STAT 721 Assignment #4 Solutions

7.22 $f(\bar{x}, \theta) = f(\theta) f(\bar{x} | \theta)$

Since each $x_i | \theta \sim N(\theta, \sigma^2)$, $\bar{x} | \theta \sim N(\theta, \frac{\sigma^2}{n})$.

(a) So $f(\bar{x} | \theta) = \frac{1}{\sqrt{2\pi\tau^2}} e^{-\frac{1}{2\tau^2}[\theta-\mu]^2} \frac{1}{\sqrt{2\pi\sigma^2/n}} e^{-\frac{1}{2\sigma^2/n}(\bar{x}-\theta)^2}$,

$-\infty < \theta < \infty, -\infty < \bar{x} < \infty$

(b)

$$m(\bar{x} | \sigma^2, \mu, \tau^2) = \int_{-\infty}^{\infty} f(\bar{x}, \theta)\, d\theta$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi\tau^2\frac{\sigma^2}{n}} \exp\left\{-\frac{1}{2\tau^2\sigma^2/n}\left[\frac{\sigma^2}{n}(\theta-\mu)^2 + \tau^2(\bar{x}-\theta)^2\right]\right\} d\theta$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi\tau^2\sigma^2/n} \exp\left\{-\frac{1}{2\tau^2\sigma^2/n}\left[\frac{\sigma^2}{n}(\theta^2 - 2\mu\theta + \mu^2) + \tau^2(\bar{x}^2 - 2\theta\bar{x} + \theta^2)\right]\right\} d\theta$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi\tau^2\sigma^2/n} \exp\left\{-\frac{1}{2\tau^2\sigma^2/n}\left[\left(\frac{\sigma^2}{n}+\tau^2\right)\theta^2 - 2\theta\left(\frac{\sigma^2}{n}\mu + \tau^2\bar{x}\right) + \frac{\sigma^2}{n}\mu^2 + \tau^2\bar{x}^2\right]\right\} d\theta$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\frac{2\pi\tau^2\sigma^2/n}{\tau^2+\sigma^2/n}}\sqrt{2\pi(\tau^2+\sigma^2/n)}} \exp\left\{-\frac{\frac{\sigma^2}{n}+\tau^2}{\tau^2\sigma^2/n}\left[\theta^2 - 2\theta\frac{\frac{\sigma^2}{n}\mu+\tau^2\bar{x}}{\frac{\sigma^2}{n}+\tau^2} + \left[\frac{\frac{\sigma^2}{n}\mu+\tau^2\bar{x}}{\frac{\sigma^2}{n}+\tau^2}\right]\right.\right.$$

$$\times \exp\left\{+\frac{1}{2}\left(\frac{\frac{\sigma^2}{n}\mu+\tau^2\bar{x}}{\frac{\sigma^2}{n}+\tau^2}\right)^2 \frac{\frac{\sigma^2}{n}+\tau^2}{\tau^2\sigma^2/n} - \frac{1}{2\tau^2\sigma^2/n}\left(\frac{\sigma^2}{n}\mu^2 + \tau^2\bar{x}^2\right)\right\}$$

$$= \frac{1}{\sqrt{2\pi(\tau^2+\sigma^2/n)}} \exp\left\{-\frac{1}{2\left(\frac{\sigma^2}{n}+\tau^2\right)}\left[\frac{\left(\frac{\sigma^2}{n}\mu^2+\tau^2\bar{x}^2\right)\left(\tau^2+\frac{\sigma^2}{n}\right)}{\tau^2\sigma^2/n} - \frac{\left(\frac{\sigma^2}{n}+\tau^2\bar{x}\right)^2}{\tau^2\sigma^2/n}\right]\right\}$$

(Here we used the fact that $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi B}} e^{-\frac{2}{B}(\theta-A)^2} d\theta = 1$ for all A and all $B>$

which was the motivation for all the above algebra to complete the squ

(continued)

$$= \frac{1}{\sqrt{2\pi(\tau^2+\sigma^2/n)}} \exp\left\{-\frac{1}{2(\frac{\sigma^2}{n}+\tau^2)} \frac{\tau^2\frac{\sigma^2}{n}\mu^2+\tau^4\bar{x}^2+(\frac{\sigma^2}{n})^2\mu^2+\frac{\sigma^2}{n}\tau^2\bar{x}^2-(\frac{\sigma^2}{n})^2\mu^2-2\frac{\sigma^2}{n}\mu\tau^2\bar{x}-\tau^4}{\tau^2\sigma^2/n}\right.$$

$$= \frac{1}{\sqrt{2\pi(\tau^2+\sigma^2/n)}} \exp\left\{-\frac{1}{2(\frac{\sigma^2}{n}+\tau^2)} \frac{\tau^2\sigma^2/n\,\bar{x}^2-(\tau^2\sigma^2/n)2\mu\bar{x}+(\tau^2\sigma^2/n)\mu^2}{\tau^2\sigma^2/n}\right.$$

$$= \frac{1}{\sqrt{2\pi(\tau^2+\sigma^2/n)}} \exp\left\{-\frac{1}{2(\tau^2+\frac{\sigma^2}{n})}(\bar{x}-\mu)^2\right\}$$

$= \text{pdf of } N(\mu, \frac{\sigma^2}{n}+\tau^2).$

(c) $\pi(\theta/\bar{x}) = f(\bar{x},\theta) / m(\bar{x})$.

So take a look at the joint pdf $f(\bar{x},\theta)$, as derived in the solution of part (b) (right before applying the operation $\int_{-\infty}^{\infty} - d\theta$), divide it by the $N(\mu, \frac{\sigma^2}{n}+\tau^2)$ pdf of $\bar{x}$, and see that the quotient is the p.d.f. of a $N(A,B)$ r.v., with $A = \frac{\frac{\sigma^2}{n}\mu+\tau^2\bar{x}}{\frac{\sigma^2}{n}+\tau^2} =$

$= \frac{\tau^2}{\tau^2+\frac{\sigma^2}{n}}\bar{x} + \frac{\sigma^2/n}{\tau^2+\sigma^2/n}\mu$ and $B = \frac{\sigma^2\tau^2/n}{\frac{\sigma^2}{n}+\tau^2}$.

7.24 $X_1, \ldots, X_n \sim$ Poisson(λ), $\lambda \sim$ gamma (α, β)

$T = \sum_{i=1}^{n} X_i \sim$ Poisson $(n\lambda)$, i.e. $P[T=t] = e^{-n\lambda}\frac{(n\lambda)^t}{t!}$, $t = 0,1,2\ldots$

The pdf of λ is $f(\lambda|\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}}\lambda^{\alpha-1}e^{-\lambda/\beta}$, $\lambda > 0$.

(a) The posterior pdf of λ given $T = \sum_{i=1}^{n} X_i = t$, $(t = 0,1,2,\ldots)$

is $g(\lambda|t) = \text{const.} \times g(\lambda)\, g(t|\lambda)$

$$= \text{const.} \times \lambda^{\alpha-1} e^{-\lambda/\beta} e^{-n\lambda} \frac{n^t \lambda^t}{t!}$$

$$= \text{const.} \times \lambda^{t+\alpha-1} e^{-(n+1/\beta)\lambda} = e^{-\lambda/[\beta/(n\beta+1)]}, \quad \lambda > 0$$

So, without working out the constant, we see that the posterior pdf of λ, given $\sum_{i=1}^{n} X_i = t$, is gamma $(t+\alpha, \beta/[n\beta+1])$.

Or, substituting into the general gamma pdf on page 624, we have

$$f(\lambda | \textstyle\sum_i X_i = t) = \frac{1}{\Gamma(t+\alpha)\left(\frac{\beta}{n\beta+1}\right)^{t+\alpha}} \lambda^{n+\alpha-1} e^{-\lambda[\beta/(n\beta+1)]}, \quad \lambda > 0.$$

(b) Using the formulas for the mean and variance of a gamma distribution (given on page 624), we have:

posterior mean, given $\sum X_i = t$, is $(t+\alpha)\left(\frac{\beta}{n\beta+1}\right)$.

& post. variance, given $\sum X_i = t$, is $(t+\alpha)\left(\frac{\beta}{n\beta+1}\right)^2$.

[You can also work this problem, getting equivalent answers, by conditioning on $\bar{X}$ $(= \sum X_i/n)$ instead of $\sum X_i$.]

7.33 As derived in Example 7.3.5, the MSE of the Bayes estimator is $\frac{np(1-p)}{(\alpha+\beta+n)^2} + \left(\frac{np+\alpha}{\alpha+\beta+n} - p\right)^2$.

We are asked to show that the MSE is a constant when $\alpha = \beta = \sqrt{n/4} = \frac{\sqrt{n}}{2}$. So we plug in these values and crank away with the algebra:

$$MSE = \frac{np(1-p)}{(\sqrt{n}+n)^2} + \left(\frac{np+\frac{\sqrt{n}}{2}}{\sqrt{n}+n} - p\right)^2$$

$$= \frac{1}{(\sqrt{n}+n)^2}\left[np - np^2 + \left(np + \frac{\sqrt{n}}{2} - np - \sqrt{n}\,p\right)^2\right]$$

$$= \frac{1}{(\sqrt{n}+n)^2}\left[np - np^2 + \left[\sqrt{n}\left(\tfrac{1}{2} - p\right)\right]^2\right]$$

$$= \frac{1}{(\sqrt{n}+n)^2}\left[np - np^2 + n\left(\tfrac{1}{4} - p + p^2\right)\right]$$

$$= \frac{1}{(\sqrt{n}+n)^2}\left[np - np^2 + \frac{n}{4} - np + np^2\right]$$

$$= \frac{n}{4(\sqrt{n}+n)^2} = \frac{1}{4(1+\sqrt{n})^2} = \text{a constant}$$

(i.e., does not depend on p, $0 \le p \le 1$

7.62 (a) $R(\theta, a\bar{X}+b) = E\left[(a\bar{X}+b-\theta)^2\right]$

$$= E\left\{\left[a\bar{X}-a\theta+b-(1-a)\theta\right]^2\right\}$$

$$= E\left\{\left[a(\bar{X}-\theta)+\left[b-(1-a)\theta\right]\right]^2\right\}$$

$$= E\left\{a^2(\bar{X}-\theta)^2+2a\left[b-(1-a)\theta\right](\bar{X}-\theta)+\left[b-(1-a)\theta\right]^2\right\}$$

$$= a^2\,\mathrm{Var}\,\bar{X}+2ab\left[b-(1-a)\theta\right](E\bar{X}-\theta)+\left[b-(1-a)\theta\right]^2$$

$$= a^2\frac{\sigma^2}{n}+\left[b-(1-a)\theta\right]^2.$$

(b) We saw earlier (in problem 7.22) that the Bayes estimator of θ is $a\bar{X}+b$ where $a = \frac{\tau^2}{\tau^2+\frac{\sigma^2}{n}} = \frac{n\tau^2}{n\tau^2+\sigma^2} = 1-\eta$ and $b = \frac{(\sigma^2/n)\mu}{\tau^2+\sigma^2/n} = \frac{\sigma^2\mu}{n\tau^2+\sigma^2} = \eta\mu$. So plug in $a = 1-\eta$ and $b=\eta\mu$ into the result of (a) to obtain that the risk function is

$$(1-\eta)^2\frac{\sigma^2}{n}+\left[\eta\mu-[1-(1-\eta)]\theta\right]^2 = (1-\eta)^2\frac{\sigma^2}{n}+\eta^2(\theta-\mu)^2.$$

(continued)

7.62 (c) The Bayes risk is $E_\pi R(\theta, \delta^\pi)$, where π is $N(\mu, \tau$

$$E_\pi\left\{(1-\eta)^2 \frac{\sigma^2}{n} + \eta^2(\theta-\mu)^2\right\} = (1-\eta)^2\frac{\sigma^2}{n} + \eta^2 E_\pi(\theta-\mu)^2$$

$$= (1-\eta)^2 \frac{\sigma^2}{n} + \eta^2\tau^2$$

$$= \left(\frac{n\tau^2}{n\tau^2+\sigma^2}\right)^2 \frac{\sigma^2}{n} + \left(\frac{\sigma^2}{n\tau^2+\sigma^2}\right)\tau^2$$

$$= \frac{1}{(n\tau^2+\sigma^2)^2}\left[\frac{n^2\tau^4\sigma^2}{n} + \sigma^4\tau^2\right]$$

$$= \frac{1}{(n\tau^2+\sigma^2)^2}\sigma^2\tau^2\left[n\tau^2+\sigma^2\right]$$

$$= \frac{\sigma^2\tau^2}{n\tau^2+\sigma^2} = \tau^2\eta.$$

(10.1) A sufficient condition for a sequence of estimators $\{T_n\}$ to be consistent is that $\text{Bias}_\theta^2(T_n) + \text{Var}_\theta(T_n) \to 0$ as $n \to \infty$.

There are a lot of ways to find a consistent estimator. Perhaps the easiest way is to find a function $h(X_1)$ such that $E_\theta h(X_1) = \theta$ and $\text{Var}_\theta\, h(X_1) < \infty$. Then $T_n = \frac{\sum_{i=1}^n h(X_i)}{n}$ is unbiased, and its variance $\frac{\text{Var}_\theta h(X_1)}{n} \to 0$ as $n \to \infty$, so $\{T_n\}$ is consistent.

(continued)

(10.1) (continued)

$$E_\theta(X_1) = \int x f(x|\theta)\,dx = \int_{-1}^{1} \tfrac{1}{2} x(1+\theta x)\,dx = \tfrac{1}{2}\int_{-1}^{1} [x + \theta x^2]\,dx$$

$$= \tfrac{1}{2}\left[\frac{x^2}{2} + \theta\frac{x^3}{3}\right]\Big|_{-1}^{1} = \tfrac{1}{2}\left[\tfrac{1}{2} + \tfrac{\theta}{3} - \left(\tfrac{1}{2} - \tfrac{\theta}{3}\right)\right] = \tfrac{1}{2}\cdot\tfrac{2}{3}\theta = \tfrac{1}{3}\theta.$$

So $E_\theta(3X_1) = \theta$.

Also $\operatorname{var}(3X_1) \le 9\operatorname{var}(X_1) \le 9\,EX_1^2 \le 9 < \infty$

since $P_\theta[-1<X<1] = 1$, for all $\theta \in (-1,1)$.

So, by the argument given earlier,

$$T_n = \frac{3\sum_{i=1}^{n} X_i}{n} \quad \text{is a consistent est of } \theta.$$

(There are, of course, many other consistent estimators of θ).

(10.3) (a) The likelihood function is

$$L(\theta|x_1,\ldots,x_n) = \left(\frac{1}{\sqrt{2\pi\theta}}\right)^n e^{-\frac{1}{2\theta}\sum(x_i-\theta)^2}$$

so $\log L(\theta) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log\theta - \frac{1}{2\theta}\sum(X_i-\theta)^2$.

Assuming that $\log L(\theta)$ is maximized at some interior point of $(0,\infty)$

(this needs to be checked), we solve $\frac{d}{d\theta}\log L(\theta) = 0$, or

$$-\frac{n}{2\theta} - \frac{1}{2\theta}\sum 2(X_i-\theta)(-1) + \frac{1}{2\theta^2}\sum(X_i-\theta)^2 = 0$$

or $$-\frac{n}{2\theta} + \frac{2}{2\theta}\left(\sum X_i - n\theta\right) + \frac{1}{2\theta^2}\left[\sum X_i^2 - 2\theta\sum X_i + n\theta^2\right] = 0$$

(continued)

10.3(a) (continued)

or (multiplying by $2\theta^2 > 0$)

$$-n\theta + 2\theta \sum x_i - 2n\theta^2 + \sum x_i^2 - 2\theta \sum x_i + n\theta^2 = 0$$

or $\quad -n\theta^2 - n\theta + \sum x_i^2 = 0$

or $\quad \theta^2 + \theta - \frac{\sum x_i^2}{n} = 0 \quad$ or $\quad \theta^2 + \theta - W_n = 0$, where $W_n = \frac{\sum x_i^2}{n}$.

This quadratic equation has the two real roots

$$\frac{-1 \pm \sqrt{1+4W_n}}{2}.$$

Since $\frac{1}{2}(-1 - \sqrt{1+4W_n}) < 0$ is outside the parameter space $[\theta \in (0,\infty)]$ and the other solution $\frac{1}{2}[\sqrt{1+4W_n} - 1] \geq 0$, the unique local extremum attained there must necessarily be the global maximum — that is the MLE is $\hat{\theta}_n = \frac{1}{2}[\sqrt{1+4W_n} - 1]$.

(b) By the theorem on asymptotic efficiency of MLE's,

$\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{\mathcal{D}} N(0, v(\theta))$, where $v(\theta) = \frac{1}{I(\theta)}$

(continued)

3 (b) (continued)

To calculate $I(\theta)$, note that

$$\log f(x|\theta) = \log\left(\frac{1}{\sqrt{2\pi\theta}}\right) e^{-\frac{1}{2\theta}(x-\theta)^2} = -\frac{1}{2}\log(2\pi) + \theta^{-1/2} - \frac{1}{2\theta}(x-\theta)^2$$

so that (after differentiation and doing some algebraic simplification)

we get that $\frac{\partial}{\partial\theta}\log f(x|\theta) = \frac{1}{2\theta^2}[x^2-\theta^2-\theta]$.

Hence $$V(\theta) = \frac{1}{E_\theta\left\{\left[\frac{\partial}{\partial\theta}\log f(x|\theta)\right]^2\right\}} = \frac{4\theta^4}{E_\theta\left\{\left[X^2-(\theta^2+\theta)\right]^2\right\}}$$

$$= \frac{4\theta^4}{E_\theta\left\{\left[X^2-(E_\theta X^2)\right]^2\right\}} = \frac{4\theta^4}{Var_\theta X^2} = \frac{4\theta^4}{E_\theta(X^4)-(E_\theta(X^2))^2}$$

noting that

$E_\theta X^2 = Var_\theta X + (E_\theta X)^2 = \theta + \theta^2$

Some further calculation yields that the denominator $= 4\theta^3 + \theta^2$,

so that $V(\theta) = \frac{4\theta^4}{4\theta^3+\theta^2} = \frac{4\theta^2}{4\theta+1}$.

So, for large n, $Var(\hat{\theta}_n) \approx \frac{4\theta^2}{n(4\theta+1)}$.

When θ is unknown, a reasonable estimate of the

approximate variance of $Var(\hat{\theta}_n)$ is $\frac{4\hat{\theta}_n^2}{n(4\hat{\theta}_n+1)}$.