STAT 721 Solutions to Assignment #3

7.1

x	MLE at x
0	1
1	1
2	2
3	3
4	3

(since $\max(\frac{1}{3}, \frac{1}{4}, 0) = \frac{1}{3}$, etc.

Another MLE is:

x	MLE at x
0	1
1	1
2	3
3	3
4	3

← note $\max\{f(x|1), f(x|2), f(x|3)\}$ is attained at both $x=2$ and $x=3$.

This example shows that a MLE need not be unique.

7.6 (a) $\prod_{i=1}^{n} f(x_i|\theta) = \theta^n (\prod x_i)^{-2} \, I_{\{x_{(1)} \geq \theta\}}$, so $X_{(1)} = \min(X_1, \ldots, X_n)$ is sufficient.

(b) $\log L(\theta) = n \log \theta - 2 \sum_{i=1}^{n} \log x_i$ if $x_{(1)} \geq \theta$ (i.e. if $0 < \theta \leq x_{(1)}$)
$= -\infty$ otherwise

Since $\log L(\theta)$ is increasing on $(0, x_{(1)}]$, max is attained at $\theta = x_{(1)}$.
Hence MLE $\hat{\theta} = X_{(1)}$.

(c) $EX = \theta \int_{\theta}^{\infty} x \cdot \frac{1}{x^2} \, dx = \theta \int_{\theta}^{\infty} \frac{dx}{x} = \theta \left[\log(\infty) - \log(\theta) \right] = \infty$.

Since $EX = \infty$, method of moments fails, i.e., there is no method of moments estimator of θ.

(7.9) $EX = \int x f(x|\theta)\,dx = \int_0^{\theta} \frac{x}{\theta}\,dx = \frac{1}{\theta} \frac{x^2}{2}\Big]_0^{\theta} = \frac{1}{\theta}\frac{\theta^2}{2} = \frac{\theta}{2}$

Method of moments = equate $\bar{X} = m_1 = \frac{\theta}{2}$.

Solution is $\theta = 2\bar{X}$. So method of moments est is $\hat{\theta}_{MM} = 2\bar{X}$.

MLE: $L(\theta) = \frac{1}{\theta^2}$ if $X_{(n)} \leq \theta$
$= 0$ elsewhere

So $L(\theta)$ reaches max at $\hat{\theta} = X_{(n)}$.
So the MLE is $\hat{\theta}_{ML} = X_{(n)}$.

The mean of $\hat{\theta}_{MM}$ is $E[2\bar{X}] = 2EX_1 = 2\frac{\theta}{2} = \theta$,

and the variance is $Var(2\bar{X}) = 4\frac{Var(X_1)}{n} = \frac{4}{n}\frac{\theta^2}{12} = \frac{\theta^2}{3n}$.

For the MLE, $E(X_{(n)}) = \int_0^{\theta} x \frac{nx^{n-1}}{\theta^n}\,dx = \frac{n}{\theta^n}\frac{\theta^{n+1}}{n+1} = \frac{n}{n+1}\theta$,

Also $E(X_{(n)}^2) = \int_0^{\theta} x^2 \frac{nx^{n-1}}{\theta^n} = \frac{n}{\theta^n}\frac{\theta^{n+2}}{n+2} = \frac{n}{n+2}\theta^2$, so

$Var(X_{(n)}) = \frac{n}{n+2}\theta^2 - \frac{n^2}{(n+1)^2}\theta^2 = \frac{n}{(n+2)(n+1)^2}\theta^2$.

<u>Comparison</u>. $\hat{\theta}_{MM}$ is unbiased $\hat{\theta}_{MLE}$ is not, so the method of moments estimator wins in terms of bias. In all other respects the MME loses to the MLE: the MME is not a function of the sufficient stat $X_{(n)}$ and so could be improved. The MME has much higher variance than the MLE — even when $n=1$, the variances are $\theta^2/3$ vs $\theta^2/12$. On balance, I prefer the MLE.

7.11 (a) $\log L(\theta) = \log\left[\theta^n (\prod x_i)^{\theta-1}\right] = n \log\theta - (\theta-1)\sum \log x_i$

$\frac{d}{d\theta}\log L(\theta) = \frac{n}{\theta} - \sum \log x_i$. Equate this to 0 and solve to obtain $\hat\theta = \frac{n}{-\sum \log x_i}$. To see that this yields a global maximum, note that $\frac{d^2}{d\theta^2}\log L(\theta) = -\frac{n}{\theta^2} < 0$.

So the MLE is $\hat\theta = \dfrac{n}{-\sum_{i=1}^{n}\log X_i}$.

[The minus sign looks screwy until one notes that $0<x<1$ implies $-\log x_i > 0$.]

To compute the variance of the MLE, we first need to figure out its distribution.

Let $Y_i = -\log X_i$. Then the pdf of Y_i is

$$g(y) = f(e^{-y})\,|e^{-y}| = \theta e^{-y(\theta-1)} e^{-y} = \theta e^{-y\theta}, \quad 0<y<\infty$$
$$= 0 \quad \text{elsewhere}$$

So $Y_1, Y_2, \ldots, Y_n$ are iid, each with exponential distribution $f(y) = \frac{1}{\beta} e^{-y/\beta}$, $y>0$, with $\beta = 1/\theta$.

Hence (using, e.g. mgf's), $\sum_{i=1}^{n} Y_i = -\sum_{i=1}^{n}\log X_i$ has Gamma distribution with $\alpha = n$, $\beta = 1/\theta$ (as listed on page 629 of text).

2.11 (a) (continued)

Write $U = \dfrac{1}{-\sum_{i=1}^{n} \log X_i}$. Then U has an inverted gamma distribution with parameters n and $\theta = 1/\beta$. Its pdf, worked out at the ~~[illegible]~~ top of p.52 of the text, is $\dfrac{1}{(n-1)!\,\beta^n}\left(\dfrac{1}{y}\right)^{n+1} e^{-1/(\beta y)}$, $0<y<\infty$, where $\beta = 1/\theta$

Then we can work out the first two moments of $\hat{\theta} = nU$ from the first two moments of U, which we can evaluate using the trick that the pdf of the inverted gamma ~~[illegible]~~ integrates to 1, no matter what the integer n is. So

$$EU = \int_0^\infty \frac{1}{(n-1)!\,\beta^n}\left(\frac{1}{y}\right)^{n+1}\left(\frac{1}{y}\right)^{-1} e^{-1/(\beta y)}\,dy$$

$$= \int_0^\infty \frac{1}{(n-1)!\,\beta^{n-1+1}}\left(\frac{1}{y}\right)^{(n-1)+1} e^{-1/(\beta y)}\,dy$$

$$= \frac{(n-2)!}{(n-1)!\,\beta}\int_0^\infty \frac{1}{[(n-1)-1]!\,\beta^{n-1}}\left(\frac{1}{y}\right)^{(n-1)+1} e^{-1/(\beta y)}\,dy$$

$$= \frac{(n-2)!}{(n-1)!\,\beta}\cdot 1 = \frac{\theta}{n-1}. \quad \text{So } E[nU] = \frac{n\theta}{n-1}$$

A similar calculation shows that $E(U^2) = \dfrac{(n-3)!}{(n-1)!\,\beta^2} = \dfrac{\theta^2}{(n-1)(n-2)}$

So $E\{[nU]^2\} = n^2 EU^2 = \dfrac{n^2\theta^2}{(n-1)(n-2)}$.

So $\mathrm{Var}(\hat{\theta}) = E[(nU)^2] - [E(nU)]^2 = \dfrac{n^2\theta^2}{(n-1)(n-2)} - \dfrac{n^2\theta^2}{(n-1)^2}$

$= \dfrac{n^2}{(n-1)^2(n-2)}\theta^2$, which is easily seen to $\to 0$ as $n \to \infty$.

(b) $EX_1 = \int_0^1 \theta x^{\theta-1} x \, dx = \int_0^1 \theta x^\theta \, dx = \theta \left.\dfrac{x^{\theta+1}}{\theta+1}\right]_0^1 = \dfrac{\theta}{\theta+1}$

So equate $\bar{X} = \dfrac{\theta}{\theta+1}$ and solve for θ.

$\bar{X}\theta + \bar{X} = \theta$, $\theta(1-\bar{X}) = \bar{X}$, $\theta = \dfrac{\bar{X}}{1-\bar{X}}$

So ~~MLE~~ method of moments estimator is $\underline{\underline{\dfrac{\bar{X}}{1-\bar{X}}}}$

(Makes sense, because $\bar{X} \in (0,1)$ with prob 1 and so $\dfrac{\bar{X}}{1-\bar{X}}$ stays in $(0,\infty)$ with prob 1).

7.38 To work this question, we need to use Cor 7.3.15 in text (necc & suff-conditions for attainment of C-R lower bound). [Note: my apologies for skipping this topic in my lectures]. In each case, we write down $a(\theta)[W(\underset{\sim}{x}) - g(\theta)] = \frac{\partial}{\partial\theta} \log L(\theta|x)$; and check whether it holds for some $g(\theta)$ and some UE. $W(\underset{\sim}{x})$ of $g(\theta)$.

(a) When $f(x|\theta) = \theta x^{\theta-1}$, $0<x<1$, $\theta>0$

$$\log L(\theta|\underset{\sim}{x}) = \log\left[\theta^n (\Pi x_i)^{\theta-1}\right] = n\log\theta + (\theta-1)\sum_{i=1}^{n}\log x_i$$

So $\frac{d}{d\theta}\log L(\theta|\underset{\sim}{x}) = \frac{n}{\theta} + \sum_{i=1}^{n}\log x_i$.

Write $a(\theta)[W(\underset{\sim}{x}) - g(\theta)] = \sum_{i=1}^{n}\log x_i + \frac{n}{\theta}$.

For the above identity to hold, $g(\theta)$ must be some non-zero multiple of $1/\theta$. Take, e.g. $g(\theta) = 1/\theta$. This forces $a(\theta) \equiv -n$ and $W(\underset{\sim}{x}) \equiv -\frac{\sum_{i=1}^{n}\log x_i}{n}$. Then, by the Cor.,

$W(\underset{\sim}{X}) = -\frac{\sum \log X_i}{n}$ attains the C-R lower bound, provided that $W(\underset{\sim}{X})$ is an unbiased est of $\frac{1}{\theta}$. (continued)

7.38(a) (continued)

$E\,W(\underset{\sim}{X}) = -E \log(X_1) = \frac{1}{\theta}$, since we showed in an earlier problem that $\log X_1 \sim$ exponential with $\beta = 1/\theta$.

7.38(b). $L(\theta|x) = \left(\frac{\log\theta}{\theta-1}\right)^n \theta^{\sum x_i}$

$$\log L(\theta|\underset{\sim}{x}) = n\log(\log\theta) - n\log(\theta-1) + \sum_1^n x_i \log\theta$$

$$\frac{\partial}{\partial\theta}\log L(\theta|\underset{\sim}{x}) = \frac{n}{\log\theta}\cdot\frac{1}{\theta} - \frac{n}{\theta-1} + \frac{\sum x_i}{\theta}$$

Try to find $g(\theta)$, $W(\underset{\sim}{x})$, $a(\theta)$ for which

$$a(\theta)\left[W(\underset{\sim}{x}) - g(\theta)\right] = \frac{n}{\theta}\left\{\frac{\sum x_i}{n} - \left[\frac{\theta}{\theta-1} - \frac{1}{\log\theta}\right]\right\}$$

Unique solution (up to a non-zero multiplicative constant) is.

$$g(\theta) = \frac{\theta}{\theta-1} - \frac{1}{\log\theta}, \quad W(\underset{\sim}{x}) = \bar{x}, \quad a(\theta) = \frac{n}{\theta}.$$

$\bar{X}$ attains the C-R lower bound for estimating $g(\theta)$ provided that $E_\theta(\bar{X}) = g(\theta)$.

$$E_\theta(\bar{X}) = E_\theta(X_1) = \frac{1}{\theta-1}\int_0^1 x\,\theta^x \log\theta\,dx = \frac{1}{\theta-1}\int_0^1 x\,d\theta^x$$

$$= \frac{1}{\theta-1}\left[x\theta^x\Big|_0^1 - \int_0^1 \theta^x\,dx\right] = \frac{1}{\theta-1}\left[\theta - \frac{\theta^x}{\log\theta}\Big]_0^1\right]$$

$$= \frac{1}{\theta-1}\left[\theta - \frac{\theta-1}{\log(\theta)}\right] = \frac{\theta}{\theta-1} - \frac{1}{\log(\theta)} = g(\theta). \quad \checkmark$$

7.40 Each of the iid X_i's have pmf

$$f(x|p) = p^x (1-p)^{1-x}, \quad x = 0, 1.$$

By the Cramér-Rao Inequality, i.i.d.-case, if $W(\underset{\sim}{X}) = W(X_1, \ldots, X_n)$ is any estimator with $E_p W(X) = p$, then

$$\operatorname{Var}_p W(\underset{\sim}{X}) \geq \frac{1}{n E_p\left[\left(\frac{\partial}{\partial p} \log(f(X_1|p))\right)^2\right]}.$$

Now

$$\frac{\partial}{\partial p} \log f(x|p) = \frac{\partial}{\partial p}\left[x \log p + (1-x)\log(1-p)\right]$$

$$= \frac{x}{p} + (1-x)\frac{1}{1-p}(-1) = \frac{x}{p} - \frac{(1-x)}{1-p}$$

$$= \frac{x(1-p) - (1-x)p}{p(1-p)} = \frac{x - xp - p + xp}{p(1-p)} = \frac{x-p}{p(1-p)}.$$

So

$$E_p\left[\frac{X_1 - p}{p(1-p)}\right]^2 = \frac{\operatorname{Var}(X_1)}{p^2(1-p)^2} = \frac{p(1-p)}{p^2(1-p)^2} = \frac{1}{p(1-p)}$$

So

$$\operatorname{Var}_p W(\underset{\sim}{X}) \geq \frac{1}{n \frac{1}{p(1-p)}} = \frac{p(1-p)}{n}.$$

But

$$\operatorname{Var}_p(\bar{X}) = \frac{1}{n}\operatorname{Var}_p(X_1) = \frac{p(1-p)}{n}.$$

Hence $\bar{X}$ attains the C.-R. lower bound and is hence the best unbiased estimator of p.

7.44 $X_1,\dots,X_n \overset{iid}{\sim} N(\theta,1)$.

We have seen before that $\bar{X}$ is a complete sufficient statistic for θ. If we can show that $E_\theta(\bar{X}^2-\frac{1}{n})=\theta^2$, then (by Lehmann-Scheffe Thm) it follows that $\bar{X}^2-\frac{1}{n}$ is the best UE of θ^2.

$$E(\bar{X}^2)=\frac{1}{n^2}E\left[\left(\sum_{i=1}^{n}X_i\right)^2\right]=\frac{1}{n^2}\left[nEX_1^2+n(n-1)E(X_1X_2)\right]$$

$$=\frac{1}{n^2}\left[n\left[\mathrm{Var}\,X_1+(EX_1)^2\right]+n(n-1)(EX_1)^2\right]$$

$$=\frac{1}{n^2}\left[n(1+\theta^2)+n(n-1)\theta^2\right]=\frac{1}{n}\left[1+\theta^2+(n-1)\theta^2\right]=\frac{1}{n}\left[n\theta^2+1\right]$$

$$=\theta^2+\frac{1}{n}, \text{ so } E\left[\bar{X}^2-\frac{1}{n}\right]=\theta^2. \quad \checkmark$$

There are many ways to compute $\mathrm{var}(\bar{X}^2)$, ranging from crude to elegant (eg. Stein's Identity). I will give a crude derivation relying on algebra and the fact that if $Z\sim N(0,1)$, then $EZ=0$, $EZ^2=1$, $EZ^3=0$ and $EZ^4=3$ (if you haven't seen $EZ^4=3$, show it using mgf or by integration by parts $\int x^4\varphi(x)\,dx=-\int x^3\,d\varphi(x)=$ etc.)

$\bar{X}-\theta\sim N(0,\frac{1}{n})$, so $\sqrt{n}(\bar{X}-\theta)\sim N(0,1)$.

We have already that $E\bar{X}=\theta$ and $E\bar{X}^2=\theta^2+\frac{1}{n}$.

Since $E\left[n^{3/2}(\bar{X}-\theta)^3\right]=0$, we have $E\left[\bar{X}^3-3\theta\bar{X}^2+3\theta^2\bar{X}-\theta^3\right]=0$

so $E(\bar{X}^3)=3\theta E(\bar{X}^2)-3\theta^2E\bar{X}+\theta^3=3\theta\left(\theta^2+\frac{1}{n}\right)-3\theta^3+\theta^3$
$=\theta^3+\frac{3\theta}{n}$

(continued)

7.44 (continued)

Also $E[n^2(\bar{X}-\theta)^4] = 3$, so

$$E[\bar{X}^4 - 4\theta\bar{X}^3 + 6\theta^2\bar{X}^2 - 4\theta^3\bar{X} + \theta^4] = \frac{3}{n^2}$$

so $E(\bar{X}^4) = \frac{3}{n^2} + 4\theta E(\bar{X}^3) - 6\theta^2 E(\bar{X}^2) + 4\theta^3 E\bar{X} - \theta^4$

$$= \frac{3}{n^2} + 4\theta\left[\theta^3 + \frac{3\theta}{n}\right] - 6\theta^2\left(\theta^2 + \frac{1}{n}\right) + 4\theta^4 - \theta^4$$

$$= \theta^4 + \frac{3}{n^2} + \frac{12\theta^2}{n} - \frac{6\theta^2}{n} = \theta^4 + \frac{6\theta^2}{n} + \frac{3}{n^2}$$

Thus $Var(\bar{X}^2 - \frac{1}{n}) = Var(\bar{X}^2) = E(\bar{X}^4) - [E(\bar{X}^2)]^2$

$$= \theta^4 + \frac{6\theta^2}{n} + \frac{3}{n^2} - \left(\theta^2 + \frac{1}{n}\right)^2$$

$$= \theta^4 + \frac{6\theta^2}{n} + \frac{3}{n^2} - \theta^4 - \frac{2\theta^2}{n} - \frac{1}{n^2} = \underline{\underline{\frac{4\theta^2}{n} + \frac{2}{n^2}}}$$

Cramer-Rao Lower bound is $\dfrac{\left[\frac{d}{d\theta}\theta^2\right]^2}{n E_\theta\left[\left(\frac{\partial}{\partial\theta}\log f(X_1|\theta)\right)^2\right]}$

Numerator is $(2\theta)^2 = 4\theta^2$.

$\log f(x|\theta) = \log\frac{1}{\sqrt{2\pi}} - \frac{1}{2}(x-\theta)^2$, $\frac{\partial}{\partial\theta}\log f(x|\theta) = -\frac{1}{2}2(x-\theta)(-1)$
$= x - \theta$

So denominator is $n E_\theta[(X_1-\theta)^2] = n\,Var(X_1) = n$.

So C-R lower bd on variance of UE of θ^2 is $\underline{\underline{\frac{4\theta^2}{n}}}$ $\left(< \frac{4\theta^2}{n} + \frac{2}{n^2}\right)$

so not achieved min.

7.46 $X_1, X_2, X_3 \overset{iid}{\sim} U[\theta, 2\theta]$

(a) $EX_1 = \frac{\theta + 2\theta}{2} = \frac{3}{2}\theta$. Equate $\bar{X} = \frac{3}{2}\hat{\theta}_{MM}$

and solve to get $\hat{\theta}_{M.M} = \frac{2}{3}\bar{X}$.

(b) First note that $0 < \theta < a < b < 2\theta \Rightarrow \frac{b}{a} < 2 \Rightarrow \frac{b}{2} < a$,

hence $P_\theta\left[\frac{X_{(3)}}{2} > X_{(1)}\right] = 0$ for all $\theta > 0$. So we will

assume that $\frac{X_{(3)}}{2} \le X_{(1)}$ (and define the MLE to be any estimator $\delta(x_1, \dots, x_n)$ that we want when $\frac{X_{(3)}}{2} > X_{(1)}$.)

Given $\frac{X_{(3)}}{2} \le X_{(1)}$, $L(\theta | x_1, \dots, x_n) = \frac{1}{\theta^3}$ if $\theta \le x_i \le 2\theta$, $i = 1, 2, 3$

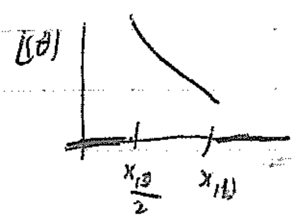
$= 0$ elsewhere

The condition $\{\theta \le x_i \le 2\theta, \; i = 1,2,3\}$ is equivalent to $\{x_{(1)} \ge \theta$ and $x_{(3)} \le 2\theta\}$,

which is equivalent to $\frac{X_{(3)}}{2} \le \theta \le X_{(1)}$ (when $\frac{X_{(3)}}{2} \le X_{(1)}$).

So $L(\theta | x_1, x_2, x_3) = \frac{1}{\theta^3}$ when $\frac{X_{(3)}}{2} \le \theta \le X_{(1)}$

$= 0$ elsewhere



so MLE is $\hat{\theta} = \frac{X_{(3)}}{2}$.

So $\hat{\theta} = \frac{X_{(3)}}{2}$ is a MLE of θ.

$E(X_{(3)}) = \theta + \frac{3}{4}\theta = \frac{7}{4}\theta$ so $E\hat{\theta} = E\left[\frac{X_{(3)}}{2}\right] = \frac{7}{8}\theta$

Take $k = \frac{8}{7}$ then $\frac{8}{7}\hat{\theta} = \frac{4}{7}X_{(3)}$ is an unb. est of θ.

(continued)

7.46 (c) The sufficient statistic is $(X_{(1)}, X_{(3)})$. Since $\frac{4}{7} X_{(3)}$ is an unbiased est. of θ which is already a function of the sufficient statistic, it cannot be improved by "Rao-Blackwellization", that is, $E\left[\frac{4}{7} X_{(3)} \middle| (X_{(1)}, X_{(3)})\right] = \frac{4}{7} X_{(3)}$ (the same estimator).

The moment estimator $\frac{2}{3}\bar{X} = \frac{2}{9}(X_1 + X_2 + X_3) = \frac{2}{9}(X_{(1)} + X_{(2)} + X_{(3)})$ is an UE of θ which is not a function of the sufficient statistic $(X_{(1)}, X_{(3)})$. So it can be improved. Let $x_{(1)}, x_{(3)}$ be two possible values of $X_{(1)}$ and $X_{(3)}$ (i.e., $\frac{x_{(3)}}{2} \le x_{(1)}$)

Then $E\left[\frac{2}{9}(X_{(1)} + X_{(2)} + X_{(3)}) \middle| X_{(1)} = x_{(1)}, X_{3} = x_{(3)}\right] =$

$$\frac{2}{9}\left(x_{(1)} + x_{(3)} + E\left[X_{(2)} \middle| X_{(1)} = x_{(1)}, X_{(3)} = x_{(3)}\right]\right)$$

$$= \frac{2}{9}\left[x_{(1)} + x_{(3)} + \frac{1}{2}(x_{(1)} + x_{(3)})\right] = \frac{2}{9}\,\frac{3}{2}(x_{(1)} + x_{(3)}) = \frac{1}{3}(x_{(1)} + x_{(3)}).$$

Here we have used the fact that the conditional dist'n of $X_{(2)}$ given $X_{(1)} = x_{(1)}$ and $X_{(3)} = x_{(3)}$ is uniform on $[x_{(1)}, x_{(3)}]$, and so the conditional expectation is $(x_{(1)} + x_{(3)})/2$.

So, by the Rao-Blackwell Theorem, $\frac{1}{3}(X_{(1)} + X_{(3)})$ is an unbiased estimator of θ with strictly smaller variance than that of $\frac{2}{3}\bar{X}$.

7.46 (d) $X_1 = 1.29$; $X_2 = 0.86$; $X_3 = 1.33$

$\hat{\theta}_{MM} = \frac{2}{9}(1.29 + 0.86 + 1.33) = \underline{\underline{0.7733}}$ ← In this case, the estimate is reasonable because all 3 X_i's lie between $\hat{\theta} = 0.7733$ and $2\hat{\theta} = 1.5467$

$X_{(1)} = 0.86$ $X_{(2)} = 1.29$, $X_{(3)} = 1.33$. Note that $\frac{X_{(3)}}{2} = 0.665 < X_{(1)} = 0.86$

so the ML est $\hat{\theta} = \frac{X_{(3)}}{2} = 0.665$ makes sense in this case. Note that all 3 observations lie between $\hat{\theta} = 0.665$ and $2\hat{\theta} = 1.33$,

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7.48 (a) C-R lower bound on variance of an UE of p is

$$\frac{1}{n E_p\left\{\left[\frac{\partial}{\partial p} \log f(X_1 | p)\right]^2\right\}}$$

$$\log f(x|p) = \log\left[p^x (1-p)^{1-x}\right] \qquad x = 0, 1$$
$$= x \log p + (1-x) \log(1-p)$$

$$\frac{\partial}{\partial p} \log f(x|p) = \frac{x}{p} + \frac{1-x}{1-p}(-1) = \frac{x}{p} - \frac{1-x}{1-p} = \frac{(1-p)x - p(1-x)}{p(1-p)}$$
$$= \frac{x-p}{p(1-p)}$$

$$E_p\left[\left(\frac{X-p}{p(1-p)}\right)^2\right] = \frac{E\{[X-p]^2\}}{p^2(1-p)^2} = \frac{\operatorname{Var} X_1}{p^2(1-p)^2} = \frac{p(1-p)}{p^2(1-p)^2} = \frac{1}{p(1-p)}.$$

So C-R lower bound is $\dfrac{1}{n \frac{1}{p(1-p)}} = \dfrac{p(1-p)}{n}$. (continued)

7.48(a) (continued)

The MLE of p: $L(p|x_1,\dots,x_n) = \prod_{i=1}^{n}\left[p^{x_i}(1-p)^{1-x_i}\right]$

$= p^{\Sigma x_i}(1-p)^{n-\Sigma x_i}$. So $\log L(p|x_1,\dots,x_n) = (\Sigma x_i)\log p + (n-\Sigma x_i)\cdot\log(1-p)$

$$\frac{d}{dp}\left[(\Sigma x_i)\log p + (n-\Sigma x_i)\log(1-p)\right] = \frac{\Sigma x_i}{p} - \frac{n-\Sigma x_i}{1-p}.$$

$\frac{d}{dp}[\;] = 0$ when $\frac{\Sigma x_i}{p} = \frac{n-\Sigma x_i}{1-p}$ or $\Sigma x_i - p\,\Sigma x_i = np - (\Sigma x_i)p$

or $p = \frac{\Sigma x_i}{n}$. One can easily check that this ~~sole~~ 0 corresponds to a unique global max of $\log L(p)$. So $\hat{p} = \frac{\sum_1^n x_i}{n}$

is the MLE.

Since $\sum_{i=1}^{n} X_i \sim \text{Binom}(n,p)$, $\text{var}(\Sigma X_i) = np(1-p)$

so $\text{var}(\hat{p}) = \frac{\text{var}(\Sigma X_i)}{n^2} = \frac{np(1-p)}{n^2} = \frac{p(1-p)}{n}$

agrees with the C-R lower bound.

(b) $X_1X_2X_3X_4 = 1 \Leftrightarrow X_1=1, X_2=1, X_3=1, X_4=1$
(~~since each~~ $X_i = 0$ or 1),

so $E_p(X_1X_2X_3X_4) = 1\cdot P_p[X_1=1, X_2=1, X_3=1, X_4=1]$

$= \prod_{i=1}^{4} P_p[X_i=1] = p\cdot p\cdot p\cdot p = p^4$.

indep of X_1, X_2, X_3, X_4

$T = \sum_{i=1}^{n} X_i$ is a complete sufficient statistic for θ. (continued).

Hence $E\{X_1 X_2 X_3 X_4 \mid \sum_{i=1}^{n} X_i\}$ is UMVU est of p^4, by the Lehmann-Scheffe Theorem.

$$E\{X_1 X_2 X_3 X_4 \mid \sum_{i=1}^{n} X_i = k\} = P[X_1=1, X_2=1, X_3=1, X_4=1 \mid \sum_{i=1}^{n} X_i = k]$$

$$= \frac{P[X_1=1, X_2=1, X_3=1, X_4=1, \sum_{i=5}^{n} X_i = k-4]}{P[\sum_{i=1}^{n} X_i = k]}$$

$$= \frac{p^4 \cdot \binom{n-4}{k-4} p^{k-4} (1-p)^{(n-4)-(k-4)}}{\binom{n}{k} p^k (1-p)^{n-k}} = \frac{\binom{n-4}{k-4}}{\binom{n}{k}}$$

$$= \frac{\dfrac{(n-4)!}{(k-4)!\,(n-k)!}}{\dfrac{n!}{k!\,(n-k)!}} = \frac{k(k-1)(k-2)(k-3)}{n(n-1)(n-2)(n-3)}, \text{ when } k = 4, 5, \ldots, n$$

Clearly, when $k = 0, 1, 2$ or 3, $E\{X_1 X_2 X_3 X_4 \mid \sum_{i=1}^{n} X_i = k\} = 0$.

Hence, with $T = \sum_{i=1}^{n} X_i$, the best unbiased est. of p^4 is

$$\frac{T(T-1)(T-2)(T-3)}{n(n-1)(n-2)(n-3)}.$$

7.52 (a) $X_1, \dots, X_n \overset{iid}{\sim}$ Poisson(λ)

$$\prod_{i=1}^{n} f(x_i|\lambda) = \prod_{i=1}^{n}\left[e^{-\lambda}\,\frac{\lambda^{x_i}}{x_i!}\right] = e^{-n\lambda}\,\frac{\lambda^{\sum_1^n x_i}}{\prod x_i!}.$$

By factorization, $\sum_{i=1}^{n} x_i$ is sufficient for λ. We also need to show that the family of distributions of $T = \sum_{i=1}^{n} X_i$ is complete. We can do that either by quoting a theorem about completeness in exponential families — or we can do the following direct verification.

Suppose that $E_\lambda[g(T)] = 0$ for all $\lambda > 0$. To show completeness we need to deduce that $g(t) = 0$ for $t = 0, 1, 2, 3, \dots$

$T \sim$ Poisson$(n\lambda)$, so $E_\lambda[g(T)] = 0$ implies

$$0 = \sum_{t=0}^{\infty} g(t) e^{-n\lambda}\frac{(n\lambda)^t}{t!} = e^{-n\lambda} \sum_{t=0}^{\infty} \frac{n^t g(t)}{t!}\lambda^t \quad \text{for all } \lambda > 0.$$

This is possible only if $\frac{n^t g(t)}{t!} = 0$ for $t = 0, 1, 2, \dots$

that is, if $g(t) = 0$ for $t = 0, 1, 2, \dots$. So T is complete.

Now $E_\lambda(\bar{X}) = E_\lambda\left(\frac{\sum_{i=1}^{n} X_i}{n}\right) = \frac{n\,EX_1}{n} = EX_1 = \lambda$. So $\bar{X}$ is a function of the complete sufficient statistic $T = \sum_1^n X_i$, which is an UE of λ. It follows from the Lehmann-Scheffe Theorem that $\bar{X}$ is the UMVUE of λ.

7.58 The pdf of X is
$$f(x|\theta) = \begin{cases} \frac{\theta}{2}, & x=-1 \\ 1-\theta, & x=0 \\ \frac{\theta}{2}, & x=+1 \end{cases}, \quad 0 \le \theta \le 1.$$

(a) When $x=-1$ or $x=+1$, the MLE is $\{\theta_0 \in [0,1] : \theta/2 \text{ is maximized at } \theta_0\} = 1$.

When $x=0$, the MLE is $\{\theta_0 \in [0,1] : 1-\theta \text{ is maximized at } \theta_0\} = 0$.

Hence the MLE is
$$\hat{\theta} = \begin{cases} 1 & \text{when } X=-1 \text{ or } +1 \\ 0 & \text{when } X=0 \end{cases}.$$

(b)
$$E_\theta T(X) = 2 P_\theta[X=1] + 0 P_\theta[X \neq 1]$$
$$= 2\frac{\theta}{2} + 0\left(1-\frac{\theta}{2}\right) = \theta \text{ for all } \theta \in [0,1]$$
Hence $T(X)$ is unbiased.

(c) Let us check to see if the MLE $\hat{\theta}(X)$ (of part (a)) is better than $T(X)$. We first verify that $\hat{\theta}(X)$ is indeed unbiased:
$$E_\theta[\hat{\theta}(X)] = 1 P_\theta[X=-1 \text{ or } +1] + 0 P_\theta[X=0]$$
$$= 1 \cdot \left[\frac{\theta}{2} + \frac{\theta}{2}\right] + 0(1-\theta) = \theta. \checkmark$$

Now we compare the variances of the two unbiased estimators $T(X)$ and $\hat{\theta}(X)$.

(continued)

7.58 (continued)

$E_\theta\{[T(X)]^2\} = 4\left(\frac{\theta}{2}\right) = 2\theta$, so

$Var_\theta(T(X)) = 2\theta - \theta^2 = 2\theta(1-\theta)$.

Also $E_\theta\{[\hat{\theta}(X)]^2\} = 1(\theta) + 0(1-\theta) = \theta$,

so $Var_\theta[\hat{\theta}(X)] = \theta - \theta^2 = \theta(1-\theta) \leq 2\theta(1-\theta) = Var_\theta[T(X)]$

with <u>strict</u> inequality for all θ with $0 < \theta < 1$.

Hence $\hat{\theta}(X)$ is a better unbiased estimator than $T(X)$.

7.59 We have already seen that, when $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$, $(\bar{X}, S^2)$ is a complete sufficient statistic for (μ, σ^2). We also note that the random variable $V \stackrel{def}{=} \frac{(n-1)S^2}{\sigma^2}$ has a χ^2_{n-1}-distribution. If we can find a function of S^2 which is an unbiased estimator of $\sigma^p = (\sigma^2)^{p/2}$, then that function will be the best unbiased estimator of σ^p, by the Lehmann–Scheffe Theorem. We now make the <u>guess</u>

(continued)

7.59 (continued) that an estimator of form const.$\times(S^2)^{p/2}$ is an unbiased estimator of σ^p. Writing $V = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$, we compute

$$E[V^{p/2}] = \int_0^\infty \frac{1}{\Gamma(\frac{n-1}{2})\, 2^{(n-1)/2}}\, v^{p/2}\, v^{\frac{n-1}{2}-1}\, e^{-v/2}\, dv$$

$$= \frac{1}{\Gamma(\frac{n-1}{2})\, 2^{(n-1)/2}} \int_0^\infty v^{\frac{n+p-1}{2}-1}\, e^{-v/2}\, dv$$

$$= \frac{1}{\Gamma(\frac{n-1}{2})\, 2^{(n-1)/2}} \cdot \Gamma\left(\frac{n+p-1}{2}\right) 2^{(n+p-1)/2}$$

$$= \frac{\Gamma(\frac{n+p-1}{2})}{\Gamma(\frac{n-1}{2})}\, 2^{p/2}$$

So $E\left[\frac{(n-1)S^2}{\sigma^2}\right]^{p/2} = E\left[\frac{(n-1)^{p/2}(S^2)^{p/2}}{\sigma^p}\right] = \frac{\Gamma(\frac{n+p-1}{2})}{\Gamma(\frac{n-1}{2})}\, 2^{p/2}$,

OR $E\left[\left(\frac{n-1}{2}\right)^{p/2}(S^2)^{p/2} \cdot \frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n+p-1}{2})}\right] = \sigma^p$

Hence $\left(\frac{(n-1)}{2}\right)^{p/2} \frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n+p-1}{2})} (S^2)^{p/2}$ is the (unique) best unbiased estimator of σ^p.

7.60 From standard theory that has previously been worked out for the Gamma model, we know that the statistic $Y = \sum_{i=1}^{n} X_i$ is sufficient and complete for the unknown parameter β. Furthermore, from standard sampling theory (e.g. use mgf's), the distribution of Y is Gamma$(n\alpha, \beta)$. Arguing just as in problem 7.59, we need only to find a function of Y which is an unbiased estimator of $1/\beta$. A reasonable guess is to try a constant multiple of $1/Y$. So we compute

$$E\left[\frac{1}{Y}\right] = \int_0^\infty \frac{1}{\Gamma(n\alpha)\,\beta^{n\alpha}} \cdot y^{n\alpha-1}\, y^{-1}\, e^{-y/\beta}\, dy$$

$$= \frac{1}{\Gamma(n\alpha)\,\beta^{n\alpha}} \int_0^\infty y^{(n\alpha-1)-1}\, e^{-y/\beta}\, dy$$

$$= \frac{1}{\Gamma(n\alpha)\,\beta^{n\alpha}}\, \Gamma(n\alpha-1)\,\beta^{n\alpha-1} = \frac{\Gamma(n\alpha-1)}{\Gamma(n\alpha)}\,\frac{1}{\beta}$$

$$= \frac{\Gamma(n\alpha-1)}{(n\alpha-1)\,\Gamma(n\alpha-1)} = \frac{1}{n\alpha-1}\cdot\frac{1}{\beta} \quad . \text{(continued)}$$

7.60 (continued)

So $E\left[(n\alpha-1)\,\frac{1}{Y}\right] = \frac{1}{\beta}$.

So the best unbiased estimator of $1/\beta$ is

$$(n\alpha-1)\,\frac{1}{Y} = (n\alpha-1)\,\frac{1}{\sum X_i} = \left(\alpha-\frac{1}{n}\right)\frac{1}{\bar{X}}.$$