STAT 723

Solution to Problem 9.4 (third try!)

(b) Recall the set-up: $X_1, \dots, X_n \sim N(0, \sigma_X^2)$,
$Y_1, \dots, Y_m \sim N(0, \sigma_Y^2)$

The earlier solution was correct down to the point where I showed that the LR test of $H_0: \sigma_Y^2 = \lambda_0 \sigma_X^2$ versus $H_1: \sigma_Y^2 \neq \lambda_0 \sigma_X^2$ rejects H_0 when

$$\left(\frac{S_1^2}{n S_1^2 + m S_2^2/\lambda_0}\right)^{n/2} \left(\frac{S_2^2}{n S_1^2 + m S_2^2/\lambda_0}\right)^{m/2} \text{ is too small,}$$

where $S_1^2 = \sum_1^n X_i^2/n$ and $S_2^2 = \sum_1^m Y_i^2/m$.

This is equivalent to rejecting H_0 when

$$\left(n + m\left(\frac{S_2^2}{S_1^2}\right)/\lambda_0\right)^{n/2} \left(n\left(\frac{S_1^2}{S_2^2}\right) + m/\lambda_0\right)^{m/2} \text{ is too large.}$$

Now, as some of you correctly pointed out to me right after last Friday's lecture, the statistic which has an $F_{n,m}$ distribution under H_0 is

$$\frac{n S_1^2/n}{\sigma_X^2} \Bigg/ \left(\frac{m S_2^2/m}{\lambda_0 \sigma_X^2}\right) = \lambda_0 \frac{S_1^2}{S_2^2}.$$

(continued)

So, writing $F = \lambda_0 S_1^2 / S_2^2$, H_0 is rejected when

$$\left(n + \frac{m}{F}\right)^{n/2} \left(\frac{n}{\lambda_0} F + \frac{m}{\lambda_0}\right)^{m/2} = \left(\frac{1}{\lambda_0}\right)^{m/2} \left(n + \frac{m}{F}\right)^{n/2} (nF + m)^{m/2}$$

is too large.

Writing $H(F) = \frac{n}{2} \log\left(n + \frac{m}{F}\right) + \frac{m}{2} \log(nF + m)$, a calculation (very similar to the one I did before) shows that $\frac{dH(F)}{dF}$ has the same sign as that of the quadratic $nF^2 + (m-n)F - m$, which is negative at $F=0$, and has two zeros (over $F \in \mathbb{R}$) at 1 and $\left(-\frac{m}{2}\right)$.

It follows that $H(F)$ is increasing for $F \in [0,1]$ and decreasing on $[1, \infty)$. It follows that the level α LR test rejects $H_0: \lambda = \lambda_0$ when either

$\frac{\lambda_0 S_1}{S_2} < f_1$ or $\frac{\lambda_0 S_1}{S_2} > f_2$, where f_1 and f_2 (with $0 < f_1 < 1 < f_2 < \infty$) are uniquely determined by the two side conditions: (i) $H(f_1) = H(f_2)$

and (ii) $\int_{f_1}^{f_2} F_{n,m}(x)\,dx = 1 - \alpha$.

(continued)

Note that f_1 and f_2 are determined only by the value of α, and not by the value of λ_0.

(c) The acceptance region of the size α LR test of $H_0: \lambda = \lambda_0$ is

$$f_1 \leq \frac{\lambda_0 S_1}{S_2} \leq f_2$$

, where f_1 and f_2 are determined by (i) and (ii). This can be re-written as $f_1 \frac{S_2}{S_1} \leq \lambda_0 \leq f_2 \cdot \frac{S_2}{S_1}$.

It follows that a $1-\alpha$ C.I. for λ is:

$\lambda \in C(\underline{x}, \underline{y})$ if: $\quad f_1 \frac{S_2}{S_1} \leq \lambda \leq f_2 \cdot \frac{S_2}{S_1}$

where f_1, f_2 are determined (from $1-\alpha$) by (i) and (ii)