STAT 721

Solution to a problem in the lecture of Nov. 9.

Let $X_1, \ldots, X_n \sim \text{Poisson}(\lambda)$, $\lambda > 0$.

Find the UMVUE of $e^{-a\lambda}$, where a is a constant, $a > 1$.

Solution:

We have seen before that $T = \sum_{i=1}^{n} X_i$ is sufficient and complete for $\{P_\lambda, \lambda > 0\}$.

By the Lehmann-Scheffe Theorem, the UMVUE must be a function of T — call it $g(T)$, so we must have

$$E_\lambda[g(T)] = e^{-a\lambda} \quad \text{for all } \lambda > 0.$$

Since we know that $T \sim \text{Poisson}(n\lambda)$, this is equivalent to

$$\sum_{t=0}^{\infty} g(t)\, e^{-n\lambda} \frac{(n\lambda)^t}{t!} = e^{-a\lambda} \quad \text{for all } \lambda > 0.$$

(continued)

or

$$\sum_{t=0}^{\infty} g(t)\,\frac{n^t\lambda^t}{t!} = e^{(n-a)\lambda} = \sum_{t=0}^{\infty} \frac{(n-a)^t\,\lambda^t}{t!} \quad \text{for all } \lambda>0.$$

The two power series above can agree for all $\lambda>0$ only if all the coefficients of λ^t coincide. That is, we must have

$$\frac{g(t)\,n^t}{t!} = \frac{(n-a)!}{t!} \qquad \text{for } t=0,1,2,\ldots$$

or $$g(t) = \left(\frac{n-a}{n}\right)^t = \left(1-\frac{a}{n}\right)^t \qquad \text{for } t=0,1,2,\ldots$$

It follows that the UMVUE of $e^{-a\lambda}$ is

$$g(T) = \left(1-\frac{a}{n}\right)^T = \left(1-\frac{a}{n}\right)^{\Sigma x_i} = \left(1-\frac{a}{n}\right)^{n\bar{x}} = \left[\left(1-\frac{a}{n}\right)^n\right]^{\bar{x}}.$$

As we have seen before, when $n=1$, the UMVUE is the unreasonable estimator $(1-a)^{x_1}$. But note that for large n, since $\lim_{n\to\infty}\left(1-\frac{a}{n}\right)^n = e^{-a}$, the UMVUE is close to the (reasonable) MLE $= e^{-a\bar{x}}$.