STAT 723 Solutions to Assignment #2

(9.1) First note $P_\theta[L(X) > \theta] = \alpha_1$ and $P_\theta[U(X) < \theta] = \alpha_2$.

$P_\theta[L(X) \le \theta \le U(X)] = 1 - P_\theta[L(X) > \theta \text{ OR } U(X) < \theta]$.

Now $P_\theta[L(X) > \theta \text{ OR } U(X) < \theta] =$

$$= P_\theta[L(X) > \theta] + P_\theta[U(X) < \theta] - P_\theta[L(X) > \theta \text{ and } U(X) < \theta]$$

$$= \alpha_1 + \alpha_2 - 0$$, with the last probability being 0 because $P_\theta[L(X) \le U(X)] = 1$.

Hence $P_\theta[L(X) \le \theta \le U(X)] = 1 - (\alpha_1 + \alpha_2) = 1 - \alpha_1 - \alpha_2$.

(9.2) When $X_1, \ldots, X_n$ are iid. $\sim N(\theta, 1)$, we have $\bar{X} \sim N(\theta, \frac{1}{n})$, from which it easily follows that $\bar{X}_n \pm 1.96/\sqrt{n}$ is a 95% C.I. for θ.

Now let $X_{n+1} \sim N(\theta, 1)$ be indep of $X_1, \ldots, X_n$. We will investigate $p = P_\theta[\bar{X}_n - 1.96/\sqrt{n} < X_{n+1} < \bar{X}_n + 1.96/\sqrt{n}]$.

We first note that p does not depend on the value of θ:

$$p = P_\theta\left[\bar{X}_n - \theta - \frac{1.96}{\sqrt{n}} < X_{n+1} - \theta < \bar{X}_n - \theta + \frac{1.96}{\sqrt{n}}\right]$$

$$= P\left[\bar{Z}_n - \frac{1.96}{\sqrt{n}} < Z_{n+1} < \bar{Z}_n + \frac{1.96}{\sqrt{n}}\right]$$, where (continued)

9,2 (continued)

where $Z_1, \ldots, Z_n, Z_{n+1}$ are iid $\sim N(0,1)$
and $\bar{Z}_n = \sum_{i=1}^{n} Z_i / n$.

Before proceeding, consider the function

$$G(c) = P\left[c - \frac{1.96}{\sqrt{n}} < Z < c + \frac{1.96}{\sqrt{n}}\right], \text{ where } Z \sim N(0,1)$$

and c varies over the real line. Then

$$G(c) = \Phi\left(c + \frac{1.96}{\sqrt{n}}\right) - \Phi\left(c - \frac{1.96}{\sqrt{n}}\right) \quad \text{and}$$

$$G'(c) = \varphi\left(c + \frac{1.96}{\sqrt{n}}\right) - \varphi\left(c - \frac{1.96}{\sqrt{n}}\right).$$

When $c = 0$, $G'(c) = 0$. When $0 < c < \frac{1.96}{\sqrt{n}}$, $\varphi(c + \frac{1.96}{\sqrt{n}})$ is decreasing in c and $\varphi(c - \frac{1.96}{\sqrt{n}})$ is increasing in c, so that $G'(c) < 0$. When $c > \frac{1.96}{\sqrt{n}}$, $c + \frac{1.96}{\sqrt{n}} > c - \frac{1.96}{\sqrt{n}} > 0$ so that $G'(c) < 0$, since $\varphi(x)$ is decreasing in x for $x > 0$. Hence $G'(c) < 0$ for all $c > 0$. Similarly, $G'(c) > 0$ for all $c < 0$. It follows that $G(c)$ reaches its maximum value over $c \in \mathbb{R}$ at $c = 0$ (where $G(0) = 2\left[\Phi\left(\frac{1.96}{\sqrt{n}}\right) - \frac{1}{2}\right]$), and furthermore that $G(c) < G(0)$ for all $c \neq 0$. (continued).

9.2 (continued)

So

$$p = P\left[\bar{Z}_n - \frac{1.96}{\sqrt{n}} < Z_{n+1} < \bar{Z}_n + \frac{1.96}{\sqrt{n}}\right]$$

$$= \int_{-\infty}^{\infty} P\left[\bar{Z}_n - \frac{1.96}{\sqrt{n}} < Z_{n+1} < \bar{Z}_n + \frac{1.96}{\sqrt{n}} \,\middle|\, \bar{Z}_n = c\right] \sqrt{n}\,\varphi(\sqrt{n}\,c)\,dc$$

$$= \int_{-\infty}^{\infty} G(c)\,\sqrt{n}\,\varphi(\sqrt{n}\,c)\,dc < G(0)\int_{-\infty}^{\infty} \sqrt{n}\,\varphi(\sqrt{n}\,c)\,dc$$

$$= G(0) = 2\left[\Phi\left(\frac{1.96}{\sqrt{n}}\right) - \frac{1}{2}\right] \le 2\left[\Phi(1.96) - \frac{1}{2}\right] = 0.95.$$

So $p < 0.95$ for all $n \ge 1$.

Remarks: (1) Note that $p \to 0$ as $n \to \infty$.

(2) The part of the proof showing that $G(c)$ reaches its unique maximum at $c = 0$ goes through if the standard normal density φ is replaced by any density function that is symmetric about 0 ~~with~~ and is strictly decreasing on the positive part of $\mathbb{R}$.

9.3 (a) The hint suggests using the MLE of β to find an upper conf. ~~c~~ limit for β. The pdf of a single X_i is $f(x) = \alpha_0 \left(\frac{x}{\beta}\right)^{\alpha_0 - 1} \frac{1}{\beta}\, I_{[0,\beta]}$,

so $L(x_1, \ldots, x_n; \beta) = \prod f(x_i) = \alpha_0^{\,n} \dfrac{(\prod x_i)^{\alpha_0 - 1}}{\beta^{n\alpha_0}}\, I_{[0 < \max x_i \le \beta]}$.

9.3 (continued)

Since $\alpha_0^n \dfrac{(\prod x_i)^{\alpha_0 - 1}}{\beta^{n\alpha_0}}$ is a decreasing function of β on the set $\beta \geq \max_i X_i$, it follows that the MLE of β is $\hat{\beta} = \max X_i$.

Since $\hat{\beta} \leq \beta$; a reasonable guess is that an upper conf. limit for β based on $\hat{\beta} = \max X_i$ could have the form $r\hat{\beta} = r \max X_i$ for some $r > 1$.

First note that $0 < c < \beta \Rightarrow P_\beta[\max X_i \leq c] = \left[\left(\frac{c}{\beta}\right)^{\alpha_0}\right]^n = \left(\frac{c}{\beta}\right)^{\alpha_0 n}$. So for $r > 1$,

$$P_\beta[\beta \leq r \max X_i] = P_\beta\left[\max X_i \geq \frac{\beta}{r}\right]$$

$$= 1 - P_\beta\left[\max X_i < \frac{\beta}{r}\right] = 1 - \left[\frac{\beta/r}{\beta}\right]^{\alpha_0 n}$$

$$= 1 - \left(\frac{1}{r}\right)^{\alpha_0 n}.$$

Hence $r \max X_i$ will be an upper confidence limit for β with confidence coef. 0.95 if we set

$$1 - \left(\frac{1}{r}\right)^{\alpha_0 n} = 0.95 \quad \text{OR} \quad \left(\frac{1}{r}\right)^{\alpha_0 n} = 0.05$$

OR $\quad r = \exp[-\log 0.05/(\alpha_0 n)]$.

9.4 (a) The joint pdf of the X_i's and Y_i's is:
$$\left(\frac{1}{\sqrt{2\pi\sigma_x^2}}\right)^n e^{-\Sigma X_i^2/(2\sigma_x^2)} \left(\frac{1}{\sqrt{2\pi\sigma_Y^2}}\right)^m e^{-\Sigma Y_i^2/(2\sigma_Y^2)}.$$

With no restriction of λ, it is easily seen that the MLE of σ_x^2 is $S_1^2 = \frac{\Sigma X_i^2}{n}$ and the MLE of σ_Y^2 is $S_2^2 = \frac{\Sigma Y_i^2}{m}$, so that the denominator of the LR statistic is
$$\left(\frac{1}{\sqrt{2\pi S_1^2}}\right)^n e^{-n/2} \left(\frac{1}{\sqrt{2\pi S_2^2}}\right)^m e^{-m/2}.$$

Now under H_0: $\lambda = \lambda_0$, i.e. $\sigma_Y^2 = \lambda_0 \sigma_x^2$, the joint pdf is
$$\left(\frac{1}{\sqrt{2\pi\sigma_x^2}}\right)^{n+m} \left(\frac{1}{\sqrt{\lambda_0}}\right)^m e^{-[(\Sigma X_i^2 + \Sigma Y_i^2/\lambda_0)]/(2\sigma_x^2)}$$

The MLE of σ_x^2 under H_0 turns out to be (omitted the details of my calculation here):
$$\hat{\sigma}^2 = \frac{\Sigma X_i^2 + \frac{\Sigma Y_i^2}{\lambda_0}}{n+m}$$

(continued)

9.4 (a) (continued)

So the LR test rejects H_0 when

$$\frac{\left(\frac{1}{\sqrt{\hat{\sigma}^2}}\right)^{n+m}\left(\frac{1}{\sqrt{\lambda_0}}\right)^{m}}{\left(\frac{1}{\sqrt{S_1^2}}\right)^{n}\left(\frac{1}{\sqrt{S_2^2}}\right)^{m}}$$ is too small.

(b) Writing $\hat{\sigma}^2 = \frac{nS_1^2 + mS_2^2/\lambda_0}{n+m}$, the LR test rejects H_0 when

$$\left(\frac{S_1^2}{\left[\frac{nS_1^2+mS_2^2/\lambda_0}{n+m}\right]}\right)^{n/2}\left(\frac{S_2^2}{\left(\frac{nS_1^2+mS_2^2/\lambda_0}{n+m}\right)}\right)^{m/2}$$ is too small;

OR equivalently when

$$\frac{(nF+m/\lambda_0)^{m/2}}{\left[(n+m/(\lambda_0 F)\right]^{n/2}}$$ is too small,

where $F = S_1^2/S_2^2$. Note $F \sim F_{n,m}$.

(c) [Omitted] Use some algebra and calculus to show that the acceptance region must have form $C_0(\lambda_0,n,m) < F < C_1(\lambda_0,n,m)$. The constants C_0, C_1 are determined from the constraints that the LR is tied at C_0, C_1 and $P_{\lambda=\lambda_0}[\text{Accept } H_0] = 1-\alpha$. Then invert in the usual to obtain the CI for λ.

9.13 (a) The Beta $(\theta,1)$ p.d.f. is

$$f_\theta(x) = \theta x^{\theta-1} \quad 0<x<1$$
$$= 0 \quad \text{elsewhere}$$

The map $y = -(\log x)^{-1}$ is easily seen to be 1-1 from $(0,1)$ onto $(0,\infty)$ with inverse map $x = e^{-1/y}$.

So $g(y) = f(x)\left|\frac{dx}{dy}\right| = \theta(e^{-1/y})^{(\theta-1)} e^{-1/y}\left(\frac{1}{y^2}\right)$

$$= \frac{\theta}{y^2} e^{-\theta/y}, \quad 0<y<\infty.$$

$$P_\theta\left[\frac{Y}{2}<\theta<Y\right] = P_\theta\left[\theta<Y<2\theta\right] = \int_{y=\theta}^{2\theta} \frac{\theta}{y^2} e^{-\theta/y}\,dy$$

$$= \int_{z=1}^{z=1/2} \theta \frac{z^2}{\theta^2} e^{-z}\left(-\frac{\theta}{z^2}\right)dz = -\int_1^{1/2} e^{-z}\,dz = +e^{-z}\Big]_1^{1/2}$$

with $z = \theta/y$

$$= e^{-1/2} - e^{-1}.$$ ← This is, by definition, the confidence coeff of $\left[\frac{Y}{2}, Y\right]$.

(b) Noting that the c.d.f. of X is $F_\theta(x) = x^\theta$, $0<x<1$, this seems to suggest that $Q = X^\theta$ may be a pivot.

To check out this guess, make the change of variables $q = x^\theta$ (so $x = q^{1/\theta}$) on $0<x<1$, to obtain that the pdf of Q is $g(q) = f(q^{1/\theta})\left|\frac{dq^{1/\theta}}{d\theta}\right|$

$$= \theta(q^{1/\theta})^{(\theta-1)} \frac{1}{\theta} q^{\left(\frac{1}{\theta}-1\right)} = 1, \quad 0<q<1.$$ (continued)

9.13 (b) (continued)

So $Q = X^{\theta} \sim U[0,1]$. Since its distribution does not depend on θ, Q is a pivotal r.v.

Now the easiest choice of subinterval of $(0,1)$ for which a $U[0,1]$ r.v. has coverage prob $e^{-1/2} - e^{-1}$ is $[e^{-1}, e^{-1/2}]$. We have

$$e^{-1/2} - e^{-1} = P\left[e^{-1} < Q < e^{-1/2}\right] = P_{\theta}\left[e^{-1} < X^{\theta} < e^{-1/2}\right]$$

$$= P_{\theta}\left[-1 < \theta \log X < -1/2\right] \overset{\text{since } \log X < 0}{=} P_{\theta}\left[-\frac{1}{\log X} > \theta > -\frac{1}{2\log X}\right]$$

$$= P_{\theta}\left[-\frac{1}{2\log X} < \theta < -\frac{1}{\log X}\right]$$

So $\left[-\frac{1}{2\log X}, -\frac{1}{\log X}\right]$ is a CI for θ with the same conf. coeff. as in part (a)

(c) Looking back at part (a), we see that the interval there is exactly the same one in part (b).

9.16 (a) $H_0: \theta = \theta_0$ vs $H_1: \theta \neq \theta_0$

A reasonable α-level test rejects when $\left|\frac{\bar{X}-\theta_0}{\sigma/\sqrt{n}}\right| > z_{\alpha/2}$,

so the acceptance region is $-z_{\alpha/2} < \frac{\bar{X}-\theta_0}{\sigma/\sqrt{n}} < z_{\alpha/2}$,

which is equivalent to $\underbrace{\bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \theta_0 < \bar{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}}$.

So the set of $\theta=\theta_0$ satisfying this is a family of $1-\alpha$ conf. intervals for θ.

(b) $H_0: \theta \geq \theta_0$ vs $H_1: \theta < \theta_0$.

A reasonable α-level test here rejects H_0 when $\frac{\bar{X}-\theta_0}{\sigma/\sqrt{n}} < -z_\alpha$. [This test is in fact UMP since the family of normal distributions has MLR]. ~~The set~~ ~~(one-sided) interval inverts to $\theta_0 \leq \bar{X} + z_\alpha\frac{\sigma}{\sqrt{n}}$.~~ ~~So the set of $\theta=\theta_0$ satisfying $\theta \in [\bar{X} + z$~~

The acceptance region $\frac{\bar{X}-\theta_0}{\sigma/\sqrt{n}} \geq -z_\alpha$ inverts to $\theta_0 \leq \bar{X} + z_\alpha\frac{\sigma}{\sqrt{n}}$. So the set of ~~$\theta \in [\bar{X} + z_\alpha\frac{\sigma}{\sqrt{n}}, \infty)$~~ $\theta \in (-\infty, \bar{X} + z_\alpha\frac{\sigma}{\sqrt{n}})$ is a family of $1-\alpha$ conf. intervals for θ.

[Here $\bar{X} + z_\alpha\frac{\sigma}{\sqrt{n}}$ is ~~an~~ a $1-\alpha$ upper conf. bound for θ]

(continued)

9.16 (c) $H_0: \theta \le \theta_0$ vs $H_1: \theta > \theta_0$

A reasonable (in fact, UMP) level-α test rejects H_0 when $\frac{\bar{X}-\theta_0}{\sigma/\sqrt{n}} > z_\alpha$. So the acceptance region $\frac{\bar{X}-\theta_0}{\sigma/\sqrt{n}} \le z_{\cancel{\alpha}}$ can be rewritten as $\theta_0 \ge \bar{X} - z_{\cancel{\alpha}}\frac{\sigma}{\sqrt{n}}$. Hence the random interval $[\bar{X} - z_{\cancel{\alpha}}\frac{\sigma}{\sqrt{n}}, \infty)$ is a $1-\alpha$ conf. interval for θ. [Here $\bar{X} - z_\alpha \frac{\sigma}{\sqrt{n}}$ is called a lower conf. bound for θ].

9.17 (a) To find a conf. interval for θ, we are going to need appropriate upper and lower statistics. The fact the ~~X~~'s are uniformly distributed suggests that $X_{(1)}$ $(= \min_{i=1..n} X_i)$ and $X_{(n)}$ $(= \max_{i=1..n} X_i)$ might work.

Also since θ is a location parameter, we might as well work with the pivotal r.v's $Y_i = X_i - \theta$, $i=1,\dots,n$, which are $\sim$ iid $U[-\frac{1}{2}, \frac{1}{2}]$.

9.17 (a)(continued)

For $0<c<1$, we will calculate the prob of the event $\{Y_{(1)} > -c$ and $Y_n < c\}$. (The choice of pairs c and $-c$, equidistant from 0, seems reasonable because the distribution of the Y_i's [i.e $U[-\frac{1}{2},\frac{1}{2}]$] is symmetric about 0).

The joint pdf of $Y_{(1)}$ and $Y_{(n)}$ is

$$\begin{cases} n(n-1)[y_n-y_1]^{n-2}, & \text{for } -\frac{1}{2}<y_1<y_2<\frac{1}{2} \\ 0 & \text{elsewhere} \end{cases}$$

Hence $P[Y_{(1)} > -c$ and $Y_n < c]$

$$= \int_{y_n=-c}^{c}\left[\int_{y_1=-c}^{y_n} n(n-1)[y_n-y_1]^{n-2}\,dy_1\right] dy_n$$

$$= \int_{y_n=-c}^{c} n(y_n+c)^{n-1}\,dy_n = (y_n+c)^n\Big]_{y_n=-c}^{y_n=c} = (2c)^n$$

This will equal $1-\alpha$ if we take $(2c)^n = 1-\alpha$, i.e, take $c = \frac{(1-\alpha)^{1/2}}{2}$ $(\in (0,1))$.

Now let's write $Y_{(1)} = X_{(1)} - \theta$ and $Y_{(n)} = X_{(n)} - \theta$ in order

(continued)

9.17 (a) (continued)

to turn this prob. statement into a confidence set for θ.

With $c = \frac{(1-\alpha)^{1/n}}{2}$, we have

$$1-\alpha = P_\theta\left[X_{(1)}-\theta > -c \text{ and } X_{(n)}-\theta < c\right]$$
$$= P_\theta\left[\theta < X_{(1)}+c \text{ and } \theta > X_{(n)}-c\right]$$
$$= P_\theta\left[X_{(n)}-c < \theta < X_{(1)}+c\right].$$

Hence the random set $\{\theta : X_{(n)}-c < \theta < X_{(1)}+c\}$ is a $1-\alpha$ confidence set for θ. However, I am not sure that such a confidence set should be called a confidence interval because there is a positive probability that the confidence set is the empty set; that is, $P_\theta[X_{(n)}-c \le X_{(1)}+c] < 1$. To see this, note that the event $X_{(n)}-c \le X_{(1)}+c$ is equivalent to $X_{(n)}-X_{(1)} \le 2c = (1-\alpha)^{1/n} < 1$, which fails with positive prob. for $0<\alpha<1$, even though $P_\theta[X_{(n)}-X_{(1)} \le 1] = 1$.

9.17 (b) If X has density $f(x|\theta) = \frac{2x}{\theta^2}$, $0<x<\theta$,
it appears that $Y = \frac{X}{\theta}$ has a positive density on $(0,1)$ and so may be pivotal: we calculate $g(y) = f(\theta y)\cdot\theta = \frac{2\theta y}{\theta^2}\cdot\theta = 2y$, $0<y<1$.

Now let $X_1, \ldots, X_n \stackrel{iid}{\sim} f(x|\theta)$, and set $Y_i = \frac{X_i}{\theta}$, $i=1,\ldots,n$. It seems reasonable to construct a C.I. based on $X_{(n)}$. Writing $Y_{(n)} = \frac{X_{(n)}}{\theta}$, we have, for every $y_0 \in (0,1)$, $P[Y_{(n)} \le y_0] = \{P[Y_1 \le y]\}^n$
$= (y_0^2)^n = y_0^{2n}$.

Given $0<1-\alpha<1$, let a and b $(0<a<b<1)$ be any two numbers satisfying $b^{2n} - a^{2n} = 1-\alpha$.
Then $P[a < Y_{(n)} < b] = b^{2n} - a^{2n} = 1-\alpha$.
Hence $1-\alpha = P_\theta\left[a < \frac{X_{(n)}}{\theta} < b\right] = P_\theta\left[\frac{1}{b} < \frac{\theta}{X_{(n)}} < \frac{1}{a}\right]$
$$= P_\theta\left[\frac{X_{(n)}}{b} < \theta < \frac{X_{(n)}}{a}\right].$$
So the random interval $\left[\frac{X_{(n)}}{b}, \frac{X_{(n)}}{a}\right]$ is a $1-\alpha$ conf. interval for θ.

9.23 (Incomplete solution) I am just going to indicate how to invert a LRT for Poisson λ, omitting comparisons with other intervals and omitting numerical calculations.

$X_1, \ldots, X_n$ has joint pdf $\prod_{i=1}^{n} \left[e^{-\lambda} \frac{\lambda^{x_i}}{x_i!} \right] =$

$= e^{-n\lambda} \frac{\lambda^{\Sigma x_i}}{\prod x_i!}$ for $\lambda > 0$ and for $x_i = 0, 1, 2, \ldots$
$i = 1, \ldots, n$.

The MLE of λ is $\hat{\lambda} = \frac{\Sigma x_i}{n}$ and so the LRT test of $H_0: \lambda = \lambda_0$ vs $H_1: \lambda \neq \lambda_0$ rejects H_0 when

$$\frac{e^{-n\lambda_0} \lambda_0^{n\hat{\lambda}}}{e^{-n\hat{\lambda}} \hat{\lambda}^{n\hat{\lambda}}} = e^{n(\hat{\lambda} - \lambda_0)} \left(\frac{\lambda_0}{\hat{\lambda}}\right)^{n\hat{\lambda}}$$

is too small,

or equivalently when $n(\hat{\lambda} - \lambda_0) - n\hat{\lambda}(\log \hat{\lambda} - \log \lambda_0)$ is too small.

Now $\hat{\lambda}$ is discrete (its possible values are all multiples of $\frac{1}{n}$), but if we extend its domain to all positive λ, then the calculation

$$\frac{d}{d\lambda}\left[n(\lambda - \lambda_0) - n\lambda(\log \lambda - \log \lambda_0)\right] = n - n(\log \lambda - \log \lambda_0) - n\lambda \frac{1}{\lambda}$$

9.23 (continued)

$$= -n(\log \lambda - \log \lambda_0)$$

shows that $n(\lambda-\lambda_0) - n\lambda(\log\lambda - \log\lambda_0)$ is increasing for $\lambda < \lambda_0$ and decreasing for $\lambda > \lambda_0$. But this is clearly also true for the discrete set of possible $\hat{\lambda}$. Hence ~~the~~ any region where the likelihood ratio exceeds a constant k, has the form $\{\hat{\lambda} < C_1 \text{ or } \hat{\lambda} > C_2\}$. Typically exact size α is not attainable. One can proceed by trial-and-error to find the k for which the corresponding $P_{\lambda=\lambda_0}[\hat{\lambda} < C_1 \text{ or } \hat{\lambda} > C_2]$ is $\leq \alpha$ and as close to α as possible. Then $P_{\lambda=\lambda_0}[C_1 \leq \hat{\lambda} \leq C_2] \geq 1-\alpha$. One can invert the acceptance region to obtain a $1-\alpha$ conf. set.

Observe $\hat{\lambda}$ and include λ_0 in the confidence set for λ if and only if for the values of C_1 and C_2 corresponding to λ_0, we have $C_1 \leq \hat{\lambda} \leq C_2$.

9.45 (a) For $\lambda_1 < \lambda_0$, $\frac{\prod f(x_i|\lambda_1)}{\prod f(x_i|\lambda_0)} = \frac{(1/\lambda_1^n)\, e^{-\Sigma x_i/\lambda_1}}{(1/\lambda_0^n)\, e^{-\Sigma x_i/\lambda_0}}$

$= \frac{\lambda_0^n}{\lambda_1^n} e^{-\Sigma x_i \left[\frac{1}{\lambda_1} - \frac{1}{\lambda_0}\right]}$. This is a monotone increasing function of $-\Sigma x_i$, so by the theorem for tests with MLR, the UMP test rejects H_0 when $-\Sigma x_i$ is too large, i.e. when $\sum_{i=1}^n x_i < k$ for k satisfying $P_{\lambda_0}[\Sigma x_i < k] = \alpha$

To find k, given α, we note that under $H_0: \lambda = \lambda_0$,

$$\frac{\Sigma x_i}{\lambda_0} \sim \text{Gamma}(\alpha = n, \beta = 1)$$

Chi-squared percentage points are more conveniently tabulated, so we use the equivalent fact that

$$\frac{2\Sigma x_i}{\lambda_0} \sim \chi^2_{2n} \quad \text{under } H_0.$$

So UMP α-level test rejects H_0 when $\frac{2\Sigma x_i}{\lambda_0} < \chi^2_{2n,\alpha}$

textbook's notation for value with prob α in <u>lower</u> tail.

(b) By a theorem, we can invert the acceptance region of the UMP test to obtain a UMA confidence interval.

The acceptance region $\frac{2\Sigma x_i}{\lambda_0} \geq \chi^2_{2n,\alpha}$

(continued)

9.45 (b) (continued)

is equivalent to $\lambda_0 \le \dfrac{2\sum X_i}{\chi^2_{2n,\alpha}}$.

Hence a UMA $1-\alpha$ conf. interval is

$$C^*(X_1,\dots,X_n): \left\{\lambda : 0 \le \lambda \le \frac{2\sum X_i}{\chi^2_{2n,\alpha}}\right\}$$

(c) The expected length of $C^*(X_1,\dots,X_n)$

is $$E_\lambda\left[\frac{2\sum X_i}{\chi^2_{2n,\alpha}} - 0\right] = \frac{2n\lambda}{\chi^2_{2n,\alpha}}.$$