STAT 723 - Solutions to Exam #1

(1) (a) By the Neyman-Pearson lemma, the MP test rejects H_0 when

$$\frac{\binom{4}{x}\left(\frac{2}{3}\right)^x\left(\frac{1}{3}\right)^{4-x}}{\binom{4}{x}\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{4-x}} \text{ is too large;}$$

or when $\left(\frac{4}{3}\right)^x\left(\frac{2}{3}\right)^4\left(\frac{2}{3}\right)^{-x}$ is too large;

or $\left(\frac{4}{3}\frac{3}{2}\right)^x$ is too large; or 2^x is too large.

So the MP level α test rejects H_0 when $X \geq k$, where k is determined by the side condition $P_{H_0}[X \geq k] = \alpha = 5/16$.

Since $P_{H_0}(X=4) = \frac{1}{16}$ and $P_{H_0}(X=3) = \binom{4}{3}\left(\frac{1}{2}\right)^4 = \frac{4}{16}$, we have $P_{H_0}[X \geq 3] = 5/16$. So the MP level $\alpha = 5/16$ test rejects when $X \geq 3$.

(b) $P_{H_1}[\text{Reject } H_0] = P_{p=2/3}[X \geq 3] =$

$$= 4\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right) + \left(\frac{2}{3}\right)^4 = \frac{32}{81} + \frac{16}{81} = \underline{\underline{\frac{48}{81}}}$$

(continued)

1 (c) We need to show that the $Bin(n=4, p)$ family has MLR in X. Let $0 < p_1 < p_2 < 1$.

Then
$$\frac{P_{p_2}[X=x]}{P_{p_1}[X=x]} = \frac{\binom{n}{x} p_2^x (1-p_2)^{4-x}}{\binom{n}{x} p_1^x (1-p_1)^{4-x}}$$

$$= \Big[\underbrace{\frac{p_2}{p_1}}_{>1} \cdot \underbrace{\frac{1-p_1}{1-p_2}}_{>1}\Big]^x \underbrace{\left(\frac{1-p_2}{1-p_1}\right)^4}_{>0}$$

is an increasing function of x (for $x = 0, 1, \ldots, n$).

By definition, the family has MLR in X. It follows, by a theorem that we studied, that the test with rejection region $X \geq k$ is UMP for testing $H_0: p \leq 1/2$ versus $H_1: p > 1/2$, where k is determined by the side condition $P_{p=1/2}[X \geq k] = \alpha = \frac{5}{16}$.

As we saw in part (a), $k = 3$. So the UMP level $\alpha = \frac{5}{16}$ test rejects H_0 when $X \geq 3$.

(2) (a) The likelihood function is

$$L(x_1,\dots,x_n|\theta) = \prod_{i=1}^{n}\left[\frac{1}{\sqrt{2\pi}}e^{-(x_i-\theta)^2/2}\right] = \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\sum_1^n (x_i-\theta)^2},$$

As θ ranges over $\mathbb{R}$, the max. of the likelihood function is obtained at the value $\hat{\theta}$ which minimizes $\sum_1^n (x_i-\theta)^2$, i.e., at $\hat{\theta}=\bar{X}$. By definition, the LR test rejects $H_0: \theta=0$ when $\dfrac{L(x_1,\dots,x_n|0)}{L(x_1,\dots,x_n|\bar{X})}$ is too small.

OR, when $e^{-\frac{1}{2}\sum x_i^2 + \sum(x_i-\bar{x})^2}$ is too small;

OR, when $\sum x_i^2 - (\sum x_i^2 - 2n\bar{x}^2 + n\bar{x}^2)$ is too large;

OR, when $n\bar{X}^2$ is too large;

OR when $(\sqrt{n}\,\bar{X})^2$ is too large.

Under $H_0: \theta=0$, the distribution of $\sqrt{n}\,\bar{X}$ $(=\bar{X}/(1/\sqrt{n}))$ is $N(0,1)$, so that the level α LR test must have the form: Reject H_0 if $\sqrt{n}\,\bar{X} > k$ or $\sqrt{n}\,\bar{X} < -k$ with $k = z_{\alpha/2}$ (since this choice of k yields $P_{H_0}[\text{Reject } H_0] = P_{H_0}[|\sqrt{n}\,\bar{X}| > k] = P[\underbrace{|Z|}_{N(0,1)} > z_{\alpha/2}] = \alpha.$).

2 (b) $P_{\theta_1}[\text{Type II error}] = P_{\theta_1}[\text{accept } H_0]$

$$= P_{\theta_1}\left[-z_{\alpha/2} \le \sqrt{n}\,\bar{X} \le z_{\alpha/2}\right]$$

$$= P_{\theta_1}\left[-z_{\alpha/2} - \sqrt{n}\,\theta_1 \le \sqrt{n}\,(\bar{X}-\theta_1) \le z_{\alpha/2} - \sqrt{n}\,\theta_1\right]$$

$$= \Phi\left[z_{\alpha/2} - \sqrt{n}\,\theta_1\right] - \Phi\left[-z_{\alpha/2} - \sqrt{n}\,\theta_1\right],$$

since the distribution of $\sqrt{n}(\bar{X}-\theta_1)$ is standard normal when $\theta = \theta_1$.

3 The random interval will be shown to be a confidence interval for θ if we can show that

$P_\theta[0 \le \theta \le C\,X_{(n)}]$ has the same value for all $\theta > 0$.

The structure of the problem suggests introducing the random variables $Y_i = X_i/\theta$ for $i = 1, \dots, n$.

Note that the Y_i's are iid each with pdf $f(y) = 3y^2$, $0<y<1$
$= 0$ elsewhere,

so that the Y_i's are pivotal (i.e., distribution does not depend on θ).

We calculate

$$P_\theta[0 \le \theta \le C X_n] = P_\theta\left[X_{(n)} \ge \frac{\theta}{C}\right] = 1 - P_\theta\left[X_{(n)} \le \frac{\theta}{C}\right] =$$

(continued)

(3) (continued)

$$= 1 - P_\theta\left[\frac{X_{(n)}}{\theta} \le \frac{1}{c}\right] = 1 - P\left[Y_{(n)} \le \frac{1}{c}\right]$$

$$= 1 - P\left[Y_1 \le \frac{1}{c}, Y_2 \le \frac{1}{c}, \ldots, Y_n \le \frac{1}{c}\right]$$

$$= 1 - \left\{P\left[Y_1 \le \frac{1}{c}\right]\right\}^n = 1 - \left[F_Y\left(\frac{1}{c}\right)\right]^n = 1 - \left(\frac{1}{c^3}\right)^n = 1 - \frac{1}{c^{3n}}.$$

Hence the random interval is a CI for θ. To find the value of C which yields confidence coefficient $1-\alpha$, we solve $1 - \alpha = 1 - \frac{1}{c^{3n}}$,

which yields $C = \alpha^{-1/(3n)}$.

(4) The first thing to note is that the test with rejection region $\{X < -3 \text{ or } X > 1\}$ has size $\alpha = P_{H_0}[X < -3 \text{ or } X > 1] = \Phi(-3) + 1 - \Phi(1)$.

So to show that this test is the most powerful of its size, it remains to show that the test ~~[illegible]~~ coincides with the MP test of H_0 vs H_1 as given by the Neyman-Pearson Lemma. (continued)

4 (continued)

By the Neyman-Pearson Lemma, the MP test reject H_0 when

$$\frac{\frac{1}{\sqrt{2\pi \cdot 2}}\, e^{-\frac{1}{4}(x-1)^2}}{\frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2}} \quad \text{is too large;}$$

OR, when $-\frac{1}{4}(x-1)^2 + \frac{1}{2}x^2$ is too large;

OR, when $-(x^2-2x+1)+2x^2$ is too large

OR, when x^2+2x is too large;

OR, when $x^2+2x+c \geq 0$ for some real c.

For this rejection region to be equivalent to $\{x<-3 \text{ or } x>1\}$, we must have that -3 and 1 are the zeros of the quadratic x^2+2x+c. Noting that $(x+3)(x-1)=x^2+2x-3$, we see that

$x^2+2x>3$ if and only if $\{x<-3 \text{ or } x>1\}$. Hence the given test is of the Neyman-Pearson form and so is the MP test of its size.

4 (continued)

Here is an alternative solution, which is equivalent to the above but keeps track of the Neyman-Pearson k:

The MP test rejects H_0 when

$$\frac{\frac{1}{\sqrt{2\pi}\sqrt{2}} e^{-\frac{1}{4}(x-1)^2}}{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}} > k$$

OR $\quad -\frac{1}{4}(x-1)^2 + \frac{1}{2}x^2 > \log(\sqrt{2}k)$

OR $\quad x^2 + 2x - 1 > 4\log(\sqrt{2}k)$

OR $\quad x^2+2x-3 = (x+3)(x-1) > 4\log(\sqrt{2}k) - 2.$

This rejection region will be equivalent to

$(x+3)(x-1) > 0 \quad$ (i.e, $x<-3$ or $x>1$),

with ~~the~~ k taken to satisfy

$4\log(\sqrt{2}k) - 2 = 0$;

that is, $k = (e/2)^{1/2}$.