STAT 723

Solutions to Assignment #4

(11.8) $\bar{Y}_{i.} \sim N(\theta_i, \frac{\sigma^2}{n_i})$ for $i=1,\ldots,k$; and the $\bar{Y}_{i.}$'s are indep., so by Cor 4.6.10, $\sum a_i \bar{Y}_{i.}$ is normally distributed with mean $\sum a_i \theta_i$ and variance $\sum a_i^2 \frac{\sigma^2}{n_i} = \sigma^2 \sum \frac{a_i^2}{n_i}$.

(11.21) (a)

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square	F statistic
Treatments	2	1228.27	614.14	110.43
Within	12	66.74	5.56	—
Total	14	1295.01	—	—

(b) The right-hand side of the last formula on page 536 is

$$= \sum_{i=1}^{k}\sum_{j=1}^{n_i}(y_{ij}-\bar{y}_{i.}) + \sum_{i=1}^{k}\sum_{j=1}^{n_i}(\bar{y}_{i.}-\bar{\bar{y}})^2$$

$$+2\sum_{i=1}^{k}\sum_{j=1}^{n_i}(y_{ij}-\bar{y}_{i.})(\bar{y}_{i.}-\bar{\bar{y}})$$

(continued)

11.21 (b) (continued)

The second of the 3 terms $= \sum_{i=1}^{k} n_i (\bar{y}_{i\cdot} - \bar{\bar{y}})^2$,

and the third term $= 2 \sum_{i=1}^{k} (\bar{y}_{i\cdot} - \bar{\bar{y}}) \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_{i\cdot})$

$$= 2 \sum_{i=1}^{k} (\bar{y}_{i\cdot} - \bar{\bar{y}}) \left[n_i \bar{y}_{i\cdot} - n_i \bar{y}_{i\cdot} \right]$$

$$= 2 \sum_{i=1}^{k} \left[(\bar{y}_{i\cdot} - \bar{\bar{y}}) \cdot 0 \right] = 0.$$

This completes the proof of Thm. 11.2.11.

(c) Omitted — not a meaningful exercise because I did not assign problem 11.12.

11.27

The problem is to find the vector $(d_1, \ldots, d_n)$ such that $E_{\alpha,\beta}(\sum d_i Y_i) = \alpha$ for all α, β — and such that the variance of $\sum d_i Y_i$ is minimized.

Now $\alpha = E\left[\sum_{i=1}^{n} d_i Y_i\right] = \sum d_i (\alpha + \beta x_i)$

$$= \alpha \left(\sum_{i=1}^{n} d_i\right) + \beta \sum_{i=1}^{n} d_i x_i \quad \text{for all } \alpha, \beta$$

$\Longrightarrow \sum d_i = 1$ and $\sum d_i x_i = 0$

As shown before, the variance of $\sum d_i Y_i$ is $\sigma^2 \sum d_i^2$.

(continued)

11.27 (continued)

So the problem is to minimize $\sum d_i^2$ subject to $\sum d_i = 1$ and $\sum d_i x_i = 0$. There must exist Lagrange multipliers λ_1 and λ_2 such that $(d_1, \ldots, d_n)$ is the (unconstrained) minimizer of

$$\sum d_i^2 + \lambda_1 \left(\sum d_i - 1\right) + \lambda_2 \sum d_i x_i .$$

The partial derivative of the above with respect to each d_j, $j = 1, \ldots, n$, must necessarily vanish, so

$$2d_j + \lambda_1 + \lambda_2 x_j = 0 \quad \text{for } j = 1, \ldots, n.$$

Equivalently, there must exist constants A and B such that the minimizing d_i's satisfy

$$d_i = A + B x_i \quad \text{for } i = 1, \ldots, n.$$

$\sum d_i = 1 \Rightarrow nA + B \sum x_i = 1$ OR $A + B\bar{x} = \frac{1}{n}$

Also $\sum d_i x_i = 0 \Rightarrow \sum (A + Bx_i) x_i = 0$ OR $A \sum x_i + B \sum x_i^2 = 0$.

We solve these 2 equations for A and B by first eliminating A.

(continued)

(11.27) Continued)

Write $nA\sum x_i + B(\sum x_i)^2 = \sum x_i$

and $nA\sum x_i + nB\sum x_i^2 = 0$

and subtract to get $nB\left(\sum x_i^2 - (\sum x_i)^2/n\right) = -\sum x_i$,

so $B = \dfrac{-\sum x_i/n}{S_{xx}} = -\dfrac{\bar{x}}{S_{xx}}$,

and $A = \frac{1}{n} - B\bar{x} = \frac{1}{n} + \dfrac{\bar{x}^2}{S_{xx}}$.

Hence $d_i = A + Bx_i = \frac{1}{n} + \dfrac{\bar{x}^2}{S_{xx}} - \dfrac{\bar{x}}{S_{xx}}x_i$

$= \frac{1}{n} - \dfrac{\bar{x}}{S_{xx}}(x_i - \bar{x})$, $i = 1, \ldots, n$.

So the BLUE is $\sum d_i Y_i = \sum\left[\frac{1}{n} - \frac{\bar{x}}{S_{xx}}(x_i - \bar{x})\right]Y_i$

$= \bar{Y} - \dfrac{\bar{x}}{S_{xx}}\sum(x_i - \bar{x})Y_i = \bar{Y} - \dfrac{\bar{x}}{S_{xx}}\sum(x_i - \bar{x})(Y_i - \bar{Y})$

$= \bar{Y} - \dfrac{\bar{x}}{S_{xx}}S_{xY} = \bar{Y} - \bar{x}\dfrac{S_{xY}}{S_{xx}} = \underline{\underline{\bar{Y} - \hat{\beta}\bar{x}}}$,

where $\hat{\beta} = S_{xY}/S_{xx}$.

11.28 Write $Q(\underset{\sim}{x}, \underset{\sim}{y}) = \sum_{i=1}^{n} (Y_i - \hat{\alpha} - \hat{\beta} x_i)^2$

At the middle of p. 551 of the text, it was shown that the MLE of σ^2 is the value of $\sigma^2 (>0)$ which maximizes

$$L(\sigma^2) = -\frac{n}{2} \log(2\pi) - \frac{n}{2} \log \sigma^2 - \frac{Q(\underset{\sim}{x}, \underset{\sim}{y})}{2\sigma^2}$$

Then $\frac{d}{d\sigma^2} L(\sigma^2) = \frac{-n}{2\sigma^2} + \frac{Q(\underset{\sim}{x}, \underset{\sim}{y})}{2(\sigma^2)^2}$,

which $= 0$ when $\sigma^2 = \frac{Q(\underset{\sim}{x}, \underset{\sim}{y})}{n}$
> 0 when $\sigma^2 < Q/n$
< 0 when $\sigma^2 > Q/n$.

Hence a global maximum is attained at $\sigma^2 = \frac{Q(\underset{\sim}{x}, \underset{\sim}{y})}{n}$

So $\hat{\sigma}^2 = \frac{Q}{n} = \frac{\sum (Y_i - \hat{\alpha} - \hat{\beta} x_i)^2}{n}$.

11.30 (a) was proved in the solution of 11.27

(b) $E\hat{\alpha} = \alpha$, because we proved in problem 11.27 that $\hat{\alpha}$ was the (best) unbiased est of α.

$$\text{Var}(\hat{\alpha}) = \sigma^2 \sum c_i^2 = \sigma^2 \sum \left[\frac{1}{n} - \frac{(x_i - \bar{x})\bar{x}}{S_{xx}}\right]^2 =$$

(continued)

11.30 (b) (continued)

$$= \sigma^2 \sum_{i=1}^{n} \left[\frac{1}{n^2} - \frac{2}{n} \frac{(x_i - \bar{x})\bar{x}}{S_{xx}} + \frac{(x_i - \bar{x})^2 \bar{x}^2}{S_{xx}^2} \right]$$

$$= \sigma^2 \left[\frac{1}{n} - 0 + \frac{\sum (x_i - \bar{x})^2 \bar{x}^2}{S_{xx}^2} = \frac{1}{n} + \frac{S_{xx}\bar{x}^2}{S_{xx}^2} \right]$$

$$= \left[\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}} = \frac{S_{xx} + n\bar{x}^2}{n S_{xx}} \right] \sigma^2$$

$$= \sigma^2 \left[\frac{\sum x_i^2 - n\bar{x}^2 + n\bar{x}^2}{n S_{xx}} \right] = \frac{\sum x_i^2}{n S_{xx}} \sigma^2 .$$

(c) Using Lemma 11.3.2 on page 551 of the text, we have $\mathrm{Cov}(\hat{\alpha}, \hat{\beta}) = \mathrm{Cov}(\sum c_i Y_i, \sum d_i Y_i)$

$\left(\text{where } c_i = \frac{x_i - \bar{x}}{S_{xx}},\ d_i = \frac{1}{n} - \frac{(x_i - \bar{x})\bar{x}}{S_{xx}} \right)$

$$= \sigma^2 \sum c_i d_i = \sigma^2 \sum_{i=1}^{n} \frac{x_i - \bar{x}}{S_{xx}} \left[\frac{1}{n} - \frac{\bar{x}}{S_{xx}} (x_i - \bar{x}) \right]$$

$$= 0 - \frac{\sigma^2 \bar{x}}{S_{xx}^2} \sum_{i=1}^{n} (x_i - \bar{x})^2 = -\frac{\bar{x}\sigma^2}{S_{xx}^2} S_{xx} = -\frac{\bar{x}}{S_{xx}} \sigma^2 .$$

11.34 (a)

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Squares	F
Regression (Slope)	1	60.357	60.357	50.70
Residual	6	7.143	1.190	—
Total	7	67.500	—	

(b) To show 11.3.36, it clearly suffices to show that the "cross-product" term vanishes, i.e. that $2\sum(\hat{y}_i-\bar{y})(y_i-\hat{y}_i)=0$.

Writing $\hat{y}_i=\hat{\alpha}+\hat{\beta}x_i$, and recalling that $\bar{y}=\hat{\alpha}+\hat{\beta}\bar{x}$, we have that $\hat{y}_i-\bar{y}=\hat{\beta}(x_i-\bar{x})$, $i=1,\ldots,n$.

Hence $\sum(\hat{y}_i-\bar{y})(\hat{y}_i-y_i)=\hat{\beta}\sum(x_i-\bar{x})(\hat{\alpha}+\hat{\beta}x_i-y_i)$

$=\hat{\beta}\left[0+\hat{\beta}\sum(x_i-\bar{x})x_i-\sum(x_i-\bar{x})y_i\right]$

$=\hat{\beta}\left[\hat{\beta}S_{xx}-S_{xy}\right]=\hat{\beta}\left[\frac{S_{xy}}{S_{xx}}S_{xx}-S_{xy}\right]=0.$ ✓

(c) $\sum(\hat{y}_i-\bar{y})^2=\sum\hat{\beta}^2(x_i-\bar{x})^2=\hat{\beta}^2S_{xx}=\frac{S_{xy}^2}{S_{xx}^2}\cdot S_{xx}=\frac{S_{xy}^2}{S_{xx}}.$

(continued)

11.34 (d) For the pairs $(Y_1, X_1), \ldots, (Y_n, X_n)$, the sample correlation coefficient is $r = \frac{S_{XY}}{\sqrt{S_{XX} S_{YY}}}$,

so $$r^2 = \frac{S_{XY}^2}{S_{XX} S_{YY}} = \frac{S_{XY}^2 / S_{XX}}{S_{YY}} = \frac{\sum (\hat{Y}_i - \bar{Y})^2}{\sum (Y_i - \bar{Y})^2}.$$

by part (c)

Also, since $\hat{Y}_i = \hat{\alpha} + \hat{\beta} x_i$ for $i = 1, \ldots, n$, we have $r_{Y,\hat{Y}} = r_{Y,X}$, since (it is easy to see that) the correlation coefficient is invariant under any (nondegenerate) linear transformation of one of the two variables (or even both variables). So $r^2_{Y,\hat{Y}} = r^2_{Y,X}$.

11.35 (a) The least squares estimator is the value of θ which minimizes $\sum (Y_i - \theta x_i^2)^2$:

Set $0 = \frac{d}{d\theta} \sum (Y_i - \theta x_i^2)^2 = 2 \sum (Y_i - \theta x_i^2)(-x_i^2)$

or $\sum x_i^2 Y_i = \theta \sum x_i^4$, so $\hat{\theta} = \frac{\sum x_i^2 Y_i}{\sum x_i^4}$

(the unique extremum must be the global ~~extremum~~ minimum).

(continued)

11.35 (b) The likelihood function is

$$L(\theta)=\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{-\frac{1}{2\sigma^2}\sum(Y_i-\theta x_i^2)^2}$$

So

$$\log L(\theta)=-\frac{n}{2}\log(2\pi\sigma^2)-\frac{1}{2\sigma^2}\sum(Y_i-\theta x_i^2)^2$$

So the MLE of θ must be the value of θ which minimizes $\sum(Y_i-\theta x_i^2)^2$. By part (a), $\hat{\theta}=\frac{\sum x_i^2 Y_i}{\sum x_i^4}$.

(c) $Y_i \sim N(\theta x_i^2, \sigma^2)$. The joint density of $Y_1, \ldots, Y_n$ is

$$\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{-\frac{1}{2\sigma^2}\sum(Y_i-\theta x_i^2)^2}$$

$$=\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^2 e^{-\frac{1}{2\sigma^2}\sum Y_i^2+\frac{\theta}{\sigma^2}\sum x_i^2 Y_i-\frac{\theta^2}{2\sigma^2}\sum x_i^4}$$

By the factorization theorem $(\sum x_i^2 Y_i, \sum Y_i^2)$ is sufficient for (θ, σ^2). [Recall that the x_i's are constants]. Also this is a regular exponential family, so the suff. stat is complete. By the Rao-Blackwell & Lehmann-Scheffe theorems (recall STAT 721), any function of the complete suff. stat $(\sum x_i^2 Y_i, \sum Y_i^2)$ which is an unbiased

(continued)

11.35 (c) (continued)

estimator of θ must be the (unique) UMVU est. of θ.

Now $E\left[\sum x_i^2 Y_i\right] = \sum x_i^2 E Y_i = \sum x_i^2 \theta x_i^2 = \theta \sum x_i^4$.

Hence $E\left[\dfrac{\sum x_i^2 Y_i}{\sum x_i^4}\right] = \theta$.

Hence $\dfrac{\sum x_i^2 Y_i}{\sum x_i^4}$ is the UMVU estimator (or "best unbiased estimator") of θ.

11.38 (a)

The least squares estimator of θ is the value of θ which minimizes $\sum (Y_i - \theta x_i)^2$.

Solving $0 = \frac{d}{d\theta} \sum (Y_i - \theta x_i)^2 = 2 \sum (Y_i - \theta x_i)(-x_i)$

yields $\hat{\theta} = \dfrac{\sum x_i Y_i}{\sum x_i^2}$.

$$\operatorname{Var} \hat{\theta} = \frac{1}{(\sum x_i)^2} \sum x_i^2 \operatorname{Var} Y_i = \frac{1}{(\sum x_i^2)^2} \sum x_i^2 \theta x_i$$

$$= \frac{\theta \sum x_i^3}{(\sum x_i^2)^2}$$

Also $\operatorname{Bias}(\hat{\theta}) = E\hat{\theta} - \theta = \dfrac{\sum x_i \theta x_i}{(\sum x_i)^2} - \theta = \theta\left[\dfrac{\sum x_i^2}{(\sum x_i)^2} - 1\right]$.

11.38 (b)

$$L(\theta) = e^{-\theta \Sigma x_i} \prod_{i=1}^{n} \frac{(\theta x_i)^{Y_i}}{Y_i!}$$

so $\log L(\theta) = -\theta \Sigma x_i + \Sigma Y_i (\log\theta + \log x_i) - \Sigma \log Y_i!$

Setting $\frac{d}{d\theta} \log L(\theta) = 0$ yields $-\Sigma x_i + \frac{\Sigma Y_i}{\theta} = 0$,

from which it is easily seen that the MLE

is $\hat{\theta} = \frac{\Sigma Y_i}{\Sigma x_i}$.

Also $\text{var}(\hat{\theta}) = \frac{1}{(\Sigma x_i)^2} \Sigma \text{var}\, Y_i = \frac{1}{(\Sigma x_i)^2} \theta \Sigma x_i = \frac{\theta}{\Sigma x_i}$.

$\text{Bias}(\hat{\theta}) = E\hat{\theta} - \theta = \frac{E[\Sigma Y_i]}{\Sigma x_i} - \theta = \frac{\theta \Sigma x_i}{\Sigma x_i} - \theta = 0$.

(c) The joint pdf of $Y_1, \ldots, Y_n$ is $e^{-\theta \Sigma x_i} \theta^{\Sigma Y_i} \frac{\prod x_i^{Y_i}}{Y_i!}$,

so by the factorization theorem ΣY_i is sufficient for θ.

It is also complete (family of distributions is regular exponential)

Hence any function of ΣY_i which is an unbiased estimator of θ is UMVU (or "best unbiased") estimator.

But the MLE of θ, $\hat{\theta} = \frac{\Sigma Y_i}{\Sigma x_i}$ (derived in part (b)) is a function of ΣY_i which is unbiased, and so it is UMVUE.

11.38 (e) (continued)

By part (b), the variance of $\hat{\theta} = \frac{\sum Y_i}{\sum x_i}$ is $\frac{\theta}{\sum x_i}$.

We will show that it attains the Cramer-Rao lower bound. We will use the form of the Cramer-Rao inequality as given in Theorem 7.3.9 at the bottom of page 335 of the text. All the regularity conditions to make its use valid apply in our case. (Note that the Y_i's do not need to be i.i.d.). The lower bound for the variance of an UE of θ is

$$\frac{1}{E_\theta\left\{\left[\frac{\partial}{\partial\theta}\log(f(\cancel{x}|\theta)\right]^2\right\}}$$

As we saw in part (b), $\frac{d}{d\theta}\log f(\cancel{x}|\theta) =$

$$= -\sum x_i + \frac{\sum Y_i}{\theta}.$$

So $\left[\frac{d}{d\theta}\log f(y|\theta)\right]^2 = (\sum x_i)^2 - \frac{2(\sum x_i)\sum Y_i}{\theta} + \frac{(\sum Y_i)^2}{\theta^2}$

(continued)

11.38 (c) (continued)

So the denominator of the C.R. lower bound is

$$E\left[\sum x_i^2 - \frac{2(\sum x_i)\sum EY_i}{\theta} + \frac{E\left[(\sum Y_i)^2\right]}{\theta^2}\right]$$

$$= \left(\sum x_i\right)^2 - \frac{2(\sum x_i)(\sum \theta x_i)}{\theta} + \frac{Var(\sum Y_i) + \{E[\sum Y_i]\}^2}{\theta^2}$$

$$= \sum x_i^2 - 2\sum x_i^2 + \frac{\theta \sum x_i + \theta^2 (\sum x_i)^2}{\theta^2} = \frac{\sum x_i}{\theta}.$$

So the C-R lower bound is $\frac{\theta}{\sum x_i}$,

and it is attained by $\hat{\theta} = \frac{\sum Y_i}{\sum x_i}$