STAT 723 Solutions to Assignment #1

(8.6) (a) The joint pdf of $X_1, \dots, X_n, Y_1, \dots, Y_m$ is

$$\frac{1}{\theta^n} e^{-\frac{1}{\theta}\sum x_i} \cdot \frac{1}{\mu^m} e^{-\frac{1}{\mu}\sum y_i} \qquad \text{all } x_i, y_i > 0.$$

The MLE's of θ and μ are $\hat{\theta} = \frac{\sum_1^n x_i}{n}$, $\hat{\mu} = \frac{\sum_1^m y_i}{m}$,

so that the denominator of the LR test statistic is

$$\frac{1}{\left(\frac{\sum x_i}{n}\right)^n} e^{-n} \cdot \frac{1}{\left(\frac{\sum y_i}{m}\right)^m} e^{-m}.$$

Under $H_0: \theta = \mu$, the MLE of $\theta\,(=\mu)$ is $\frac{\sum x_i + \sum y_i}{n+m}$,

so the numerator of the LR test stat is

$$\frac{1}{\left(\frac{\sum x_i + \sum y_i}{n+m}\right)^{n+m}} e^{-n-m}.$$

So the LR test rejects H_0 when $\frac{(\text{const.})\left(\sum x_i\right)^n \left(\sum y_i\right)^m}{\left(\sum x_i + \sum y_i\right)^{n+m}}$

is too small.

(b) With $T = \frac{\sum x_i}{\sum x_i + \sum y_i}$, the LR test statistic

is clearly equivalent to $T^n(1-T)^m$. Note that since $t^n(1-t)^m$ is increasing for $t \in (0, n/(n+m))$ and decreasing for $t > n/(n+m)$, then rejecting H_0 when $T^n(1-T)^m$ is $\le c$ is equivalent to rejecting when $T \le c_1$ or $T \le c_2$ for some c_1, c_2.

8.6 (c) Write $T = \dfrac{1}{1 + (\Sigma Y_i)/(\Sigma X_i)}$.

Under H_0, ΣX_i and ΣY_i are indep., with

$\dfrac{\Sigma X_i}{\theta} \sim$ gamma with $\alpha = n$, $\beta = 1$

and $\dfrac{\Sigma Y_i}{\theta} \sim$ gamma, $\alpha = m$, $\beta = 1$.

Equivalently, $\dfrac{2\Sigma X_i}{\theta} \sim \chi^2_{2n}$, $\dfrac{2\Sigma Y_i}{\theta} \sim \chi^2_{2m}$,

so that $\dfrac{\Sigma Y_i / m}{\Sigma X_i / n} \sim F_{m,n}$.

So the null distribution of $F = \dfrac{1}{1 + \frac{m}{n} T}$ is $F_{m,n}$,

or the null distribution of T is the distribution of

$\left(\frac{1}{F} - 1\right)\frac{n}{m}$ where $F \sim F_{m,n}$. You can easily work out the p.d.f. of T under H_0 by making a 1-1 change of variables in the $F_{m,n}$ p.d.f.

<u>Note</u>. Writing the null density of T as $f_0(t)$, the α-level L-R tests rejects H_0 when $T < c_1$ or $T > c_2$,

where c_1 and c_2 are uniquely determined by the 2 side

conditions: (i) $c_1^n (1-c_1)^m = c_2^n (1-c_2)^m$

and (ii) $\int_0^{c_1} f_0(t)\,dt + \int_{c_1}^{\infty} f_0(t) = \alpha$.

It can be shown that the two integrals above are typically <u>not</u> each $= \alpha/2$.

(8.8) (a) Note: in the textbook's notation $N(\theta, a\theta)$ means a normal with $\mu = \theta$ and variance $\sigma^2 = a\theta$.

The joint pdf of $X_1, \dots, X_n$ is

$$L(\theta, a) = \left(\frac{1}{\sqrt{2\pi a\theta}}\right)^n e^{-\frac{1}{2a\theta}\Sigma(X_i - \theta)^2}$$

We need to find $\hat{a}, \hat{\theta}$ which maximize

$$\log L(\theta, a) = -\frac{n}{2}\left(\log(2\pi) + \log a + \log\theta\right) - \frac{1}{2a\theta}\Sigma(X_i - \theta)^2.$$

We will solve $\frac{d}{da}\log L = 0$, $\frac{\partial}{\partial\theta}\log L = 0$. If there is a unique soln $(\hat{\theta}, \hat{a})$, then since it (a prior) clear that the sup cannot be found ~~at the~~ as the boundary of the parameter space is approached, then the soln $(\hat{\theta}, \hat{a})$ must be a unique global maximum.

$$\frac{d}{d\theta}L = 0 \Rightarrow -\frac{n}{2\theta} - \frac{2}{2a\theta}\Sigma(X_i - \theta)(-1) + \frac{1}{2a\theta^2}\Sigma(X_i - \theta)^2 = 0$$

$$\frac{\partial}{\partial a}L = 0 \Rightarrow -\frac{n}{2a} + \frac{1}{2a^2\theta}\Sigma(X_i - \theta)^2 = 0.$$

The solution to the 2^{nd} equation is $a = \frac{\Sigma(X_i - \theta)^2}{n\theta}$

Substituting this into the first equation yields $\theta = \bar{x}$.

Hence the MLE of (θ, a) is $\left(\bar{X}, \frac{\Sigma(X_i - \hat{\theta})^2}{n\bar{X}}\right) = \left(\bar{X}, \frac{S^2}{\bar{X}}\right)$.

So the denom. of the LR statistic is

$$\frac{1}{\sqrt{2\pi(S^2/\bar{X})\bar{X}}} e^{-\frac{1}{2(S^2/\bar{X})\bar{X}} nS^2} = \frac{1}{\sqrt{2\pi S^2}} e^{-n/2}$$

(continued)

8.8 (a) (continued)

Under $H_0: a=1$, the log likelihood function is

$$\log L(\theta) = -\frac{n}{2}\left[\log(2\pi) + \log\theta\right] - \frac{1}{2\theta}\sum(X_i-\theta)^2$$

$$\frac{d}{d\theta}\log L = 0 \Rightarrow -\frac{n}{2} + \frac{1}{\theta}\sum(X_i-\theta) + \frac{1}{2\theta^2}\sum(X_i-\theta)^2 = 0$$

or $\sum(X_i-\theta)^2 + 2\theta\left(\sum X_i - n\theta\right) - n\theta = 0$

or

$$\sum X_i^2 - 2\theta\sum X_i + n\theta^2 + 2\theta\sum X_i - 2n\theta^2 - n\theta = 0$$

or

$$n\theta^2 + n\theta - \frac{\sum X_i^2}{2} = 0, \text{ so } \theta = \frac{-1 \pm \sqrt{1+4(\sum X_i^2)/n}}{2}.$$

The restriction $\theta > 0$ (so Var > 0) forces the MLE of θ under H_0 to be the positive root:

$$\hat{\theta} = -1 + \sqrt{1+4(\sum X_i)^2/n}.$$

So the LR test rejects H_0 when λ is too small, where

$$\lambda = \frac{\left(\frac{1}{\sqrt{2\pi\hat{\theta}}}\right)^n e^{-\frac{1}{2\hat{\theta}}\sum(X_i-\hat{\theta})^2}}{\left(\frac{1}{\sqrt{2\pi S^2}}\right)^n e^{-n/2}}$$

where $\hat{\theta}$ is given above. This test statistic can be simplified a bit.

(b) This is similar to (a) (and probably just as tedious) so I am omitting the solution.

(8.15) By the N-P lemma, ~~a~~ MP test rejects H_0 when

$$\left[\left(\frac{1}{\sqrt{2\pi\sigma_1^2}}\right)^n e^{-\frac{1}{2\sigma_1^2}\sum X_i^2}\right] \Big/ \left(\frac{1}{\sqrt{2\pi\sigma_0^2}}\right)^n e^{-\frac{1}{2\sigma_0^2}\sum X_i^2} \text{ too large,}$$

Equivalently, reject H_0 when $e^{\left(-\frac{1}{2\sigma_1^2}+\frac{1}{2\sigma_0^2}\right)\sum X_i^2}$ is too large.

Since $\sigma_0 < \sigma_1 \Rightarrow -\frac{1}{2\sigma_1^2}+\frac{1}{2\sigma_0^2} > 0$,

the MP test rejects when $\sum X_i$ is too large.

Under H_0, $\frac{\sum X_i^2}{\sigma_0^2} \sim \chi^2_n$, so $P_0\left[\frac{\sum X_i^2}{\sigma_0^2} > \chi^2_{n,\alpha}\right] = \alpha$

So the MP level α test rejects H_0 when $\sum X_i^2 > \sigma_0^2 \chi^2_{n,\alpha}$.

(8.18) The power against θ is

$$P_\theta\left[\frac{\bar X-\theta_0}{\sigma/\sqrt{n}} > c \text{ or } < -c\right] = P_\theta\left[\frac{\bar X-\theta}{\sigma/\sqrt{n}} + \frac{(\theta-\theta_0)}{\sigma/\sqrt{n}} > c \text{ or } < -c\right]$$

$$= P\left[\underset{N(0,1)}{Z} > c - \frac{(\theta-\theta_0)}{\sigma/\sqrt{n}}\right] + P\left[Z < -c - \frac{(\theta-\theta_0)}{\sigma/\sqrt{n}}\right]$$

$$= 1 - \Phi\left(c - \frac{(\theta-\theta_0)}{\sigma/\sqrt{n}}\right) + \Phi\left(-c - \frac{(\theta-\theta_0)}{\sigma/\sqrt{n}}\right).$$

(b) $\alpha = .05 \Rightarrow c = z_{.025} = 1.960$. At $\theta = \theta_0 + \sigma$, $\frac{\theta-\theta_0}{\sigma} = 1$.
We need to find the smallest value of n such that

$$\Phi(1.96 - \sqrt{n}) - \Phi(-1.96 - \sqrt{n}) \le 0.25$$

I wrote a little program to compute this, and found $n=7$ yields Type II error prob of 0.2764.

8.20

x	1	2	3	4	5	6	7
$f(x\|H_0)$	.01	.01	.01	.01	.01	.01	.94
$f(x\|H_1)$	.06	.05	.04	.03	.02	.01	.79
$r(x) = f(x\|H_1)/f(x\|H_0)$	6	5	4	3	2	1	0.840

By the N-P lemma, the most powerful test has the rejection region $r(x) > C$. The four points $x = 1, 2, 3, 4$ have the highest values of $r(x)$ and a total type I error probability of $\alpha = .01 + .01 + .01 + .01 = .04$. So the most powerful level $\alpha = .04$ test rejects H_0 when $r(x) > 2$, i.e. when $X \in \{1, 2, 3, 4\}$.

The prob of Type II error for this test is

$$P_{H_1}\left[X \notin \{1,2,3,4\}\right] = .06 + .05 + .04 + .03$$

$$= \underline{\underline{0.18}}.$$

8.22 (a) By N-P lemma, MP test rejects H_0 when

$$\frac{\left(\frac{1}{4}\right)^{\Sigma x_i}\left(\frac{3}{4}\right)^{10-\Sigma x_i}}{\left(\frac{1}{2}\right)^{\Sigma x_i}\left(\frac{1}{2}\right)^{10-\Sigma x_i}} = \frac{\left(\frac{3}{4}\right)^{10}\left(\frac{1}{3}\right)^{\Sigma x_i}}{\left(\frac{1}{2}\right)^{10}} \text{ is too large.}$$

Equivalently, reject H_0 when $\sum_{i=1}^{10} x_i$ is too small.

By trial and error $P_{1/2}\left[\Sigma x_i \le 2\right] = \frac{56}{1024} = .0547$.

So MP size 0.0547 test is: Rej H_0 if $\sum_{i=1}^{n} X \le 2$.

The power of this test is $P_{1/4}\left[X \le 2\right] = \left(\frac{3}{4}\right)^{10} + 10\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^{9} + 45\left(\frac{1}{4}\right)^{2}\left(\frac{3}{4}\right)^{8}$
= (do the arithmetic).

(b) Size $\alpha = P_{1/2}\left[\sum_{i=1}^{10} x_i \ge 6\right] = \sum_{x=6}^{10}\binom{10}{x}\left(\frac{1}{2}\right)^{10}$

$= \frac{1}{1024}\left[1+10+45+120+210\right] = \frac{386}{1024}$

(c) Exact non-randomized α-level tests exist when $\alpha = P_{1/2}\left[\sum_{i=1}^{10} x_i \le c\right]$

for some c. That is the possible α's are

$0, \frac{1}{1024}, \frac{11}{1024}, \frac{56}{1024}, \frac{176}{1024}, \ldots\ldots, \frac{1023}{1024}$ and 1.

8.25 (a) $X \sim N(\theta, \sigma^2)$. Fix $\theta_0 < \theta_1$. Then $f(x|\theta_1)/f(x|\theta_0) =$

$$= e^{-\frac{1}{2\sigma^2}\{(X-\theta_1)^2 - (X-\theta_0)^2\}} = \exp\left\{-\frac{1}{2\sigma^2}\left[X^2 - 2\theta_1 x + \theta_1^2 - x^2 + 2\theta_0 x + \theta_0^2\right]\right\}$$

$$= e^{-\frac{1}{2\sigma^2}(\theta_1^2 - \theta_0)^2} \; e^{\frac{1}{\sigma^2}(\theta_1 - \theta_0)x}$$, which is monotone incr. in X.

(b) $\theta_0 < \theta_1$, then $f(x|\theta_1)/f(x|\theta_0) = e^{-\theta_1} \frac{\theta_1^x}{x!} \Big/ \left[e^{-\theta_0} \theta_0^x / x!\right]$

$$= e^{-\theta_1 + \theta_0} \left(\frac{\theta_1}{\theta_0}\right)^x$$, which is increasing in x.

(c) $\theta_0 < \theta_1$, then $f(x|\theta_1)/f(x|\theta_0) = \binom{n}{x}\theta_1^x (1-\theta_1)^{n-x} \Big/ \left[\binom{n}{x}\theta_0^x (1-\theta_0)^{n-x}\right]$

$$= \left(\frac{1-\theta_1}{1-\theta_0}\right)^n \left[\frac{\theta_1}{1-\theta_1} \Big/ \frac{\theta_0}{1-\theta_0}\right]^x$$, which is increasing in x.

8.28 (a) Fix $\theta_0 < \theta_1$. Then $\frac{f(x|\theta_1)}{f(x|\theta_0)}$

$$= \frac{e^{(X-\theta_1)}}{\left(1 + e^{(X-\theta_1)}\right)^2} \frac{\left(1 + e^{(X-\theta_0)}\right)^2}{e^{X-\theta_0}} = e^{\theta_0 - \theta_1} \left[\frac{1 + e^{(X-\theta_0)}}{1 + e^{(X-\theta_1)}}\right]^2.$$

Since $\frac{1 + e^{(X-\theta_0)}}{1 + e^{(X-\theta_1)}}$ is positive for all x, it suffices to show that $\frac{d}{dx}\left[\frac{1 + e^{(X-\theta_0)}}{1 + e^{(X-\theta_1)}}\right] \geq 0$ for all x.

The above derivative has the same sign as

$$\left[1 + e^{(X-\theta_1)}\right] e^{(X-\theta_0)} - \left(1 + e^{(X-\theta_0)}\right) e^{(X-\theta_1)}$$

$$= e^{(x-\theta_0)} - e^{(X-\theta_1)} = e^x e^{(-\theta_0 + \theta_1)} > 0 \text{ for all } x.$$

Thus the family has MLR in X. (continued)

8.28 (continued)

(b) Since the family has MLR, the MP size α test must have rejection region of the form $X > c$.

$$P_0[X > c] = \int_c^\infty \frac{e^x}{(1+e^x)^2}\,dx \overset{v = 1+e^{-x}}{=} \int_{1+e^c}^\infty \frac{dv}{v^2}$$

$$= -\frac{1}{v}\Big]_{1+e^c}^\infty = \frac{1}{1+e^c} = \alpha.$$

So $c = \log\left(\frac{1-\alpha}{\alpha}\right)$, and MP level α test rejects H_0

when $X > \log\left(\frac{1-\alpha}{\alpha}\right)$.

When $\alpha = .2$, reject when $X > \log 4 = 1.3863$.

$$P_{\theta=1}[\text{Reject } H_0] = \int_{\log 4}^\infty \frac{e^{(x-1)}}{(1+e^{(x-1)})^2}\,dx = \int_{y=\log 4 - 1}^\infty \frac{e^y}{(1+e^y)^2}\,dy$$

$$= \frac{1}{1+e^{(\log 4 - 1)}} = \frac{1}{1+4/e} = 0.4046$$

So the Type II error probability is $1 - 0.4046 = 0.5954$.

(c) One of the theorems was that when a family has MLR in X, then the UMP size α test of $H_0: \theta \le \theta_0$ vs $H_1: \theta < \theta_0$ has rejection region of form $X > c$, with c determined by $P_{\theta_0}[X > c] = \alpha$.

8.31 (a) For $\lambda_0 < \lambda_1$,

$$\prod_{i=1}^{n} f(x_i|\lambda_1) \Big/ \prod_{i=1}^{n} f(x_i|\lambda_0) = e^{-n(\lambda_1-\lambda_0)} \left(\frac{\lambda_1}{\lambda_0}\right)^{\Sigma x_i}$$

is monotone increasing in Σx_i. Since this family therefore has MLR, it follows that the test which rejects H_0 when $\sum_{i=1}^{n} X_i \geq c$ is UMP level α for testing $H_0: \lambda \leq \lambda_0$ vs. $H_1: \lambda > \lambda_0$, where $\alpha = P_{\lambda_0}[\Sigma X_i \geq c]$.

(b) ~~[illegible]~~ $\sum_{i=1}^{n} X_i \sim$ Poisson $(n\lambda)$. When n is large, by the CLT, $\dfrac{\Sigma X_i - n\lambda}{\sqrt{n\lambda}}$ is approx $\sim N(0,1)$. So $P_{\lambda=1}[\Sigma X_i \geq c] =$

$$P_{\lambda=1}\left[\frac{\Sigma X_i - n}{\sqrt{n}} \geq \frac{c-n}{\sqrt{n}}\right] \approx P\left[Z \geq \frac{c-n}{\sqrt{n}}\right] = .05$$

This will be achieved by taking $\dfrac{c-n}{\sqrt{n}} \approx 1.645$.

Similarly to achieve

$$P_{\lambda=2}[\Sigma X_i \geq c] = P\left[\frac{\Sigma X_i - 2n}{\sqrt{2n}} \geq \frac{c-2n}{\sqrt{2n}}\right] \approx 0.9,$$

set $\dfrac{c-2n}{\sqrt{2n}} \approx -1.282$.

We now solve these two equations for n.

(continued)

(8.31) (b) continued

$$c - n \approx 1.645\sqrt{n}$$

and ~~$c-2n = 1.645\ldots\sqrt{n}$~~

$$c - 2n \approx -1.282\sqrt{2}\,\sqrt{n}.$$

So $n \approx (1.645 + 1.282\sqrt{2})\sqrt{n}$

or $n \approx (1.645+1.282\sqrt{2})^2 = 11.95$

So take $n = 12$. (Note: this n is too small for the normal approx. to be very accurate. Calculations with exact Poisson probabilities may lead to a somewhat different value of the required n.)

(8.38) (a) The distribution of $(\bar{X}-\theta_0)/\sqrt{S^2/n}$ under $H_0: \theta = \theta_0$ is t_{n-1}. Hence

$$P_{\theta_0}\left[|\bar{X}-\theta_0| > t_{n-1,\alpha/2}\sqrt{S^2/n}\right] = P_{\theta_0}\left[\frac{\bar{X}-\theta_0}{S/\sqrt{n}} > t_{n-1,\alpha/2} \text{ or } < -t_{n-1,\alpha/2}\right]$$

$$= \alpha/2 + \alpha/2 = \alpha.$$

(continued)

8.38 (b)

$$L(\underset{\sim}{x}|\theta,\sigma^2) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{-\frac{1}{2\sigma^2}\sum(x_i-\theta)^2}$$

Since $(\bar{x}, \widehat{\sigma^2})$ is the MLE of (θ, σ^2) (where $\bar{x} = \frac{\sum x_i}{n}$, $\widehat{\sigma^2} = \frac{\sum(x_i - \bar{x})^2}{n}$), the ~~denom~~ denominator of the LR-test statistic is

$$L(\underset{\sim}{x}|\bar{x}, \widehat{\sigma^2}) = \left(\frac{1}{\sqrt{2\pi\widehat{\sigma^2}}}\right)^n e^{-n/2}.$$

The numerator of the LR. test statistic is

$$\sup_{\sigma^2>0} L(\bar{x}|\theta_0, \sigma^2).$$

It is easy to see that this supremum is attained when $\sigma^2 = \widehat{\sigma_*^2} \overset{\text{def}}{=} \frac{\sum(x_i-\theta_0)^2}{n}$, ~~so~~ so that the numerator is

$$\left(\frac{1}{\sqrt{2\pi\widehat{\sigma_*^2}}}\right)^n e^{-n/2};$$

So the LRT when

$$\left(\frac{1}{\sqrt{2\pi\widehat{\sigma_*^2}}}\right)^n \Big/ \left(\frac{1}{\sqrt{2\pi\widehat{\sigma^2}}}\right)^n$$ is too small;

(continued)

8.38 (b) (continued)

OR, equivalently, when $\frac{\hat{\sigma}_*^2}{\hat{\sigma}_0^2}$ is too large.

OR $\frac{\sum(x_i-\theta_0)^2}{\sum(x_i-\bar{x})^2}$ is too large.

OR, when $\frac{\sum(x_i-\bar{x})^2 + n(\bar{x}-\theta_0)^2}{\sum(x_i-\bar{x})^2}$ is too large;

OR $\frac{(\bar{x}-\theta_0)^2}{S^2/n}$ is too large, (where $S^2 = \frac{\sum(x_i-\bar{x})^2}{n-1}$)

OR $\frac{|\bar{x}-\theta_0|}{S/\sqrt{n}}$ is too large.

Since $\frac{\bar{X}-\theta_0}{S/\sqrt{n}} \sim t_{n-1}$ under $H_0: \theta=\theta_0$,

the size α LR test rejects H_0 when

$$\frac{|\bar{X}-\theta_0|}{\sqrt{S^2/n}} > t_{n-1,\alpha/2}. \checkmark$$