STAT 721

Here is the second STAT 721 midterm, when I taught the course 2 years ago, along with solutions.

Note: the exam two years ago asked for a proof of completeness in problem 3 that turned out to ~~be~~ require a long and difficult demonstration. I will try to avoid setting a disastrously difficult problem on this year's Exam #2.

– J. Collins

STAT 721 —Midterm Exam No. 2 — December 2, 2005

NAME Solutions

1. Let $X_1, \ldots, X_n$ be a random sample from a Poisson(λ) distribution, where λ is unknown ($\lambda > 0$).

(a) Find the MLE of $\lambda^2 e^{-\lambda}$.

$$L(\lambda) = \prod_{i=1}^{n} \left[e^{-\lambda} \frac{\lambda^{x_i}}{x_i!} \right] = e^{-n\lambda} \cdot \frac{\lambda^{\Sigma x_i}}{\prod x_i!}$$

$$\log L(\lambda) = -n\lambda + (\Sigma x_i) \log \lambda - \log(\prod x_i!)$$

$\frac{d}{d\lambda} \log L(\lambda) = -n + \frac{\Sigma x_i}{\lambda} = 0$ has unique sol'n $\hat{\lambda} = \frac{\Sigma x_i}{n} = \bar{x}$.

Since $\frac{d^2}{d\lambda^2} \log L(\lambda) = -\Sigma x_i / \lambda^2 < 0$, the unique extremum must be a global max.

Since $\bar{X}$ is MLE of λ, by an invariance theorem for MLE's, we have that $\bar{X}^2 e^{-\bar{X}}$ is the MLE of $\lambda^2 e^{-\lambda}$.

(b) Find the best (i.e., uniformly minimum variance) unbiased estimator of $\lambda^2 e^{-\lambda}$.

$T = \sum_{i=1}^{n} X_i$ is a complete, suff stat for λ by the Factorization theorem and the one-dim. exponential family representation.

There are two possible solutions, both using the Lehmann-Scheffe Theorem:

(1) An unbiased est of $\lambda^2 e^{-\lambda}$ is $2 I_{\{X_1 = 2\}}$, since $E\, I_{\{X_1=2\}} = P[X_1 = 2] = \frac{\lambda^2}{2} e^{-\lambda}$.

For $t \geq 2$, $E\left[2 I_{\{X_1 = 2\}} \mid T = t\right] = 2\, P[X_1 = 2 \mid T = t] = 2\, P[X_1 = 2]\, P\left[\sum_{i=2}^{n} X_i = t-2\right]$

$$= \left[2 e^{-\lambda} \frac{\lambda^2}{2}\, e^{-(n-1)\lambda} [(n-1)\lambda]^{t-2} / (t-2)!\right] \Big/ \left[e^{-n\lambda} \frac{(n\lambda)^t}{t!}\right]$$

$$= \frac{(n-1)^{t-2}}{(t-2)!} \frac{t!}{n^t} = \frac{t(t-1)(n-1)^{t-2}}{n^t}$$

So the UMVU est is $\frac{T(T-1)(n-1)^{T}}{n^T}$ where $T = \sum_{i=1}^{n} X_i$.

1 (Continued)

2 (Another solution)

The UMVU est. must be a function of the complete sufficient statistic $T = \sum_{i=1}^{n} X_i$, so we try to solve

$$E_\lambda[\delta(T)] = \lambda^2 e^{-\lambda}.$$

or $$\sum_{t=0}^{\infty} \delta(t)\, e^{-n\lambda} \frac{(n\lambda)^t}{t!} = \lambda^2 e^{-\lambda}$$

or $$\sum_{t=0}^{\infty} \delta(t) \frac{n^t \lambda^t}{t!} = \lambda^2 e^{-(n-1)\lambda} = \lambda^2 \sum_{t=0}^{\infty} \frac{(n-1)^t \lambda^t}{t!}$$
$$= \sum_{t=0}^{\infty} \frac{(n-1)^t \lambda^{t+2}}{t!} = \sum_{t=2}^{\infty} \frac{(n-1)^{t-2} \lambda^t}{(t-2)!}.$$

In order for these two power series to agree for all $\lambda > 0$, we must have $\delta(0) = 0$; $\delta(1) = 0$; and for $t \geq 2$,

$$\delta(t) \frac{n^t}{t!} = \frac{(n-1)^{t-2}}{(t-2)!}, \text{ so } \delta(t) = \frac{t(t-1)(n-1)^{t-2}}{n^t}.$$

So $\delta(T) = \dfrac{T(T-1)(n-1)^{T-2}}{n^T}$ is the UMVU est of $\lambda^2 e^{-\lambda}$.

③

2. Let $X_1, \ldots, X_n$ be a random sample from a Bernoulli(p) distribution with unknown p, $0 < p < 1$.

(a) Compute the Cramér-Rao lower bound on the variance of an unbiased estimator of p^3.

$$f(x|p) = p^x (1-p)^{1-x}, \quad x = 0, 1$$

$$\log f(x|p) = x \log p + (1-x) \log (1-p),$$

$$\frac{\partial}{\partial p} \log f(x|p) = \frac{x}{p} - \frac{(1-x)}{1-p} = \frac{x-p}{p(1-p)}$$

$$I(p) = E_p\left\{\left[\frac{\partial}{\partial p} \log f(x|p)\right]^2\right\} = E_p\left[\frac{(X-p)^2}{p^2(1-p)^2}\right] = \frac{p(1-p)}{p^2(1-p)^2} = \frac{1}{p(1-p)}$$

$\tau(p) = p^3$, $\tau'(p) = 3p^2$, so the C-R lower bound on the var. of an UE of p^3 is $\frac{[\tau'(p)]^2}{n\, I(p)} = \frac{(3p^2)^2}{n\,(1/(p(1-p)))} = \frac{9p^5(1-p)}{n}$.

(b) Find the best (i.e., uniformly minimum variance) unbiased estimator of p^3.

Let $\delta = \begin{cases} 1 & \text{if } X_1 = X_2 = X_3 = 1 \\ 0 & \text{otherwise.} \end{cases}$

Then $E_p[\delta(X_1, \ldots, X_n)] = P_p[X_1 = X_2 = X_3 = 1] = p^3$, so δ is an UE of p^3.

By the usual exponential family representation, $T = \sum_{i=1}^n X_i$ is complete suff. stat for p.

So the UMVU est of p^3 is $E[\delta \mid T = t]$.

Fix $t \geq 3$. Then $E_p[\delta \mid T = t] = \dfrac{P[X_1 = 1, X_2 = 1, X_3 = 1]\, P[\sum_{i=4}^n X_i = t-3]}{P[\sum_{i=1}^n X_i = t]}$

$$= \frac{p^3 \binom{n-3}{t-3} p^{t-3} (1-p)^{(n-3)-(t-3)}}{\binom{n}{t} p^t (1-p)^{n-t}} = \frac{\binom{n-3}{t-3}}{\binom{n}{t}} = \frac{t(t-1)(t-2)}{n(n-1)(n-2)}$$

So $\dfrac{T(T-1)(T-2)}{n(n-1)(n-2)}$ is the UMVU est of p^3, where $T = \sum_{i=1}^n X_i$.

3. Let $X_1, \ldots, X_n$ be a random sample from the distribution with pdf

$$f(x|\beta,\theta) = \begin{cases} \frac{1}{\beta}e^{-(x-\theta)/\beta}, & \text{if } \theta \le x < \infty, \\ 0 & \text{elsewhere,} \end{cases}$$

where β ($\beta > 0$) and θ ($-\infty < \theta < \infty$) are both unknown parameters.

(a) Find a complete sufficient statistic for (θ, β).

$$\prod_{i=1}^{n} f(x_i|\beta,\theta) = \frac{1}{\beta^n} e^{-\sum_{i=1}^{n}(x_i-\theta)/\beta} I_{\{X_{(1)} \ge \theta\}},$$

so by the Factorization Theorem, $(X_{(1)}, \sum_{i=1}^{n} X_i)$ is sufficient for (θ,β). Another equivalent statistic which is also sufficient (by way of a 1-1 mapping) is $(X_{(1)}, \sum_{i=1}^{n}(X_i - X_{(1)}))$.

See appendix A.1 to A.4 for proof of completeness of this sufficient statistic. (I didn't expect anyone to get it on the test! It took me hours to do it).

(b) Find the best (i.e., uniformly minimum variance) unbiased estimator of β.

We will find (by inspection) a function δ of the complete suff. stat which satisfy $E_{\theta,\beta}(\delta) = \beta$ for all $\theta \in \mathbb{R}$ and β.

Since $X_{(1)} - \theta$ is exponential, mean β/n, we have: $E(X_{(1)}) = \theta + \beta/n$.

Since $\sum_{i=1}^{n}(X_i - \theta)$ is exponential, mean $n\beta$, $E(\sum_{i=1}^{n} X_i) = n\theta + n\beta$,

or $E(\bar{X}) = \theta + \beta$. So $E[\bar{X} - X_{(1)}] = \theta + \beta - (\theta + \frac{\beta}{n}) = \beta\left(\frac{n-1}{n}\right)$.

So $E\left\{\frac{n}{n-1}[\bar{X} - X_{(1)}]\right\} = \beta$.

Since $\frac{n}{n-1}(\bar{X} - X_{(1)})$ is a function of the complete suff. stat $(X_{(1)}, \sum^{n} X_i)$ which is an unbiased est of β, it follows from Lehmann–Scheffe Thm that it is UMVU est of β.

APPENDIX

Proof of "completeness" in problem 3:

WLOG, we can replace the sufficient statistic $\left(X_{(1)}, \sum_{i=1}^{n} X_i\right)$ with the equivalent sufficient statistic $\left(X_{(1)}, \sum_{i=1}^{n}(X_i - X_{(1)})\right)$.

First note that the joint pdf of the order statistics $(X_{(1)}, X_{(2)}, \ldots, X_{(n)})$ is

$$f(x_{(1)}, x_{(2)}, \ldots, x_{(n)}) = \frac{1}{n!}\frac{1}{\beta^n} e^{-\sum_1^n (x_{(i)}-\theta)/\beta} \quad \text{if } \theta < x_{(1)} < x_{(2)} < \cdots < x_{(n)}$$
$$= 0 \quad \text{otherwise}.$$

Also it is easy to see that the marginal pdf of $X_{(1)}$ is:

$$f(x_{(1)}) = \frac{n}{\beta} e^{-n(x_{(1)}-\theta)/\beta} \quad \text{if } x_{(1)} > \theta$$
$$= 0 \quad \text{otherwise}.$$

Hence the conditional density of $(X_{(2)}, X_{(3)}, \ldots, X_{(n)})$ given that $X_{(1)} = x_{(1)}$ $(>\theta)$ is

$$f(x_{(2)}, \ldots, x_{(n)} \mid X_{(1)} = x_{(1)}) = \frac{f(x_{(1)}, x_{(2)}, \ldots, x_{(n)})}{f(x_{(1)})}$$

$$\underset{\text{after some algebra}}{=} \begin{cases} \dfrac{1}{(n-1)!}\dfrac{1}{\beta^{n-1}} e^{-\sum_{i=2}^{n}(x_{(i)}-x_{(1)})} & \text{if } \theta < x_{(1)} < x_{(2)} < \cdots < x_{(n)} \\ 0 & \text{otherwise} \end{cases}$$

Hence the cond. distribution of $(X_{(2)}, \ldots, X_{(n)})$, given $X_{(1)} = x_{(1)}$,
is the same as the unconditional distribution of

$$(x_{(1)} + Y_{(1)}, x_{(1)} + Y_{(2)}, \ldots, x_{(1)} + Y_{(n-1)}),$$

where $Y_1, Y_2, \ldots, Y_{n-1}$ denotes a random sample of size $n-1$ from an exponential distr. with mean β.

So the conditional distr of $(X_{(2)} - X_{(1)}, X_{(3)} - X_{(1)}, \ldots, X_{(n)} - X_{(1)})$ given $X_{(1)} = x_{(1)}$ is the same as the uncond. distr. of $(Y_{(1)}, Y_{(2)}, \ldots, Y_{(n-1)})$.
Thus the cond. distr. of $\sum_{i=2}^{n} (X_{(i)} - X_{(1)})$ given $X_{(1)} = x_{(1)}$ is the same as the uncond. distr of $\sum_{i=1}^{n-1} Y_{(i)}$. Equivalently, the conditional distr. of $\sum_{i=1}^{n} (X_i - X_{(1)})$ given $X_{(1)} = x_{(1)}$ is the same as the unconditional distr. of $\sum_{i=1}^{n-1} Y_i$, which is Gamma $(\beta, n-1)$ since the Y_i's are iid exponential with mean β.
Since this is true for all values of $x_{(1)}$ $(> \beta)$, we have shown now that ~~those~~ the sufficient statistics $X_{(1)}$ and $T \stackrel{def}{=} \sum_{i=1}^{n} (X_i - X_{(1)})$ are independent. [Note: I worked too hard to get this. If we fix the value of β, the independence follows directly from Basu's Theorem.]

Furthermore, we have shown that the joint pdf of the sufficient statistics $X_{(1)}$ and T is:

$$f(x,t) = \frac{n}{\beta} e^{-n(x-\theta)/\beta} \frac{1}{\Gamma(n-1)\beta^{n-1}} t^{n-2} e^{-t/\beta}, \quad \text{if } \theta < x < \infty \text{ and } 0 < t < \infty$$
$$= 0, \quad \text{otherwise.}$$

Now suppose that $g(x,t)$ is a function such that

(1) $E_{\theta,\beta}\, g(X_{(1)}, T) = 0$ for all $\theta \in \mathbb{R}$ and all $\beta > 0$.

To complete the proof that $(X_{(1)}, T)$ is complete, it suffices to show that (1) implies

(2) $g(x,t) = 0$ for all $x \in \mathbb{R}$ and $t > 0$

(except perhaps on a set of prob. 0)

Now (1) can be written as

(3) $\displaystyle\int_{x=\theta}^{\infty}\int_{t=0}^{\infty} g(x,t)\, \frac{n}{\beta} e^{-n(x-\theta)/\beta} \frac{1}{\Gamma(n-1)} t^{n-2} e^{-t/\beta}\, dt\, dx$

for all $\theta \in \mathbb{R}$ and $\beta > 0$.

or

(4) $\displaystyle e^{n\theta/\beta}\int_{x=\theta}^{\infty}\left[\int_{t=0}^{\infty} g(x,t)\, \frac{1}{\Gamma(n-1)} t^{n-2} e^{-t/\beta}\, dt\right] \frac{n}{\beta} e^{-nx/\beta}\, dx = 0$

for all θ and all $\beta > 0$

~~we~~ Since $e^{n\theta/\beta} > 0$ for all real θ, it follows that (4) is equivalent to

(5) $\displaystyle\int_{x=\theta}^{\infty} H(x,\beta)\, e^{-nx/\beta}\, dx = 0 \quad \forall\theta \text{ and } \forall\beta>0,$

where $H(x,\beta) \overset{\text{def.}}{=} \displaystyle\int_{t=0}^{\infty} g(x,t)\,\frac{1}{\Gamma(n-1)}\, t^{n-2} e^{-t/\beta}\, dt.$

Next, fix $\beta > 0$, so that the integrand in (5) is a function of x alone. By the Fund. Thm of Calculus, we can differentiate (5) with resp. to θ to obtain

(6) $\quad -H(\theta,\beta)\, e^{-n\theta/\beta} = 0 \quad \forall\theta\in\mathbb{R}$, for fixed $\beta>0$.

It follows that $H(x,\beta) = 0 \quad \forall x\in\mathbb{R}$ and $\forall\beta>0$; that is, that

(7) $\displaystyle\int_{t=0}^{\infty} g(x,t)\,\frac{1}{\Gamma(n-1)}\, t^{n-2}\, e^{-t/\beta}\, dt = 0$

for all $x\in\mathbb{R}$ and for all $\beta>0$.

Now fix $x\in\mathbb{R}$, so that $g(x,t)$ in the integrand is a function of t alone. Since (7) holds for all $\beta>0$, it follows from the completeness of the one-dimension Gamma $(n-1,\beta)$ family that

$$g(x,t) = 0 \quad \text{for } \underline{\text{fixed}}\ x\in\mathbb{R} \text{ and for all } t>0.$$

Since this is true for every $x\in\mathbb{R}$, it follows that $g(x,t)=0$ for all $x\in\mathbb{R}$ and all $t>0$. $\square$