STAT 723 Solutions to Assignment #1

(8.6) (a) The joint pdf of $X_1, \ldots, X_n, Y_1, \ldots, Y_m$ is

$$\frac{1}{\theta^n} e^{-\frac{1}{\theta}\Sigma x_i} \cdot \frac{1}{\mu^m} e^{-\frac{1}{\mu}\Sigma y_i} \qquad \text{all } x_i, y_i > 0.$$

The MLE's of θ and μ are $\hat{\theta} = \frac{\sum_1^n x_i}{n}$, $\hat{\mu} = \frac{\sum_1^m y_i}{m}$,

so that the denominator of the LR test statistic is

$$\frac{1}{\left(\frac{\Sigma x_i}{n}\right)^n} e^{-n} \cdot \frac{1}{\left(\frac{\Sigma y_i}{m}\right)^m} e^{-m}.$$

Under $H_0: \theta = \mu$, the MLE of $\theta (= \mu)$ is $\frac{\Sigma x_i + \Sigma y_i}{n+m}$,

so the numerator of the LR test stat is

$$\frac{1}{\left(\frac{\Sigma x_i + \Sigma y_i}{n+m}\right)^{n+m}} e^{-n-m},$$

So the LR test rejects H_0 when $\frac{(\text{const.})(\Sigma x_i)^n (\Sigma y_i)^m}{(\Sigma x_i + \Sigma y_i)^{n+m}}$

is too small.

(b) With $T = \frac{\Sigma X_i}{\Sigma X_i + \Sigma Y_i}$, the LR test statistic

is clearly equivalent to $T^n (1-T)^m$. Note that since $t^n(1-t)^m$ is increasing for $t \in (0, n/(n+m))$ and decreasing for $t > n/(n+m)$, then rejecting H_0 when $T^n(1-T)^m$ is $\leq c$ is equivalent to rejecting when $T \leq c_1$ or $T \leq c_2$ for some c_1, c_2.

8.6 (c) Write $T = \dfrac{1}{1+(\Sigma Y_i)/(\Sigma X_i)}$.

Under H_0, ΣX_i and ΣY_i are indep, with

$\dfrac{\Sigma X_i}{\theta} \sim$ gamma with $\alpha = n$, $\beta = 1$

and $\dfrac{\Sigma Y_i}{\theta} \sim$ gamma, $\alpha = m$, $\beta = 1$.

Equivalently, $\dfrac{2\Sigma X_i}{\theta} \sim \chi^2_{2n}$, $\dfrac{2\Sigma Y_i}{\theta} \sim \chi^2_{2m}$,

so that $\dfrac{\Sigma Y_i / m}{\Sigma X_i / n} \sim F_{m,n}$.

So the null distribution of $F = \dfrac{1}{1+\frac{m}{n}T}$ is $F_{m,n}$,

or the null distribution of T is the distribution of $\left(\frac{1}{F}-1\right)\frac{n}{m}$ where $F \sim F_{m,n}$. You can easily work out the p.d.f. of T under H_0 by making a 1-1 change of variables in the $F_{m,n}$ p.d.f.

<u>Note</u>. Writing the null density of T as $f_0(t)$, the α-level L-R tests rejects H_0 when $T < c_1$ or $T > c_2$, where c_1 and c_2 are uniquely determined by the 2 side conditions: (i) $c_1^n(1-c_1)^m = c_2^n(1-c_2)^m$

and (ii) $\int_0^{c_1} f_0(t)\,dt + \int_{c_1}^{\infty} f_0(t) = \alpha$.

It can be shown that the two integrals above are typically <u>not</u> each $= \alpha/2$.

8.8 (a) Note: in the textbook's notation $N(\theta, a\theta)$ means a normal with $\mu = \theta$ and variance $\sigma^2 = a\theta$.

The joint pdf of $X_1, \ldots X_n$ is

$$L(\theta, a) = \left(\frac{1}{\sqrt{2\pi a\theta}}\right)^n e^{-\frac{1}{2a\theta}\Sigma(X_i-\theta)^2}$$

We need to find $\hat{a}, \hat{\theta}$ which maximize

$$\log L(\theta, a) = -\frac{n}{2}\left(\log(2\pi) + \log a + \log\theta\right) - \frac{1}{2a\theta}\Sigma(X_i-\theta)^2.$$

We will solve $\frac{\partial}{\partial a}\log L = 0$, $\frac{\partial}{\partial\theta}\log L = 0$. If there is a unique soln $(\hat{\theta}, \hat{a})$, then since it (a prior) clear that the sup cannot be found ~~at the~~ as the boundary of the parameter space is approached, then the soln $(\hat{\theta}, \hat{a})$ must be a unique global maximum.

$$\frac{\partial}{\partial\theta}L = 0 \Rightarrow -\frac{n}{2\theta} - \frac{2}{2a\theta}\Sigma(X_i-\theta)(-1) + \frac{1}{2a\theta^2}\Sigma(X_i-\theta)^2 = 0$$

$$\frac{\partial}{\partial a}L = 0 \Rightarrow -\frac{n}{2a} + \frac{1}{2a^2\theta}\Sigma(X_i-\theta)^2 = 0.$$

The solution to the 2^{nd} equation is $a = \frac{\Sigma(X_i-\theta)^2}{n\theta}$

Substituting this into the first equation yields $\theta = \bar{x}$.

Hence the MLE of (θ, a) is $\left(\bar{X}, \frac{\Sigma(X_i-\hat{\theta})^2}{n\bar{X}}\right) = \left(\bar{X}, \frac{S^2}{\bar{X}}\right)$.

So the denom. of the LR statistic is

$$\frac{1}{\sqrt{2\pi(S^2/\bar{X})\bar{X}}} e^{-\frac{1}{2(S^2/\bar{X})\bar{X}} nS^2} = \frac{1}{\sqrt{2\pi S^2}} e^{-n/2}$$

(continued)

8.8 (a) (continued)

Under $H_0: a=1$, the log likelihood function is

$$\log L(\theta) = -\frac{n}{2}\left[\log(2\pi) + \log\theta\right] - \frac{1}{2\theta}\sum(x_i-\theta)^2$$

$$\frac{d}{d\theta}\log L = 0 \Rightarrow -\frac{n}{2} + \frac{1}{\theta}\sum(x_i-\theta) + \frac{1}{2\theta^2}\sum(x_i-\theta)^2 = 0$$

or $\sum(x_i-\theta)^2 + 2\theta\left(\sum x_i - n\theta\right) - n\theta = 0$

or

$$\sum x_i^2 - 2\theta\sum x_i + n\theta^2 + 2\theta\sum x_i - 2n\theta^2 - n\theta = 0$$

or

$$n\theta^2 + n\theta - \frac{\sum x_i^2}{2} = 0, \text{ so } \theta = \frac{-1 \pm \sqrt{1+4(\sum x_i^2)/n}}{2}.$$

The restriction $\theta > 0$ (so Var > 0) forces the MLE of θ under H_0 to be the positive root:

$$\hat\theta = -1 + \sqrt{1+4(\sum x_i)^2/n}.$$

So the LR test rejects H_0 when λ is too small, where

$$\lambda = \frac{\left(\frac{1}{\sqrt{2\pi\hat\theta}}\right)^n e^{-\frac{1}{2\hat\theta}\sum(x_i-\hat\theta)^2}}{\left(\frac{1}{\sqrt{2\pi s^2}}\right)^n e^{-n/2}}$$

where $\hat\theta$ is given above. This test statistic can be simplified a bit.

(b) This is similar to (a) (and probably just as tedious) so I am omitting the solution.

(4 1/2)

(8.14) Using a Normal approximation, the test rejects H_0 if

$$\frac{\sum X_i - n(.49)}{\sqrt{n(.49)(.51)}} > z_{.01} = 2.326 .$$

The power of this test against $H_1: p = .51$ is approximately

$$P_{p=.51}\left[\frac{\sum X_i - n(.51)}{\sqrt{n(.49)(.51)}} + \frac{n(.02)}{\sqrt{n(.49)(.51)}} > 2.326\right]$$

$$\approx P\left[Z > 2.326 - \frac{\sqrt{n}(.02)}{\sqrt{(.49)(.51)}}\right].$$

To make the power $\approx .99$, we solve

$$2.326 - \frac{\sqrt{n}\,(.02)}{\sqrt{(.49)(.51)}} \approx -2.326$$

or $$\frac{\sqrt{n}\,(.02)}{\sqrt{(.49)(.51)}} \approx 4.652$$

or $$n \approx \frac{(4.652)^2(.49)(.51)}{(.02)^2} \approx \underline{13,520}.$$

(8.15) By the N-P lemma, ~~the~~ MP test rejects H_0 when

$$\left[\left(\frac{1}{\sqrt{2\pi\sigma_1^2}}\right)^n e^{-\frac{1}{2\sigma_1^2}\sum X_i^2}\right] \Big/ \left(\frac{1}{\sqrt{2\pi\sigma_0^2}}\right)^n e^{-\frac{1}{2\sigma_0^2}\sum X_i^2} \text{ too large.}$$

Equivalently, reject H_0 when $e^{\left(-\frac{1}{2\sigma_1^2}+\frac{1}{2\sigma_0^2}\right)\sum X_i^2}$ is too large.

Since $\sigma_0 < \sigma_1 \Rightarrow -\frac{1}{2\sigma_1^2}+\frac{1}{2\sigma_0^2} > 0$,

the MP test rejects when $\sum X_i$ is too large.

Under H_0, $\frac{\sum X_i^2}{\sigma_0^2} \sim \chi^2_n$, so $P_0\left[\frac{\sum X_i^2}{\sigma_0^2} > \chi^2_{n,\alpha}\right] = \alpha$

So the MP level α test rejects H_0 when $\sum X_i^2 > \sigma_0^2 \chi^2_{n,\alpha}$.

(8.18) The power against θ is

$$P_\theta\left[\frac{\bar{X}-\theta_0}{\sigma/\sqrt{n}} > c \text{ or } < -c\right] = P_\theta\left[\frac{\bar{X}-\theta}{\sigma/\sqrt{n}} + \frac{(\theta-\theta_0)}{\sigma/\sqrt{n}} > c \text{ or } < -c\right]$$

$$= P\left[\underbrace{Z}_{N(0,1)} > c - \frac{(\theta-\theta_0)}{\sigma/\sqrt{n}}\right] + P\left[Z < -c - \frac{(\theta-\theta_0)}{\sigma/\sqrt{n}}\right]$$

$$= 1 - \Phi\left(c - \frac{(\theta-\theta_0)}{\sigma/\sqrt{n}}\right) + \Phi\left(-c - \frac{(\theta-\theta_0)}{\sigma/\sqrt{n}}\right).$$

(b) $\alpha = .05 \Rightarrow c = z_{.025} = 1.960$. At $\theta = \theta_0 + \sigma$, $\frac{\theta-\theta_0}{\sigma} = 1$.

We need to find the smallest value of n such that

$$\Phi(1.96 - \sqrt{n}) - \Phi(-1.96 - \sqrt{n}) \leq 0.25$$

I wrote a little program to compute this, and found $n = 7$ yields Type II error prob of 0.2964.

8.20

x	1	2	3	4	5	6	7
$f(x\|H_0)$	.01	.01	.01	.01	.01	.01	.94
$f(x\|H_1)$	.06	.05	.04	.03	.02	.01	.79
$r(x)=f(x\|H_1)/f(x\|H_0)$	6	5	4	3	2	1	0.840

By the N-P lemma, the most powerful test has the rejection region $r(x) > C$. The four points $x=1,2,3,4$ have the highest value of $r(x)$ and a total type I error probability of $\alpha = .01+.01+.01+.01 = .04$. So the most powerful level $\alpha = .04$ test rejects H_0 when $r(x) > 2$, i.e. when $x \in \{1,2,3,4\}$.

The prob of Type II error for this test is

$$P_{H_1}[X \in \{1,2,3,4\}] = .06+.05+.04+.03$$

$$= \underline{\underline{0.18}}.$$

8.22 (a) By N-P lemma, MP test rejects H_0 when

$$\frac{\left(\frac{1}{4}\right)^{\Sigma x_i}\left(\frac{3}{4}\right)^{10-\Sigma x_i}}{\left(\frac{1}{2}\right)^{\Sigma x_i}\left(\frac{1}{2}\right)^{10-\Sigma x_i}} = \frac{\left(\frac{3}{4}\right)^{10}\left(\frac{1}{3}\right)^{\Sigma x_i}}{\left(\frac{1}{2}\right)^{10}} \text{ is too large.}$$

Equivalently, reject H_0 when $\sum_{i=1}^{10} X_i$ is too small.

By trial and error $P_{1/2}\left[\Sigma X_i \le 2\right] = \frac{56}{1024} = .0547$.

So MP size 0.0547 test is: Rej. H_0 if $\sum_{i=1}^{n} X \le 2$.

The power of this test is $P_{1/4}\left[X \le 2\right] = \left(\frac{3}{4}\right)^{10} + 10\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^9 + 45\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^8$
$=$ (do the arithmetic).

(b) Size $\alpha = P_{1/2}\left[\sum_{i=1}^{10} X_i \ge 6\right] = \sum_{x=6}^{10}\binom{10}{x}\left(\frac{1}{2}\right)^{10}$

$$= \frac{1}{1024}\left[1+10+45+120+210\right] = \frac{386}{1024}$$

(c) Exact non-randomized α-level tests exist when $\alpha = P_{1/2}\left[\sum_{i=1}^{10} X_i \le c\right]$

for some c. That is the possible α's are

$0, \frac{1}{1024}, \frac{11}{1024}, \frac{56}{1024}, \frac{176}{1024}, \ldots\ldots, \frac{1023}{1024}$ and 1.

8.25 (a) $X \sim N(\theta, \sigma^2)$. Fix $\theta_0 < \theta_1$. Then $f(x|\theta_1)/f(x|\theta_0) =$

$$= e^{-\frac{1}{2\sigma^2}\left[(x-\theta_1)^2 - (x-\theta_0)^2\right]} = \exp\left\{-\frac{1}{2\sigma^2}\left[x^2 - 2\theta_1 x + \theta_1^2 - x^2 + 2\theta_0 x - \theta_0^2\right]\right\}$$

$$= e^{-\frac{1}{2\sigma^2}(\theta_1^2 - \theta_0^2)} \, e^{\frac{1}{\sigma^2}(\theta_1 - \theta_0)x}$$, which is monotone incr. in x.

(b) $\theta_0 < \theta_1$, then $f(x|\theta_1)/f(x|\theta_0) = e^{-\theta_1} \frac{\theta_1^x}{x!} \Big/ \left[e^{-\theta_0} \theta_0^x / x!\right]$

$$= e^{-\theta_1 + \theta_0} \left(\frac{\theta_1}{\theta_0}\right)^x$$, which is increasing in x.

(c) $\theta_0 < \theta_1$, then $f(x|\theta_1)/f(x|\theta_0) = \binom{n}{x}\theta_1^x (1-\theta_1)^{n-x} \Big/ \left[\binom{n}{x}\theta_0^x (1-\theta_0)^{n-x}\right]$

$$= \left(\frac{1-\theta_1}{1-\theta_0}\right)^n \left[\frac{\theta_1}{1-\theta_1} \Big/ \frac{\theta_0}{1-\theta_0}\right]^x$$, which is increasing in x.

8.28 (a) Fix $\theta_0 < \theta_1$. Then $\frac{f(x|\theta_1)}{f(x|\theta_0)} =$

$$= \frac{e^{(x-\theta_1)}}{\left(1+e^{(x-\theta_1)}\right)^2} \frac{\left(1+e^{(x-\theta_0)}\right)^2}{e^{x-\theta_0}} = e^{\theta_0 - \theta_1} \left[\frac{1+e^{(x-\theta_0)}}{1+e^{(x-\theta_1)}}\right]^2.$$

Since $\frac{1+e^{(x-\theta_0)}}{1+e^{(x-\theta_1)}}$ is positive for all x, it suffices to show that $\frac{d}{dx}\left[\frac{1+e^{(x-\theta_0)}}{1+e^{(x-\theta_1)}}\right] \geq 0$ for all x.

The above derivative has the same sign as

$$\left[1+e^{(x-\theta_1)}\right] e^{(x-\theta_0)} - \left(1+e^{(x-\theta_0)}\right) e^{(x-\theta_1)}$$

$$= e^{(x-\theta_0)} - e^{(x-\theta_1)} = e^x e^{(-\theta_0+\theta_1)} > 0$$ for all x.

Thus the family has MLR in x. (continued)

8.28 (continued)

(b) Since the family has MLR, the MP size α test must have rejection region of the form $X > c$.

$$P_0[X>c] = \int_c^\infty \frac{e^x}{(1+e^x)^2}\,dx \overset{v=1+e^{-x}}{=} \int_{1+e^c}^\infty \frac{dv}{v^2}$$

$$= -\frac{1}{v}\Big]_{1+e^c}^{\infty} = \frac{1}{1+e^c} = \alpha.$$

So $c = \log\left(\frac{1-\alpha}{\alpha}\right)$, and MP level α test rejects H_0 when $X > \log\left(\frac{1-\alpha}{\alpha}\right)$.

When $\alpha = .2$, reject when $X > \log 4 = 1.3863$.

$$P_{\theta=1}[\text{Reject } H_0] = \int_{\log 4}^\infty \frac{e^{(x-1)}}{(1+e^{(x-1)})^2}\,dx = \int_{y=\log 4-1}^\infty \frac{e^y}{(1+e^y)^2}\,dy$$

$$= \frac{1}{1+e^{(\log 4-1)}} = \frac{1}{1+4/e} = 0.4046$$

So the Type II error probability is $1 - 0.4046 = 0.5954$.

(c) One of the theorems was that when a family has MLR in X, then the UMP size α test of $H_0: \theta \le \theta_0$ vs $H_1: \theta < \theta_0$ has rejection region of form $X > c$, with c determined by $P_{\theta_0}[X>c] = \alpha$.

8.29 (a) The likelihood ratio for $\theta=0$ vs $\theta>0$ is

$$\frac{f(x|\theta)}{f(x|0)} = \frac{1+x^2}{1+(x-\theta)^2}.$$

Now $\frac{d}{dx}\left[\frac{1+x^2}{1+(x-\theta)^2}\right]$ has the same sign as

$$h(x) \stackrel{\text{def}}{=} \left[1+(x-\theta)^2\right]\cdot 2x - (1+x^2)\,2(x-\theta).$$

Note that $h(0) = 0 - 2(-\theta) > 0$,

$h(\theta) = 2\theta - (1+\theta^2)\cdot 0 = 2\theta > 0$,

but that $h(2\theta) = (1+\theta^2)(4\theta) - (1+4\theta^2)(2\theta)$

$= 2\theta(1-2\theta^2)$, which is <0 whenever $\theta > \frac{1}{\sqrt{2}}$.

Hence the likelihood ratio cannot be monotone in x.

(b) By the N-P lemma, $\phi(x) = 1$ if $\underbrace{\frac{1+x^2}{1+(x-1)^2}}$ $> k$.

We will find k so that the inequality is equivalent to $1 < x < 3$.

Write the rejection region as $1+x^2 > k\left[1+(x^2-2x+1)\right]$

OR $(k-1)x^2 - 2kx + (2k-1) \leq 0$.

So $x=1$ and $x=3$ must be the two real solutions to the quadratic equation

$(k-1)x^2 - 2kx + (2k-1) = 0$ (continued)

8.29 (b) continued

The solutions are

$$x = \frac{2k \pm \sqrt{4k^2 - 4(k-1)(2k-1)}}{2(k-1)}.$$

This implies that

$\frac{2k}{2(k-1)}$ must be equidistant from 1 and 3.

Hence $k=2$. As a check, the two equations to the quadratic equation with $k=2$ are seen to be 2 ± 1, i.e. 1 and 3.

It remains to check that $x^2 - 4x + 3 < 0$ for all $x \in (1,3)$. We need only check at one such x: at $x=2$, we obtain $4-8+3<0$.

Hence the MP test is $\phi(x) = 1$ if $1 < x < 3$
$= 0$ otherwise.

The type I error prob. is $P_0[1 < X < 3] =$

$$= \int_1^3 \frac{1}{\pi}\frac{1}{1+x^2}\,dx = \frac{1}{\pi}\left[\tan^{-1}(3) - \tan^{-1}(1)\right] = \frac{1}{\pi}\left[1.2490 - \frac{\pi}{4}\right]$$
$$= \underline{\underline{0.1476}}.$$

The Type II error prob is $1 - P_1[1 < X < 3] =$

$$= 1 - \int_1^3 \frac{1}{\pi}\frac{1}{1+(x-1)^2}\,dx = \int_{y=0}^{2} \frac{1}{\pi}\frac{1}{1+y^2}\,dy = 1 - \frac{1}{\pi}\tan^{-1}(2)$$
$$= \underline{\underline{0.6476}}$$

8.29 (c) We will show that the test is not UMP.

We need only show that the level $\alpha = 0.1476$ test, $[\phi(x) = 1 \text{ if } 1 < x < 3]$ is not MP against some alternative $H_1: \theta = \theta_1 \neq 1$. Consider $\theta_1 = 2$.

The MP level α test of $H_0: \theta = 0$ vs $H_1: \theta = 2$ rejects when $\frac{1+x^2}{1+(x-2)^2} > k$ for some k.

We will show that there is no $k > 0$ for which this rejection region coincides with $1 < x < 3$. Suppose so. Then the rejection region is

$$1 + x^2 > k\left[1 + x^2 - 4x + 4\right]$$

OR $(k-1)x^2 - 4kx + (5k-1) < 0$.

So 1 and 3 must be $\frac{4k \pm \sqrt{16k^2 - 4(k-1)(5k-1)}}{2(k-1)}$

Hence $\frac{4k}{2(k-1)} = 2$, which is impossible for all $k > 0$.

Since no such k exists, $\phi(x) = 1$ if $1 < x < 3$ is not MP level $\alpha = .1476$ against $H_1: \theta = 2$. Hence it cannot be UMP against $H_1: \theta > 0$.

8.30

(a) Fix $0 < \theta_1 < \theta_2 < \infty$.

Then $\dfrac{f(x|\theta_2)}{f(x|\theta_1)} = \dfrac{\theta_2}{\theta_1} \dfrac{\theta_1^2 + x^2}{\theta_2^2 + x^2}$

Then $\dfrac{d}{dx}\left[\dfrac{f(x|\theta_2)}{f(x|\theta_1)}\right]$ has the same sign as

$$(\theta_2^2 + x^2)(2x) - (\theta_1^2 + x^2)\, 2x = 2x\,(\theta_2^2 - \theta_1^2)$$

So $\dfrac{f(x|\theta_2)}{f(x|\theta_1)}$ is decreasing when $x < 0$, and increasing when $x > 0$.

So the family does not have monotone LR.

(b) $f(x|\theta) = \dfrac{\theta}{\pi} \dfrac{1}{\theta^2 + x^2} = \dfrac{\theta}{\pi} \dfrac{1}{\theta^2 + (|x|)^2}$,

so by the Factorization Theorem, $|X|$ is sufficient for θ.

The pdf of $Y = |X|$ is

$$f(y|\theta) = \frac{2\theta}{\pi} \frac{1}{\theta^2 + y^2}, \quad 0 < y < \infty, \quad \theta > 0$$
$$= 0, \quad y < 0.$$

So, by a similar calculation to part (a), with $0 < \theta_1 < \theta_2 < \infty$

$\dfrac{d}{dy}\left[\dfrac{f(y|\theta_2)}{f(y|\theta_2)}\right]$ has the same sign as $2y\,(\theta_2^2 - \theta_1^2)$, for all $y > 0$.

But $2y\,(\theta_2^2 - \theta_1^2) > 0$ for all $y > 0$, so the family has MLR.

(8.33) (a) Find k so that $P_0[Y_n \geq 1 \text{ or } Y_1 \geq k] = \alpha$.

$P_0[Y_n \geq 1] = 0$, so choose k so that

$P_0[Y_1 \geq k] = P_0[X_1 \geq k, \ldots, X_n \geq k] = (1-k)^n = \alpha$.

So $k = 1 - \alpha^{1/n}$.

(b) First note that $0 < \alpha < 1 \Rightarrow 0 < k < 1$.

When $\theta \geq k$, $P_\theta[Y_n \geq 1 \text{ or } Y_1 \geq k] \geq P[Y_1 \geq k] = 1$,

so ~~that~~ the power is $= 1$ when $\theta \geq k$.

So consider the case $0 < \theta < k$ (< 1). Then

$P_\theta[Y_1 \geq k \text{ or } Y_n \geq 1] = P_\theta[Y_1 \geq k] + P_\theta[Y_n \geq 1] - P_\theta[Y_1 \geq k \text{ and } Y_n \geq 1]$

$= P_\theta[Y_1 \geq k] + P_\theta[Y_n \geq 1] - \{P_\theta(Y_1 \geq k) - P_\theta[(Y_1 \geq k) \text{ and } (Y_n \leq 1)]\}$

$= P_\theta[Y_n \geq 1] + P_\theta[(Y_1 \geq k) \text{ and } (Y_n \leq 1)]$

Now $P_\theta[Y_n \geq 1] = 1 - P_\theta[Y_n \leq 1] = 1 - P_\theta[X_1 \leq 1, \ldots, X_n \leq 1]$

$= 1 - (1-\theta)^n$,

and $P_\theta[(Y_1 \geq k) \text{ and } (Y_n \leq 1)] = P_\theta[\text{all } X_i\text{'s are in } (k, 1)]$

$= (1-k)^n$. (continued)

8.33 (b) (continued). So when $0<\theta<k$ (<1),

we have $P_\theta[\text{Reject } H_0] = 1-(1-\theta)^n + (1-k)^n$
$= 1-(1-\theta)^n + \alpha$.

So the power function is

$$P_\theta[\text{Reject } H_0] = 1-(1-\theta)^n + \alpha \quad \text{if } 0 \le \theta \le k$$
$$= 1 \quad \text{if } \theta \ge k.$$

(c) (Informal sketch of proof). For testing $H_0: \theta = 0$ against a particular alternative $\theta = \theta_1 > 0$, the likelihood ratio is $\dfrac{f(x_1,\ldots,x_n|\theta_1)}{f(x_1,\ldots,x_n|0)} = \dfrac{I_{[y_1>\theta_1]}\, I_{[y_n<\theta_1+1]}}{I_{[y_1>0]}\, I_{[y_n<1]}}$.

The only possible values of the likelihood ratio are ∞, 1 and 0. [The undefined $0/0$ case only occurs on sets with prob 0 under both H_0 and H_1, and so can be ignored.]. What the N-P lemma says in such a case is that a MP level α test of $H_0: \theta=0$ vs $H_1: \theta=\theta_1$ is any test that includes all points where the ratio is ∞ in the rejection region, and also contains any subset of the region where the (cont.)

8.33 (c) (continued)

ratio is 1, provided that the prob, under H_0, of that subset is α. [So for $H_0: \theta = 0$ vs $H_1: \theta = \theta_1$, there are an uncountably infinite number of MP level α tests, all tied in power against θ_1].

Now a careful consideration of the particular level α test described at the beginning of the problem shows that this test is one of the tests that is MP level α against $H_1: \theta = \theta_1$ — and this is true for all $\theta_1 > 0$. Hence the test is UMP level α against $H_1: \theta > 0$.

(d) [This look like a defective problem]. Take $n = 1$ and $k = .9$. Then $P_0[X_1 > k] = .1$ and the power against $\theta > 1$ is 1, since $P_\theta[X_1 > .9] = 1$ when $\theta > .9$.

8.34 (a) "θ is a location parameter" means that the distribution of $T-\theta$ under parameter θ is the same as the distribution of T under parameter 0.

Now fix $\theta_1 \le \theta_2$. Then

$$P_{\theta_1}[T > c] = P_{\theta_1}[T-\theta_1 > c-\theta_1] = P_0[T > c-\theta_1]$$
$$= P_{\theta_2}[T-\theta_2 > c-\theta_1]$$
$$= P_{\theta_2}[T > c + \underbrace{(\theta_2-\theta_1)}_{\ge 0}] \le P_{\theta_2}[T > c].$$

(b) Solution deferred — I am still trying to find a proof.

8.37 Solution omitted because I ran out of time. This problem is routine, and I think that I have already done all (or a large part of it) as an example in class.