STAT 721 Assignment #1 Solutions

(2.2) (a) $y = x^2$ is 1-1 from $0<x<1$ to $0<y<1$.

Inverse map is $x = \sqrt{y}$, $0<y<1$

So $f_Y(y) = f_X(\sqrt{y}) \left| \frac{d\sqrt{y}}{dy} \right|$, $0<y<1$

$= 1 \cdot \left| \frac{1}{2} y^{-1/2} \right|$

$= \frac{1}{2\sqrt{y}}$, $0<y<1$

(b) $y = -\log x$ is a 1-1 map from $(0,1)$ onto $(0,\infty)$

Inverse map is $x = e^{-y}$. Also $\left| \frac{d}{dy} e^{-y} \right| = \left| -e^{-y} \right| = e^{-y}$,

so $f_Y(y) = f_X(e^{-y}) \cdot e^{-y}$, $0<y<\infty$

$= \frac{(n+m+1)!}{n!\,m!} e^{-yn} (1-e^{-y})^m e^{-y}$

$= \frac{(n+m+1)!}{n!\,m!} e^{-(n+1)y} (1-e^{-y})^m$, $0<y<\infty$.

(c) $y = e^x$ is a 1-1 map of $(0,\infty)$ onto $(1,\infty)$.

Inverse map is $x = \log y$, and $\left| \frac{d(\log y)}{dy} \right| = \frac{1}{y}$, so

$f_Y(y) = \frac{1}{\sigma^2} \log y \; e^{-(\log y/\sigma)^2/2} \cdot \frac{1}{y}$, $1<y<\infty$.

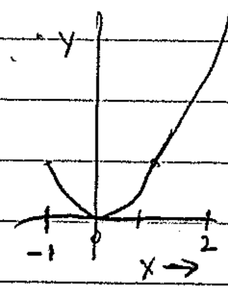
2.3 $y=\frac{x}{x+1}$ is a 1-1 map of $\{0,1,2,3,\ldots\}$

onto $\{0,\frac{1}{2},\frac{2}{3},\frac{3}{4},\ldots\}$

Given that y is a possible value of Y, the corresponding x is obtained by solving $y=\frac{x}{x+1}$ for x, which yields $x=\frac{y}{1-y}$

So $\underline{f_Y(y)}=P[Y=y]=P[X=\frac{y}{1-y}]=\underline{\frac{1}{3}\left(\frac{2}{3}\right)^{\frac{y}{y-1}}}$ for $\underline{y\in\{0,\frac{1}{2},\frac{2}{3},\frac{3}{4},\ldots}$

2.7 (a) Note that $y=x^2$ is <u>not</u> a 1-1 map of $[-1,2]$ onto its range $[0,4]$, so we can not use the standard change of variables formula. One way to proceed is to calculate $F_Y(y)=P[Y\le y]=P[X^2\le y]$ for all possible y, then differentiate.



<u>Case 1</u>, $0\le y\le 1$. Then $P[Y\le y]=P[X^2\le y]=P[-\sqrt{y}<X<\sqrt{y}]$

$$=\int_{-\sqrt{y}}^{\sqrt{y}}\frac{2}{9}(x+1)\,dx.$$ Note $P[Y\le 1]=\int_{-1}^{1}\frac{2}{9}(x+1)\,dx$

$$=\frac{2}{9}\left[\frac{x^2}{2}+x\right]_{-1}^{1}=$$
$$=\frac{2}{9}\,\frac{3}{2}-\frac{2}{9}\left(-\frac{1}{2}\right)=\frac{4}{9}.$$

<u>Case 2</u> $1\le y\le 4$. Then

$$P[Y\le y]=P[Y\le 1]+P[1\le Y\le y]=\frac{4}{9}+P[1\le X^2\le y]$$

$$=\frac{4}{9}+P[1\le X\le\sqrt{y}]\,.$$ (Continued)

2.7(a) (continued)

So $F_Y(y) = F_X(\sqrt{y}) - F_X(-\sqrt{y}) \qquad 0<y<1$

$= \frac{4}{9} + F_X(\sqrt{y}) - F_X(1) \qquad 1<y<4.$

So, when $0<y<1$, $f_Y(y) = \frac{d}{dy}\left[F_X(\sqrt{y}) - F_X(-\sqrt{y})\right]$

$= f_X(\sqrt{y})\,\frac{1}{2}y^{-1/2} - f_X(-\sqrt{y})\left(-\frac{1}{2}y^{-1/2}\right)$

$= \frac{2}{9}(\sqrt{y}+1)\,\frac{1}{2}\,\frac{1}{\sqrt{y}} + \frac{2}{9}(-\sqrt{y}+1)\,\frac{1}{2}\,\frac{1}{\sqrt{y}}$

$= \frac{2}{9}\,\frac{1}{\sqrt{y}},$

And when $1<y<4$, $f_Y(y) = f_X(\sqrt{y})\,\frac{1}{2}y^{-1/2} = \frac{2}{9}(\sqrt{y}+1)\frac{1}{2}\frac{1}{\sqrt{y}} = \frac{1}{9}\left(1+\frac{1}{\sqrt{y}}\right)$

So $f_Y(y) = \frac{2}{9}\,\frac{1}{\sqrt{y}}, \quad 0 \le y < 1$

$= \frac{1}{9}\left(1+\frac{1}{\sqrt{y}}\right), \quad 1<y\le 4.$

2.23 (a) For $0<y<1$, $P[Y\le y] = P[X^2 \le y] = P[-\sqrt{y} < X < \sqrt{y}]$

$= F_X(\sqrt{y}) - F_X(-\sqrt{y}),$

so $f_Y(y) = \frac{d}{dy}\left[F_X(\sqrt{y}) - F_X(-\sqrt{y})\right] = f_X(\sqrt{y})\,\frac{1}{2}y^{-1/2} - f_X(-\sqrt{y})\left(-\frac{1}{2}y^{-1/2}\right)$

$= \frac{1}{2}y^{-1/2}\left[f_X(\sqrt{y}) + f_X(-\sqrt{y})\right] = \frac{1}{2\sqrt{y}}\left[\frac{1}{2}(1+\sqrt{y}) + \frac{1}{2}(1-\sqrt{y})\right]$

$= \frac{1}{2\sqrt{y}}$

So $f_Y(y) = \frac{1}{2\sqrt{y}} \quad 0<y<1$

$= 0 \quad$ elsewhere

(continued)

2.23 (continued)

(b) $EY = \int_0^1 y \frac{1}{2\sqrt{y}}\,dy = \frac{1}{2}\int_0^1 y^{1/2}\,dy = \frac{1}{2}\,\frac{2}{3}\,y^{3/2}\Big]_0^1 = \underline{\underline{\frac{1}{3}}}$

$EY^2 = \int_0^1 y^2\,\frac{1}{2}y^{-1/2}\,dy = \frac{1}{2}\int_0^1 y^{3/2}\,dy = \frac{1}{2}\,\frac{2}{5}\,y^{5/2}\Big]_0^1 = \frac{1}{5}$

so $Var(Y) = E(Y^2) - (EY)^2 = \frac{1}{5} - \frac{1}{9} = \underline{\underline{\frac{4}{45}}}$

2.30 (a) $M_X(t) = Ee^{tX}$

When $t=0$, $M_X(t) = 1$

When $t \neq 0$,

$M_X(t) = Ee^{tX} = \int_0^c \frac{1}{c} e^{tx}\,dx = \frac{1}{c}\,\frac{1}{t}\,e^{tx}\Big]_0^c = \frac{1}{ct}\left(e^{tc}-1\right).$

(b) When $t \neq 0$,

$M_X(t) = Ee^{tX} = \frac{2}{c^2}\int_0^c x e^{tx}\,dx = \frac{2}{c^2}\int_0^c x\,d\left(\frac{1}{t}e^{tx}\right)$

$= \frac{2}{c^2}\left\{\left[x\,\frac{1}{t}e^{tx}\right]_0^c - \int_0^c \frac{1}{t}e^{tx}\,dx\right\}$

$= \frac{2}{c^2}\left\{\frac{c}{t}e^{tc} - \frac{1}{t^2}\int_0^c de^{tx}\right\} = \frac{2}{c^2}\left\{\frac{c}{t}e^{tc} - \frac{1}{t^2}\left(e^{tc}-1\right)\right\}$

(c) $\cancel{Ee^{tX} = \frac{1}{2\beta}\left[\int_{-\infty}^{\alpha} e^{tx} e^{-(x-\alpha)/\beta}\,dx + \int_{\alpha}^{\infty} e^{tx} e^{-(\alpha-x)/\beta}\,dx\right]}$

$Ee^{tX} = \frac{1}{2\beta}\int_{-\infty}^{\alpha} e^{tx} e^{\left(\frac{x-\alpha}{\beta}\right)}\,dx + \int_{\alpha}^{\infty} e^{tx} e^{-\frac{(x-\alpha)}{\beta}}\,dx$

~~Simplify this by making change of variables $y = \frac{\alpha - x}{\beta}$~~

in the first integral, $z = \frac{x-\alpha}{\beta}$ in the second integral, combine into a single integral over $(0,\infty)$, then evaluate.

Details and answer omitted. This is a double exponential distribution (see bottom of page 623).

2.30 (d) $Ee^{tX} = \sum_{x=0}^{\infty} e^{tx} P[X=x]$

$$= \sum_{x=0}^{\infty} \binom{r+x-1}{x} p^r \left[(1-p)e^t\right]^x$$

$$= \sum_{x=0}^{\infty} \binom{r+x-1}{x} \left\{1-(1-p)e^t\right\}^r \left[(1-p)e^t\right]^x \quad \frac{p^r}{\left[1-(1-p)e^t\right]^r}$$

$$= \left[\frac{p}{1-(1-p)e^t}\right]^r \sum_{x=0}^{\infty} \frac{r+x-1}{x} Q^r (1-Q)^x$$

(where $Q = 1-(1-p)e^t$)

$$= \left[\frac{p}{1-(1-p)e^t}\right]^r,$$

~~The series converges (and so the calculation makes sense)~~ provided that $(1-p)e^t < 1$ or $t < \log(1/(1-p))$

3.22 (a) $E[X(X-1)] = \sum_{x=0}^{\infty} x(x-1)\, P[X=x]$

$$= \sum_{x=2}^{\infty} x(x-1)\, e^{-\lambda} \frac{\lambda^x}{x!} = \sum_{x=2}^{\infty} e^{-\lambda} \frac{\lambda^2 \lambda^{x-2}}{(x-2)!}$$

$$= e^{-\lambda} \lambda^2 \sum_{y=0}^{\infty} \frac{\lambda^y}{y!} = e^{-\lambda}\lambda^2 e^{\lambda} = \lambda^2$$

So $EX^2 - EX = \lambda^2$. Since $EX = \lambda$, $E(X^2) - \lambda = \lambda^2$

So $EX^2 = \lambda^2 + \lambda$ and $Var(X) = E(X^2) - (EX)^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$.

(b) One way to compute $Var(X) = E(X^2) - (EX)^2$ is to take $M_X(t) = \left(\frac{p}{1-(1-p)e^t}\right)^r$ (derived in an earlier problem) and calculate $E(X^2) = \left.\frac{d^2}{dt^2} M_X(t)\right|_{t=0}$.

For (c) and (d), use the usual trick of writing the integrand in $\int x f(x)\,dx$ and $\int x^2 f(x)\,dx$ as a constant times a gamma(α,β) [beta(α,β)] density (with different parameter values).

(e) If $X \sim DE(\mu,\sigma)$, then (by defn) $Y = \frac{X-\mu}{\sigma}$ is $DE(0,1)$

So $EX = \mu + \sigma EY$ and $Var(X) = \sigma^2 Var Y$.

$EY = \int_{-\infty}^{0} \frac{1}{2} y e^{y}\,dy + \int_0^{\infty} \frac{1}{2} y e^{-y}\,dy = 0$, so $EX = \mu + \sigma \cdot 0 = \mu$

$Var(Y) = EY^2 = \int_{-\infty}^{0} \frac{1}{2} y^2 e^{y}\,dy + \int_0^{\infty} \frac{1}{2} y^2 e^{-y}\,dy = 2\cdot\frac{1}{2}\int_0^{\infty} y^2 e^{-y}\,dy = \left.-y^2 e^{-y}\right|_0^{\infty} + \int_0^{\infty} 2y e^{-y}\ldots$

3.22 (e) (continued)

$$= 2\int_0^\infty y e^{-y}\,dy = 2, \text{ so } Var(X) = (Var\,Y)\,\sigma^2 = 2\sigma^2.$$

3.23

(a) $f(x)$ is ≥ 0, and $\int_{-\infty}^{\infty} f(x)\,dx = \int_0^\infty \frac{\beta\alpha^\beta}{x^{\beta+1}}\,dx$

$$= \beta\alpha^\beta \int_\alpha^\infty x^{-\beta-1}\,dx = \beta\alpha^\beta \int_\alpha^\infty d\left[-\frac{1}{\beta}x^{-\beta}\right]$$

$$= \beta\alpha^\beta\left[-\frac{1}{\beta}x^{-\beta}\right]_\alpha^\infty = \beta\alpha^\beta\left[0 + \frac{1}{\beta}\alpha^{-\beta}\right] = 1. \checkmark$$

(b) $$EX = \int_\alpha^\infty \frac{\beta\alpha^\beta x}{x^{\beta+1}}\,dx = \int_\alpha^\infty \frac{\beta\alpha^\beta}{x^{(\beta-1)+1}}\,dx$$

$$= \frac{\beta\alpha}{\beta-1}\int_\alpha^\infty \frac{(\beta-1)\alpha^{\beta-1}}{x^{(\beta-1)+1}}\,dx = \frac{\beta\alpha}{\beta-1}$$

Note that the integrand here is a pdf, integrating to 1, if and only if $\beta - 1 > 0$ (or $\beta > 1$). If $\beta \leq 1$, then $EX = \infty$. This is because $\int_\alpha^\infty \frac{1}{x^\nu}\,dx < \infty$ if $\nu > 1$
$= \infty$ if $\nu \leq 1$.

Similarly, $$E(X^2) = \frac{\beta\alpha^2}{\beta-2}\int_\alpha^\infty \frac{(\beta-2)\alpha^{\beta-2}}{x^{(\beta-2)+1}}\,dx = \frac{\beta\alpha^2}{\beta-2} \quad \text{if } \beta > 2$$
$$= \infty \quad \text{if } \beta \leq 2$$

(continued)

3.23 (b) (continued)

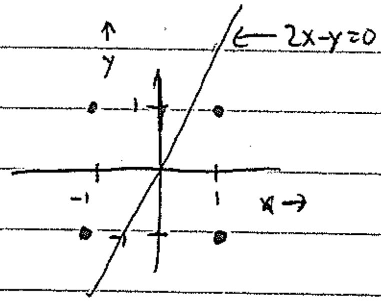
So $Var(X) = E(X^2) - (EX)^2 = \frac{\beta\alpha^2}{\beta-2} - \frac{\beta^2\alpha^2}{(\beta-1)^2}$ if $\beta > 2$

$= \infty$ if $1 \leq \beta \leq 2$

$=$ undefined if $\beta \leq 1$

(c) Already shown above in part (b). The authors of the textbook take "variance does not exist" to mean "variance $= \infty$" or "variance is undefined."

4.1 (a) $P(X^2+Y^2=2)=1$, so $P[X^2+Y^2<1]=0$.

(b) 

$P[2X-Y>0] = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

(c)

x	y	|x+y|
1	1	2
1	−1	0
−1	1	0
−1	−1	2

$P[|X+Y|<2] = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

4.4 (a) $1 = C\int_{y=0}^{1}\int_{x=0}^{2}(x+2y)\,dx\,dy = C\int_0^1\left[\frac{x^2}{2}+2xy\right]_0^2 dy = C\int_0^1[2+4y]\,dy$

$= C\left[2y+2y^2\right]_0^1 = 4C$, so $C=\frac{1}{4}$.

(b) For $0<x<2$, $f(x) = \int_0^1 \frac{1}{4}(x+2y)\,dy = \frac{1}{4}(xy+y^2)\Big|_0^1 = \frac{1}{4}(x+1)$

check $\int_0^2 \frac{1}{4}(x+1)\,dx = \frac{1}{4}\left[\frac{x^2}{2}+x\right]_0^2 = \frac{1}{4}(2+2) = 1.$ ✓

4.4 (continued)

(c) For $0 \le x_0 \le 2$ and $0 \le y_0 \le 1$,

$$F_{X,Y}(x_0, y_0) = \int_{y=0}^{y_0} \int_{x=0}^{x_0} \frac{1}{4}(x+2y)\,dx\,dy = \frac{1}{4}\int_0^{y_0} \left[\frac{x^2}{2} + 2xy\right]_0^{x_0} dy$$

$$= \int_0^{y_0} \frac{1}{4}\left[\frac{x_0^2}{2} + 2x_0 y\right] dy = \frac{1}{4}\left[\frac{x_0^2 y}{2} + x_0 y^2\right]_0^{y_0}$$

$$= \frac{1}{4}\left[\frac{x_0^2 y_0}{2} + x_0 y_0^2\right] = \frac{1}{8} x_0 y_0 \left[x_0 + 2y_0\right].$$

[check: when $x_0 = 2, y_0 = 1$, $F(2,1) = \frac{1}{8}\, 2 \cdot 1 \cdot 4 = 1$]

So $F_{X,Y}(x_0, y_0) = \frac{1}{8} x_0 y_0 [x_0 + 2y_0]$ $0 \le x_0 \le 2$, $0 \le y_0 \le 1$

$= 0$ $x_0 \le 0$ OR $y_0 < 0$

$= 1$ $x_0 > 2$ and $y_0 > 1$.

(d) $z = \frac{9}{(x+1)^2}$ is a 1–1 decreasing map from $0 < x < 2$ to $1 < z < 9$

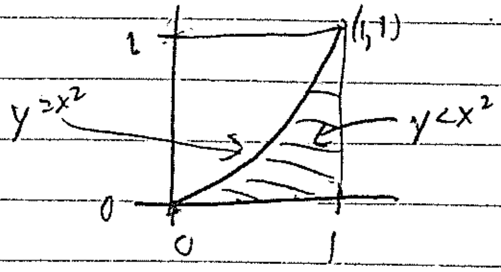
Inverse map $(x+1)^2 = \frac{9}{z}$, $x + 1 = \frac{3}{\sqrt{z}}$, $x = 3z^{-1/2} - 1$

So $f_Z(z) = \frac{1}{4}(x+1)\left|\frac{dx}{dz}\right|$

$$= \frac{1}{4}\, 3z^{-1/2} \left|-\frac{3}{2} z^{-3/2}\right| = \frac{9}{8} z^{-2} = \frac{9}{8z^2}, \quad 1 < z < 9$$

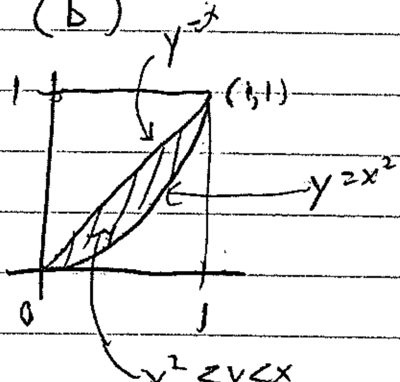
$= 0$ otherwise

4.5 (a)



$$P[X>\sqrt{Y}]=P[Y<X^2]$$
$$=\int_{x=0}^{1}\left[\int_{y=0}^{x^2}(x+y)\,dy\right]dx$$
$$=\int_0^1\left[xy+\frac{y^2}{2}\right]_0^{x^2}dx=\int_0^1\left(x^3+\frac{x^4}{2}\right)dx$$
$$=\frac{1}{4}+\frac{1}{2}\,\frac{1}{5}=\underline{\underline{\frac{7}{20}}}.$$

(b)



$$P[X^2<Y<X]=\int_{x=0}^{1}\int_{y=x^2}^{x}2x\,dy\,dx$$
$$=\int_{x=0}^{1}\left[2xy\right]_{x^2}^{x}dx=\int_0^1\left(2x^2-2x^3\right)dx$$
$$=\frac{2}{3}-\frac{2}{4}=\underline{\underline{\frac{1}{6}}}$$

4.20 The mapping $(X_1,X_2)\to(Y_1,Y_2)$ takes $\mathbb{R}^2$ onto $R^+\times[-1,1]$

since $Y_2^2=\dfrac{X_1^2}{X_1^2+X_2^2}\le 1$

The inverse mapping is: $X_1=Y_2\sqrt{Y_1}$

and $X_2^2=Y_1-X_1^2=Y_1-Y_1Y_2^2=Y_1(1-Y_2^2)$ so $X_2=\pm\sqrt{Y_1}\sqrt{1-Y_2^2}$

This inverse map is <u>not</u> 1-1, so we have to use a method explained in the text (which I did not cover in the lecture), namely split the domain $\mathbb{R}^2$ into several disjoint pieces $Q_1,\ldots$ (continued) such that the mapping $(x_1,x_2)\to(y_1,y_2)$ is 1-1 from each of the A_i's to its range, then combine using equation (4.36) at the bottom of page (6) of the text. In the problem at hand it is clear that the mapping from each of the 4 quadrants of the (x_1,x_2)-plane is 1-1. Consider the first quadrant $A_1=\{(x_1,x_2);\ x_1>0, x_2>0\}$. Then $(x_1,x_2)\to(y_1,y_2)$ is 1-1 from A_1 onto $B_1=\{y_1>0,\ 0<y_2<1\}$, with inverse map $x_1=y_2y_1^{1/2}$, $x_2=y_1^{1/2}(1-y_2^2)^{1/2}$. The Jacobian is

$$J=\begin{vmatrix}\frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2}\\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2}\end{vmatrix}=\begin{vmatrix} y_2\frac{1}{2}y_1^{-1/2} & y_1^{1/2}\\ \frac{1}{2}y_1^{-1/2}(1-y_2^2)^{1/2} & y_1^{1/2}\cdot\frac{1}{2}(1-y_2^2)^{-1/2}(-2y_2)\end{vmatrix}$$

$$=-\frac{1}{2}y_2^2(1-y_2^2)^{-1/2}-\frac{1}{2}(1-y_2^2)^{1/2}=-\frac{1}{2}(1-y_2^2)^{-1/2}\left[y_2^2+(1-y_2^2)\right]$$

$$=-\frac{1}{2}(1-y_2^2)^{-1/2},\quad \text{so } |J|=\frac{1}{2}(1-y_2^2)^{-1/2}.$$

Since $f_{X_1,X_2}(x_1,x_2)=\dfrac{1}{2\pi\sigma^2}e^{-\frac{1}{2\sigma^2}(x_1^2+x_2^2)}$,

the piece of the joint density of (Y_1,Y_2) corresponding to (x_1,x_2) in first quadrant is

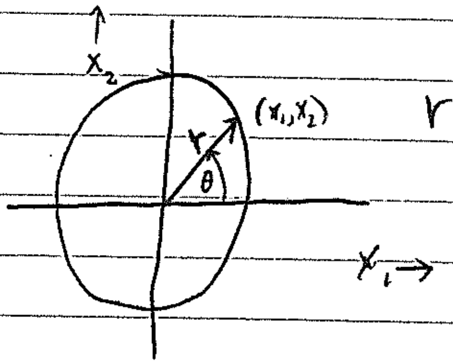
$$\frac{1}{2\pi\sigma^2}e^{-\frac{1}{2\sigma^2}y_1}|J|=\frac{1}{2\pi\sigma^2}e^{-\frac{y_1}{2\sigma^2}}\cdot\frac{1}{2}(1-y_2^2)^{-1/2}.$$

Combining similar calculations for the other 3 quadrants yields (via formula 4.3.6), the joint pdf of Y_1 and Y_2,

$$f_{Y_1,Y_2}(y_1,y_2) = \frac{1}{\pi\sigma^2} e^{-y_1/(2\sigma^2)} (1-y_2^2)^{-1/2},$$

$$y_1 > 0, \ -1 < y_2 < 1$$

(b) $f_{Y_1,Y_2}(y_1,y_2)$ factors as a (function of y_1) $\times$ (function of y_2), hence Y_1 and Y_2 are independent.



$r = \sqrt{y_1}$, $\theta = \cos^{-1}(y_2)$

The joint density of (X_1,X_2) is constant on each circle $x_1^2+x_2^2=r^2$. This make it intuitively clear that the radius $R = \sqrt{Y_1}$ should be independent of the angle $\theta = \cos^{-1}(Y_2)$ and hence that (Y_1, Y_2) are independent.

4.27 The joint p.d.f. of X and Y is

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}\left[(x-\mu)^2+(y-\gamma)^2\right]} \qquad \begin{array}{l} -\infty < x < \infty \\ -\infty < y < \infty \end{array}$$

(continued)

(4.27) (continued)

$$U = X+Y, \quad V = X-Y, \quad \text{so } X = \frac{U+V}{2}, \quad Y = \frac{U-V}{2}$$

so clearly the mapping $(x,y) \to (u,v)$ is 1-1 from $\mathbb{R}^2$ onto $\mathbb{R}^2$, with Jacobian $J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$, so $|J| = \frac{1}{2}$.

So the joint pdf of U and V is

$$f_{U,V}(u,v) = \frac{1}{4\pi\sigma^2} e^{-\frac{1}{2\sigma^2}\left[\left(\frac{u+v}{2}-\mu\right)^2 + \left(\frac{u-v}{2}-\gamma\right)^2\right]} \qquad \begin{array}{l} -\infty < u < \infty \\ -\infty < v < \infty \end{array}$$

$$= \frac{1}{4\pi\sigma^2} e^{-\frac{1}{8\sigma^2}\left[(u+v-2\mu)^2 + (u-v-2\gamma)\right]^2}$$

Now we do some algebra with the exponent:

$$(u+v-2\mu)^2 + (u-v-2\gamma)^2 = u^2 + 2uv + v^2 - 4\mu(u+v) + 4\mu^2 + u^2 - 2uv + v^2 - 4\gamma(u-\gamma) + 4\gamma^2$$

$$= 2u^2 + 2v^2 - 4u(\mu+\gamma) - 4v(\mu-\gamma) + 4\mu^2 + 4\gamma^2$$

$$= 2\left\{[u-(\mu+\gamma)]^2 + [v-(\mu-\gamma)]^2\right\} \quad \left(\text{note that } (\mu+\gamma)^2 + (\mu-\gamma)^2 = 2(\mu^2+\gamma^2)\right)$$

Hence

$$f_{U,V}(u,v) = \frac{1}{2\pi(2\sigma^2)} e^{-\frac{1}{2(2\sigma^2)}\left\{[u-(\mu+\gamma)]^2 + [v-(\mu-\gamma)]^2\right\}}$$

$$= \frac{1}{\sqrt{2\pi(2\sigma^2)}} e^{-\frac{1}{2(2\sigma^2)}[u-(\mu+\gamma)]^2} \cdot \frac{1}{\sqrt{2\pi(2\sigma^2)}} e^{-\frac{1}{2(2\sigma^2)}[v-(\mu-\gamma)]^2}$$

This factors into (a fn. of u) $\times$ (a fn. of v). Hence U and V are indep.
Furthermore, we observe that $U \sim N(\mu+\gamma, 2\sigma^2)$, $V \sim N(\mu-\gamma, 2\sigma^2)$.

4.30 (a) $Y|X=x \sim N(x, x^2)$, so $E[Y|X=x] = x$, so $E[Y|X] = X$

so $E(Y) = E_X E[Y|X] = EX = \int_0^1 x\,dx = \frac{1}{2}$.

Also $E[Y^2|X=x] = \mu_x^2 + \sigma_x^2 = x^2 + x^2 = 2x^2$,

so $E[Y^2|X] = 2X^2$, so $E(Y^2) = E[E[Y^2|X]] = 2E(X^2) = 2\int_0^1 x^2 dx$

$= 2/3$

Hence $\mathrm{var}(Y) = E[Y^2] - (EY)^2 = \frac{2}{3} - (\frac{1}{2})^2 = \frac{5}{12}$.

$E[XY|X=x] = x E[Y|X=x] = x \cdot x = x^2$, so $E[XY|X] = X^2$

so $E[XY] = E\,E[XY|X] = EX^2 = \frac{1}{3}$,

So $\mathrm{Cov}(X,Y) = E[XY] - (EX)(EY) = \frac{1}{3} - \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{12}$

(b) The joint p.d.f of X and Y is

$$f_{X,Y}(x,y) = f(x) f(y|x) = 1 \cdot \frac{1}{\sqrt{2\pi x^2}} e^{-\frac{1}{2x^2}(y-x)^2}, \quad 0<x<1,\ -\infty<y<\infty$$

Define $Q = Y/X$ and $R = X$.

$q = \frac{y}{x}$, $r = x \Rightarrow x = r$, $y = qr$, so the change of variables $(x,y) \to (q,r)$ is 1-1 from $(0,1)\times\mathbb{R} \to \mathbb{R}\times(0,1)$

The Jacobian is $J = \begin{vmatrix} \frac{\partial x}{\partial q} & \frac{\partial x}{\partial r} \\ \frac{\partial y}{\partial q} & \frac{\partial y}{\partial r} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ r & q \end{vmatrix} = -r$, so $|J| = r$.

(continued)

4.30 (b) (continued)

So the joint pdf of Q and R is

$$f_{Q,R}(q,r) = \frac{1}{\sqrt{2\pi r^2}}\, e^{-\frac{1}{2r^2}(qr-r)^2} \cdot r$$

$$= \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}(q-1)^2}, \quad \begin{array}{l} 0<r<1 \\ -\infty<q<\infty. \end{array}$$

~~Since the marginal pdf $f_R(r)$ is known to be $U[0,1]$, the~~

The marginal pdf of Q is

$$f_Q(q) = \int_0^1 \frac{1}{\sqrt{2\pi}}\, e^{-(q-1)^2}\, dr = \frac{1}{\sqrt{2\pi}}\, e^{-(q-1)^2}, \quad -\infty<q<\infty$$

Since the marginal pdf of R is known to be $f_R(r) = 1 \quad 0<r<1$,

we see that

$$f_{Q,R}(q,r) = f_Q(q)\, f_R(r) \quad \begin{array}{l} \text{for all } r,\ 0<r<1 \\ \text{and all } q,\ -\infty<q<\infty. \end{array}$$

Hence $Q = \frac{Y}{X}$ and $R = X$ are independent.

(Furthermore $Y \sim N(1,1)$).