the x_i's are $=0$.]

(c) Since $\hat\beta$ is a linear combination of indep. normal random variables, it must have a normal distribution (e.g. use mgf's to see this).

Its mean is $E(\hat\beta)=\beta$, and its variance is

$$\operatorname{var}(\hat\beta) = \frac{1}{(\sum x_i)^2}\operatorname{Var}(\sum x_i Y_i) \underset{\uparrow}{=} \frac{1}{(\sum x_i^2)^2}\sum x_i^2 \operatorname{Var} Y_i = \frac{1}{(\sum x_i^2)^2}\sum x_i^2\sigma^2$$

by independence

$$= \frac{\sigma^2 \sum x_i^2}{(\sum x_i^2)^2} = \frac{\sigma^2}{\sum x_i^2}$$

So $\hat\beta \sim N(\beta, \sigma^2/(\sum_i^n x_i^2))$.

(7.22) $f(\bar{x},\theta) = f(\theta)\, f(\bar{x}|\theta)$

Since each $x_i|\theta \sim N(\theta,\sigma^2)$, $\bar{x}|\theta \sim N(\theta, \frac{\sigma^2}{n})$.

(a) So $f(\bar{x}|\theta) = \frac{1}{\sqrt{2\pi\tau^2}}\, e^{-\frac{1}{2\tau^2}[\theta-\mu]^2}\, \frac{1}{\sqrt{2\pi\sigma^2/n}}\, e^{-\frac{1}{2\sigma^2/n}(\bar{x}-\theta)^2}$,

$-\infty<\theta<\infty, -\infty<\bar{x}<\infty.$

(b)

$$m(\bar{x}|\sigma^2,\mu,\tau^2) = \int_{-\infty}^{\infty} f(\bar{x},\theta)\, d\theta$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi\tau^2\frac{\sigma^2}{n}} \exp\left\{-\frac{1}{2\tau^2\sigma^2/n}\left[\frac{\sigma^2}{n}(\theta-\mu)^2+\tau^2(\bar{x}-\theta)^2\right]\right\} d\theta$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi\tau^2\sigma^2/n} \exp\left\{-\frac{1}{2\tau^2\sigma^2/n}\left[\frac{\sigma^2}{n}(\theta^2-2\mu\theta+\mu^2)+\tau^2(\bar{x}^2-2\theta\bar{x}+\theta^2)\right]\right\} d\theta$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi\tau^2\sigma^2/n} \exp\left\{-\frac{1}{2\tau^2\sigma^2/n}\left[\left(\frac{\sigma^2}{n}+\tau^2\right)\theta^2-2\theta\left(\frac{\sigma^2}{n}\mu+\tau^2\bar{x}\right)+\frac{\sigma^2}{n}\mu^2+\tau^2\bar{x}^2\right]\right\} d\theta$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\frac{2\pi\tau^2\sigma^2/n}{\tau^2+\sigma^2/n}}\sqrt{2\pi(\tau^2+\sigma^2/n)}} \exp\left\{-\frac{\frac{\sigma^2}{n}+\tau^2}{\tau^2\sigma^2/n}\left[\theta^2-2\theta\,\frac{\frac{\sigma^2}{n}\mu+\tau^2\bar{x}}{\frac{\sigma^2}{n}+\tau^2}+\left[\frac{\frac{\sigma^2}{n}\mu+\tau^2\bar{x}}{\frac{\sigma^2}{n}+\tau^2}\right]^2\right],\right.$$

$$\left.\times \exp\left\{+\frac{1}{2}\left(\frac{\frac{\sigma^2}{n}\mu+\tau^2\bar{x}}{\frac{\sigma^2}{n}+\tau^2}\right)^2\frac{\frac{\sigma^2}{n}+\tau^2}{\tau^2\sigma^2/n}-\frac{1}{2\tau^2\sigma^2/n}\left(\frac{\sigma^2}{n}\mu^2+\tau^2\bar{x}^2\right)\right\} d\right.$$

$$= \frac{1}{\sqrt{2\pi(\tau^2+\sigma^2/n)}} \exp\left\{-\frac{1}{2(\frac{\sigma^2}{n}+\tau^2)}\left[\frac{\left(\frac{\sigma^2}{n}\mu^2+\tau^2\bar{x}^2\right)\left(\tau^2+\frac{\sigma^2}{n}\right)-\left(\frac{\sigma^2}{n}+\tau^2\bar{x}\right)^2}{\tau^2\sigma^2/n}\right]\right\}$$

(Here we used the fact that $\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi B}}\, e^{-\frac{2}{B}(\theta-A)^2}\, d\theta = 1$ for all A and all $B>0$ – which was the motivation for all the above algebra to "complete the squa

(continued)

$$= \frac{1}{\sqrt{2\pi\,(\tau^2+\sigma^2/n)}}\exp\left\{-\frac{1}{2(\frac{\sigma^2}{n}+\tau^2)}\;\frac{\tau^2\frac{\sigma^2}{n}\mu^2+\tau^4\bar{x}^2+\left(\frac{\sigma^2}{n}\right)^2\mu^2+\frac{\sigma^2}{n}\tau^2\bar{x}^2-\left(\frac{\sigma^2}{n}\right)^2\mu^2-2\frac{\sigma^2}{n}\mu\tau^2\sigma^2-\tau^4\bar{x}}{\tau^2\sigma^2/n}\right.$$

$$= \frac{1}{\sqrt{2\pi\,(\tau^2+\sigma^2/n)}}\exp\left\{-\frac{1}{2(\frac{\sigma^2}{n}+\tau^2)}\;\frac{\tau^2\sigma^2/n\;\bar{x}^2-(\tau^2\sigma^2/n)2\mu\bar{x}+(\tau^2\sigma^2/n)\mu^2}{\tau^2\sigma^2/n}\right\}$$

$$= \frac{1}{\sqrt{2\pi\,(\tau^2+\sigma^2/n)}}\exp\left\{-\frac{1}{2(\tau^2+\frac{\sigma^2}{n})}(\bar{x}-\mu)^2\right\}$$

$$= \text{pdf of } N\left(\mu, \frac{\sigma^2}{n}+\tau^2\right).$$

(c) $\pi(\theta/\bar{x}) = f(\bar{x},\theta)\,/\,m(\bar{x})$.

So take a look at the joint pdf $f(\bar{x},\theta)$, as derived in the solution of part (b) (right before applying the operation $\int_{-\infty}^{\infty} - d\theta$), divide it by the $N(\mu, \frac{\sigma^2}{n}+\tau^2)$ pdf of $\bar{x}$, and see that the quotient is the p.d.f. of a $N(A,B)$ r.v., with $A = \dfrac{\frac{\sigma^2}{n}\mu+\tau^2\bar{x}}{\frac{\sigma^2}{n}+\tau^2} =$

$$= \frac{\tau^2}{\tau^2+\frac{\sigma^2}{n}}\bar{x} + \frac{\sigma^2/n}{\tau^2+\sigma^2/n}\mu \quad\text{and}\quad B = \frac{\sigma^2\tau^2/n}{\frac{\sigma^2}{n}+\tau^2}.$$

7.24 $X_1, \dots, X_n \sim \text{Poisson}(\lambda)$, $\lambda \sim \text{gamma}(\alpha, \beta)$

$T = \sum_{i=1}^{n} X_i \sim \text{Poisson}(n\lambda)$, i.e. $P[T=t] = e^{-n\lambda}\frac{(n\lambda)^t}{t!}$, $t = 0,1,2\dots$

The pdf of λ is $f(\lambda|\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}}\lambda^{\alpha-1}e^{-\lambda/\beta}$, $\lambda > 0$.

(a) The posterior pdf of λ given $T = \sum_{i=1}^{n} X_i = t$, $(t = 0,1,2,\dots)$

is $g(\lambda|t) = \text{const.} \times g(\lambda)\, g(t|\lambda)$

$$= \text{const.} \times \lambda^{\alpha-1} e^{-\lambda/\beta}\, e^{-n(\lambda)} \frac{n^t \lambda^t}{t!}$$

$$= \text{const.} \times \lambda^{t+\alpha-1} e^{-(n+1/\beta)\lambda} = e^{-\lambda/[\beta/(n\beta+1)]}, \quad \lambda > 0$$

So, without working out the constant, we see that the posterior pdf of λ, given $\sum_{i=1}^{n} X_i = t$, is gamma $(t+\alpha, \beta/[n\beta+1])$.

Or, substituting into the general gamma pdf on page 624, we have

$$f(\lambda|\textstyle\sum X_i = t) = \frac{1}{\Gamma(t+\alpha)\left(\frac{\beta}{n\beta+1}\right)^{t+\alpha}} \lambda^{n+\alpha-1} e^{-\lambda[\beta/(n\beta+1)]}, \quad \lambda > 0.$$

(b) Using the formulas for the mean and variance of a gamma distribution (given on page 624), we have:

posterior mean, given $\sum X_i = t$, is $(t+\alpha)\left(\frac{\beta}{n\beta+1}\right)$

& post. variance, given $\sum X_i = t$, is $(t+\alpha)\left(\frac{\beta}{n\beta+1}\right)^2$.

[You can also work this problem, getting equivalent answers, by conditioning on $\bar{X}$ $(= \sum X_i/n)$ instead of $\sum X_i$.]

7.33 As derived in Example 7.3.5, the MSE of the Bayes estimator is $\dfrac{np(1-p)}{(\alpha+\beta+n)^2} + \left(\dfrac{np+\alpha}{\alpha+\beta+n} - p\right)^2$.

We are asked to show that the MSE is a constant when $\alpha = \beta = \sqrt{n/4} = \frac{\sqrt{n}}{2}$. So we plug in these values and crank away with the algebra:

$$MSE = \frac{np(1-p)}{(\sqrt{n}+n)^2} + \left(\frac{np+\frac{\sqrt{n}}{2}}{\sqrt{n}+n} - p\right)^2$$

$$= \frac{1}{(\sqrt{n}+n)^2}\left[np - np^2 + \left(np + \frac{\sqrt{n}}{2} - np - \sqrt{n}\,p\right)^2\right]$$

$$= \frac{1}{(\sqrt{n}+n)^2}\left[np - np^2 + \left[\sqrt{n}\left(\tfrac{1}{2} - p\right)\right]^2\right]$$

$$= \frac{1}{(\sqrt{n}+n)^2}\left[np - np^2 + n\left(\tfrac{1}{4} - p + p^2\right)\right]$$

$$= \frac{1}{(\sqrt{n}+n)^2}\left[np - np^2 + \frac{n}{4} - np + np^2\right]$$

$$= \frac{n}{4(\sqrt{n}+n)^2} = \frac{1}{4(1+\sqrt{n})^2} = \text{a constant}$$

(i.e., does not depend on p, $0 < p < 1$.

7.49 (a) First, work out the distribution of Y.

For $y > 0$, $P[Y \geq y] = \prod_{i=1}^{n}\left[P[X_i \geq y]\right] = \prod_{i=1}^{n}\left[1 - (1 - e^{-y/\lambda})\right]$

$= e^{-(y/\lambda)n} = e^{-y/(\lambda/n)}$. So the cdf of Y is $1 - e^{-y/(\lambda/n)}$, y;

Hence Y has an exponential distr. with mean λ/n,

(continued)

7.49 (a) (continued)

Since $E(Y) = \frac{\lambda}{n}$, we have $E[nY] = n\frac{\lambda}{n} = \lambda$.

So $nY = n \min\{X_1, \ldots, X_n\}$ is an unbiased est of λ based only on Y.

(b) $\prod_{i=1}^{n} f(x_i; \lambda) = \prod\left[\frac{1}{\lambda} e^{-x_i/\lambda}\right] = \frac{1}{\lambda^n} e^{-\sum x_i/\lambda}$, so $\sum X_i$ is a sufficient statistic for λ. Also $\sum X_i$ is complete (exponential family). Hence the best UE of λ will be a function of $\sum X_i$. Since $\bar{X} = \frac{\sum X_i}{n}$ is a function of $\sum X_i$ and since $E\bar{X} = \lambda$, we conclude that $\bar{X}$ is the unique best (i.e. minimum variance) unbiased estimator of λ, by the Lehmann-Scheffe Theorem. It is strictly better than nY since $nY \not\equiv \bar{X}$. (Note that no calculation of variances is necessary to draw this conclusion).

A more mundane way to show that $\bar{X}$ beats nY, without using sufficiency and completeness, etc., is simply to calculate $Var(nY) = n^2 Var Y = n^2\left(\frac{\lambda}{n}\right)^2 = \lambda^2$ and note that this is larger than $var \bar{X} = \frac{Var X_1}{n} = \frac{\lambda^2}{n}$,

(continued)

7.49 (c) $n = 12$; $Y = \min\{X_1, \dots, X_{12}\} = 50.1$, so $nY = 12(50.1)$
$= \underline{601.2}$ hours

(This is way out of range of the bulk of the data, but this is what one would expect from an estimator whose variance is 12 times that of $\bar{X}$.

$$\bar{X} = \frac{\sum x_i}{12} = \frac{1497.9}{12} = \underline{124.825 \text{ hours}}$$ (a more reasonable estimate).

7.62 (a) $R(\theta, a\bar{X}+b) = E\left[(a\bar{X}+b-\theta)^2\right]$

$$= E\left\{\left[a\bar{X} - a\theta + b - (1-a)\theta\right]\right\}^2$$

$$= E\left\{\left[a(\bar{X}-\theta) + \left[b-(1-a)\theta\right]\right]^2\right\}$$

$$= E\left\{a^2(\bar{X}-\theta)^2 + 2a\left[b-(1-a)\theta\right](\bar{X}-\theta) + \left[b-(1-a)\theta\right]^2\right\}$$

$$= a^2 \operatorname{Var}\bar{X} + 2ab\left[b-(1-a)\theta\right](E\bar{X}-\theta) + \left[b-(1-a)\theta\right]^2$$

$$= a^2\frac{\sigma^2}{n} + \left[b-(1-a)\theta\right]^2.$$

(b) We saw earlier (in problem 7.22) that the Bayes estimator of θ is $a\bar{X}+b$ where $a = \frac{\tau^2}{\tau^2+\frac{\sigma^2}{n}} = \frac{n\tau^2}{n\tau^2+\sigma^2} = 1-\eta$

and $b = \frac{(\sigma^2/n)\mu}{\tau^2+\sigma^2/n} = \frac{\sigma^2\mu}{n\tau^2+\sigma^2} = \eta\mu$. So plug in $a = 1-\eta$

and $b = \eta\mu$ into the result of (a) to obtain that the risk function is

$$(1-\eta)^2\frac{\sigma^2}{n} + \left[\eta\mu - [1-(1-\eta)]\theta\right]^2 = (1-\eta)^2\frac{\sigma^2}{n} + \eta^2(\theta-\mu)^2.$$

(continued)

7.62 (c) The Bayes risk is $E_\pi R(\theta, \delta^\pi)$, where π is $N(\mu, \tau^2)$.

$$E_\pi \left\{ (1-\eta)^2 \frac{\sigma^2}{n} + \eta^2 (\theta-\mu)^2 \right\} = (1-\eta)^2 \frac{\sigma^2}{n} + \eta^2 E_\pi (\theta-\mu)^2$$

$$= (1-\eta)^2 \frac{\sigma^2}{n} + \eta^2 \tau^2$$

$$= \left(\frac{n\tau^2}{n\tau^2+\sigma^2} \right)^2 \frac{\sigma^2}{n} + \left(\frac{\sigma^2}{n\tau^2+\sigma^2} \right) \tau^2$$

$$= \frac{1}{(n\tau^2+\sigma^2)^2} \left[\frac{n^2\tau^4\sigma^2}{n} + \sigma^4\tau^2 \right]$$

$$= \frac{1}{(n\tau^2+\sigma^2)^2} \sigma^2\tau^2 \left[n\tau^2 + \sigma^2 \right]$$

$$= \frac{\sigma^2\tau^2}{n\tau^2+\sigma^2} = \tau^2 \eta.$$